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AN ALEXSANDROV TYPE THEOREM FOR k-CONVEX FUNCTIONS

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In this note we show that k-convex functions on $\mathbb{R}^n$ are twice differentiable almost everywhere for every positive integer $k > n/2$. This generalises Alexsandrov's classical theorem for convex functions.

1. INTRODUCTION

A classical result of Alexsandrov [**1**] asserts that convex functions in $\mathbb{R}^n$ are twice differentiable *almost everywhere*, (see also [**3, 8**] for more modern treatments). It is well known that Sobolev functions $u \in W^{2,p}$, for $p > n/2$ are twice differentiable almlost everywhere. The following weaker notion of convexity known as k-convexity was introduced by Trudinger and Wang [**12, 13**]. Let $\Omega \subset \mathbb{R}^n$ be an open set and $C^2(\Omega)$ be the class of continuously twice differentiable functions on Ω. For $k = 1, 2, \dots, n$ and a function $u \in C^2(\Omega)$, the k-Hessian operator, F_k, is defined by

$$F_k[u] := S_k\big(\lambda(\nabla^2 u)\big)\,, \tag{1.1}$$

where $\nabla^2 u = (\partial_{ij} u)$ denotes the Hessian matrix of the second derivatives of u, $\lambda(A) = (\lambda_1, \lambda_2, \dots, \lambda_n)$ the vector of eigenvalues of an $n \times n$ matrix $A \in \mathbb{R}^{n\times n}$ and $S_k(\lambda)$ is the k-th elementary symmetric function on $\mathbb{R}^n$, given by

$$S_k(\lambda) := \sum_{i_1<\cdots<i_k} \lambda_{i_1} \cdots \lambda_{i_k}\,. \tag{1.2}$$

Alternatively we may write

$$F_k[u] = [\nabla^2 u]_k, \tag{1.3}$$

where $[A]_k$ denotes the sum of the $k \times k$ principal minors of an $n \times n$ matrix A, which may also be called the k-trace of A. The study of k-Hessian operators was initiated by Caffarelli, Nirenberg and Spruck [**2**] and Ivochkina [**6**] and further developed by Trudinger and Wang [**10, 12, 13, 14, 15**].

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A function $u \in C^2(\Omega)$ is called *k-convex* in Ω if $F_j[u] \geqslant 0$ in Ω for $j = 1, 2, \dots, k$; that is, the eigenvalues $\lambda(\nabla^2 u)$ of the Hessian $\nabla^2 u$ of u lie in the closed convex cone given by

$$\Gamma_k := \left\{\lambda \in \mathbb{R}^n : S_j(\lambda) \geqslant 0,\ j = 1, 2, \dots, k\right\}. \tag{1.4}$$

(see [**2**] and [**13**] for the basic properties of Γ_k.) We notice that $F_1[u] = \Delta u$, is the Laplacian operator and 1-convex functions are subharmonic. When $k = n$, $F_n[u] = \det(\nabla^2 u)$, the Monge-Ampère operator and n-convex functions are convex. To extend the definition of k-convexity for non-smooth functions we adopt a *viscosity* definition as in [**13**]. *An upper semi-continuous function $u : \Omega \to [-\infty, \infty)$ ($u \not\equiv -\infty$ on any connected component of Ω) is called k-convex if $F_j[q] \geqslant 0$, in Ω for $j = 1, 2, \dots, k$, for every quadratic polynomial q for which the difference $u - q$ has a finite local maximum in Ω.* Henceforth, we shall denote the class of k-convex functions in Ω by $\Phi^k(\Omega)$. When $k = 1$ the above definition is equivalent to the usual definition of subharmonic function, see for example ([**5**, Section 3.2]) or ([**7**, Section 2.4]). Thus $\Phi^1(\Omega)$ is the class of subharmonic functions in Ω. We notice that $\Phi^k(\Omega) \subset \Phi^1(\Omega) \subset L^1_{\mathrm{loc}}(\Omega)$ for $k = 1, 2, \dots, n$, and a function $u \in \Phi^n(\Omega)$ if and only if it is convex on each component of Ω. Among other results Trudinger and Wang [**13**] (Lemma 2.2) proved that $u \in \Phi^k(\Omega)$ if and only if

$$\int_\Omega u(x)\Big(\sum_{i,j}^{n} a^{ij}\partial_{ij}\phi(x)\Big)\,dx \geqslant 0 \tag{1.5}$$

for all smooth compactly supported functions $\phi \geqslant 0$, and for all constant $n \times n$ symmetric matrices $A = (a^{ij})$ with eigenvalues $\lambda(A) \in \Gamma_k^*$, where Γ_k^* is the dual cone defined by

$$\Gamma_k^* := \left\{\lambda \in \mathbb{R}^n \ :\ \langle \lambda, \mu\rangle \geqslant 0 \ \text{ for all } \ \mu \in \Gamma_k\right\}. \tag{1.6}$$

In this note we prove the following Alexsandrov type theorem for k-convex functions.

Theorem 1.1. *Let $k > n/2$, $n \geqslant 2$ and $u : \mathbb{R}^n \to [-\infty, \infty)$ ($u \not\equiv -\infty$ on any connected subsets of $\mathbb{R}^n$), be a k-convex function. Then u is twice differentiable almost everywhere. More precisely, we have the Taylor's series expansion for $\mathcal{L}^n$ x almost everywhere,*

$$\left|u(y) - u(x) - \big\langle \nabla u(x)\, y - x\big\rangle - \frac{1}{2}\big\langle \nabla^2 u(x)(y - x)\ y - x\big\rangle\right| = o\big(|y - x|^2\big), \tag{1.7}$$

as $y \to x$.

In Section 3 (see Theorem 3.2.), we also prove that the absolutely continuous part of the k-Hessian measure (see [**12, 13**]) $\mu_k[u]$, associated to a k-convex function for $k > n/2$ is represented by $F_k[u]$. For the Monge-Ampère measure $\mu[u]$ associated to a convex function u, a similar result is obtained in [**16**].

To conclude this introduction we note that it is equivalent to assume only $F_k[q] \geqslant 0$, in the definition of k-convexity [**13**]. Moreover Γ_k may also be characterised as the closure of the positivity set of S_k containing the positive cone Γ_n, [**2**].

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2. Notations and preliminary results

Throughout the text we use the following standard notations. $|\cdot|$ and $\langle\cdot,\cdot\rangle$ will stand for the Euclidean norm and inner product in $\mathbb{R}^n$, and $B(x,r)$ will denote the open ball in $\mathbb{R}^n$ of radius r centred at x. For measurable $E\subset\mathbb{R}^n$, $\mathcal{L}^n(\mathrm{E})$ will denote its Lebesgue measure. For a smooth function u, the gradient and Hessian of u are denoted by $\nabla u=(\partial_1 u,\dots,\partial_n u)$ and $\nabla^2 u=(\partial_{ij}u)_{1\leqslant i,j\leqslant n}$ respectively. For a locally integrable function f, the distributional gradient and Hessian are denoted by $Df=(D_1 f,\dots,D_n f)$ and $D^2 u=(D_{ij}u)_{1\leqslant i,j\leqslant n}$ respectively.

For the convenience of the readers, we cite the following Hölder and gradient estimates for k-convex functions, and the weak continuity result for k-Hessian measures, [**12, 13**].

Theorem 2.1. ([**13**, Theorem 2.7].) *For $k>n/2$, $\Phi^k(\Omega)\subset C^{0,\alpha}_{\mathrm{loc}}(\Omega)$ with $\alpha:=2-n/k$ and for any subdomain $\Omega'\subset\subset\Omega$, $u\in\Phi^k(\Omega)$, there exists $C>0$, depending only on n and k such that*

$$\sup_{\substack{x,y\in\Omega'\\ x\neq y}} d_{x,y}^{n+\alpha}\frac{|u(x)-u(y)|}{|x-y|^\alpha}\leqslant C\int_{\Omega'}|u|, \tag{2.1}$$

where $d_x:=\operatorname{dist}(x\,,\partial\Omega')$ and $d_{x,y}:=\min\{d_x\,,\,d_y\}$.

Theorem 2.2. ([**13**, Theorem 4.1].) *For $k=1,\dots,n$, and $0<q<nk/(n-k)$, the space of k-convex functions $\Phi^k(\Omega)$ lies in the local Sobolev space $W^{1,q}_{\mathrm{loc}}(\Omega)$. Moreover, for any $\Omega'\subset\subset\Omega''\subset\subset\Omega$ and $u\in\Phi^k(\Omega)$ there exists $C>0$, depending on n, k, q, Ω' and Ω'', such that*

$$\left(\int_{\Omega'}|Du|^q\right)^{1/q}\leqslant C\int_{\Omega''}|u|. \tag{2.2}$$

Theorem 2.3. ([**13**, Theorem 1.1].) *For any $u\in\Phi^k(\Omega)$, there exists a Borel measure $\mu_k[u]$ in Ω such that*

(i) *$\mu_k[u](V)=\int_V F_k[u](x)\,dx$ for any Borel set $V\subset\Omega$, if $u\in C^2(\Omega)$ and*

(ii) *if $(u_m)_{m\geqslant1}$ is a sequence in $\Phi^k(\Omega)$ converging in $L^1_{\mathrm{loc}}(\Omega)$ to a function $u\in\Phi^k(\Omega)$, the sequence of Borel measures $\left(\mu_k[u_m]\right)_{m\geqslant1}$ converges weakly to $\mu_k[u]$.*

Let us recall the definition of the dual cones, [**11**]

$$\Gamma_k^*:=\left\{\lambda\in\mathbb{R}^n:\langle\lambda,\mu\rangle\geqslant0\ \text{ for all }\ \mu\in\Gamma_k\right\},$$

which are also closed convex cones in $\mathbb{R}^n$. We notice that $\Gamma_j^*\subset\Gamma_k^*$ for $j\leqslant k$ with $\Gamma_n^*=\Gamma_n=\{\lambda\in\mathbb{R}^n:\lambda_i\geqslant0,\ \ j=1,2,\dots,n\}$, Γ_1^* is the ray given by

$$\Gamma_1^*=\left\{t(1,\dots,1):t\geqslant0\right\},$$

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and Γ_2^* has the following interesting characterisation,

$$\Gamma_2^* = \left\{\lambda \in \Gamma_n : |\lambda|^2 \leqslant \frac{1}{n-1}\Big(\sum_{i=1}^n \lambda_i\Big)^2\right\}. \tag{2.3}$$

We use this explicit representation of Γ_2^* to establish that the distributional derivatives $D_{ij}u$ of the k-convex function u are signed Borel measures for $k \geqslant 2$, (see also [**13**]).

Theorem 2.4. *Let $2 \leqslant k \leqslant n$ and $u : \mathbb{R}^n \to [-\infty, \infty)$, be a k-convex function. Then there exist signed Borel measures $\mu^{ij} = \mu^{ji}$ such that*

$$\int_{\mathbb{R}^n} u(x)\partial_{ij}\phi(x)\,dx = \int_{\mathbb{R}^n} \phi(x)\,d\mu^{ij}(x), \quad \text{for } i,\, j = 1, 2, \ldots, n, \tag{2.4}$$

for all $\phi \in C_c^\infty(\mathbb{R}^n)$.

Proof: Let $k \geqslant 2$ and $u \in \Phi^k(\mathbb{R}^n)$. Since $\Phi^k(\mathbb{R}^n) \subset \Phi^2(\mathbb{R}^n)$ for $k \geqslant 2$, it is enough to prove the theorem for $k = 2$. Let u be a 2-convex function in $\mathbb{R}^n$. For $A \in \mathbb{S}^{n\times n}$, the space of $n \times n$ symmetric matrices, define the distribution $T_A : C_c^2(\mathbb{R}^n) \to \mathbb{R}$, by

$$T_A(\phi) := \int_{\mathbb{R}^n} u(x) \sum_{i,j}^n a^{ij}\partial_{ij}\phi(x)\,dx.$$

By (1.5), $T_A(\phi) \geqslant 0$ for $A \in \mathbb{S}^{n\times n}$ with eigenvalues $\lambda(A) \in \Gamma_2^*$, and $\phi \geqslant 0$. Therefore, by Riesz representation (see for example [**9**, Theorem 2.14] or [**3**, Theorem 1, Section 1.8]), there exist a Borel measure μ^A in $\mathbb{R}^n$, such that

$$T_A(\phi) = \int_{\mathbb{R}^n} \phi \sum_{i,j}^n a^{ij}D_{ij}u\,dx = \int_{\mathbb{R}^n} \phi\,d\mu^A, \tag{2.5}$$

for all $\phi \in C_c^2(\mathbb{R}^n)$ and all $n \times n$ symmetric matrices A with $\lambda(A) \in \Gamma_2^*$. In order to prove that the second order distributional derivatives $D_{ij}u$ of u to be signed Borel measures, we need to make special choices for the matrix A. By taking $A = I_n$, the identity matrix, $\lambda(A) \in \Gamma_1^* \subset \Gamma_2^*$, we obtain a Borel measure μ^{I_n} such that

$$\int_{\mathbb{R}^n} \phi \sum_{i=1}^n D_{ii}u\,dx = \int_{\mathbb{R}^n} \phi\,d\mu^{I_n}, \tag{2.6}$$

for all $\phi \in C_c^2(\mathbb{R}^n)$. Therefore, the trace of the distributional Hessian D^2u, is a Borel measure. For each $i = 1, \ldots, n$, let A_i be the diagonal matrix with all entries 1 but the i-th diagonal entry being 0. Then by the characterisation of Γ_2^* in (2.3), it follows that $\lambda(A_i) \in \Gamma_2^*$. Hence there exist a Borel measure μ^i in $\mathbb{R}^n$ such that

$$\int_{\mathbb{R}^n} \phi \sum_{j\neq i}^n D_{jj}u\,dx = \int_{\mathbb{R}^n} \phi\,d\mu^i, \tag{2.7}$$

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for all $\phi \in C_c^2(\mathbb{R}^n)$. From (2.6) and (2.7) it follows that the diagonal entries $D_{ii}u = \mu^{I_n} - \mu^i := \mu^{ii}$ are signed Borel measure and

$$\int_{\mathbb{R}^n} u\partial_{ij}\phi\, dx = \int_{\mathbb{R}^n} \phi\, d\mu^{ii}, \tag{2.8}$$

for all $\phi \in C_c^2(\mathbb{R}^n)$. Let $\{e_1, \dots, e_n\}$ be the standard orthonormal basis in $\mathbb{R}^n$ and for $a, b \in \mathbb{R}^n$, $a \otimes b := (a^i b^j)$, denotes the $n \times n$ rank-one matrix. For $0 < t < 1$ and $i \neq j$, let us define $A_{ij} := I_n + t[e_i \otimes e_j + e_j \otimes e_i]$. By a straight forward calculation, it is easy to see that the vector of eigenvalues is $\lambda(A_{ij}) = (1-t, 1+t, 1, \dots, 1) \in \Gamma_2^*$, for $0 < t < \big(n/2(n-1)\big)^{1/2}$. Note that for this choice of A_{ij}

$$\sum_{k,l=1}^{n} a^{kl}\partial_{kl}\phi = \sum_{k=1}^{n} \partial_{kk}\phi + 2t\, \partial_{ij}\phi\,.$$

Thus for $i \neq j$, (2.5) and (2.6) yields

$$\begin{aligned}\int_{\mathbb{R}^n} u\, \partial_{ij}\phi\, dx &= \frac{1}{2t}\left[\int_{\mathbb{R}^n} u \sum_{k,l=1}^{n} a^{kl}\partial_{kl}\phi\, dx - \int_{\mathbb{R}^n} u \sum_{k=1}^{n} \partial_{kk}\phi\, dx\right] \\ &= \frac{1}{2t}\left[\int_{\mathbb{R}^n} \phi\, d\mu^{A_{ij}} - \int_{\mathbb{R}^n} \phi\, d\mu^{I_n}\right] \\ &= \int_{\mathbb{R}^n} \phi\, d\mu^{ij}\,, \end{aligned}\tag{2.9}$$

where

$$\mu^{ij} := \frac{1}{2t}(\mu^{A_{ij}} - \mu^{I_n}) = \frac{1}{2t}\left(\mu^{A_{ij}} - \sum_{k=1}^{n} \mu^{kk}\right).$$

Therefore $D_{ij}u = \mu^{ij}$ are signed Borel measures and satisfy the identity (2.4). ∎

A function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ is said to have *locally bounded variation* in $\mathbb{R}^n$ if for each bounded open subset Ω' of $\mathbb{R}^n$,

$$\sup\left\{\int_{\Omega'} f \operatorname{div}\phi\, dx : \phi \in C_c^1(\Omega'; \mathbb{R}^n),\ \big|\phi(x)\big| \leqslant 1 \text{ for all } x \in \Omega'\right\} < \infty\,.$$

We use the notation $BV_{\mathrm{loc}}(\mathbb{R}^n)$ to denote the space of such functions. For the theory of *functions of bounded variation* readers are referred to [**4, 17, 3**].

THEOREM 2.5. *Let $n \geqslant 2$, $k > n/2$ and $u : \mathbb{R}^n \to [-\infty, \infty)$, be a k-convex function. Then u is differentiable almost everywhere $\mathcal{L}^n$ and $\partial_i u \in BV_{\mathrm{loc}}(\mathbb{R}^n)$, for all $i = 1, \dots, n$.*

PROOF: Observe that for $k > n/2$, we can take $n < q < nk/(n-k)$ and by the gradient estimate (2.2), we conclude that k-convex functions are differentiable $\mathcal{L}^n$ almost

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everywhere x. Let $\Omega' \subset\subset \mathbb{R}^n$, $\phi = (\phi^1, \dots, \phi^n) \in C_c^1(\Omega'; \mathbb{R}^n)$ such that $|\phi(x)| \leqslant 1$ for $x \in \Omega'$. Then by integration by parts and the identity (2.4), we have for $i = 1, \dots, n$,

$$\begin{aligned}\int_{\Omega'} \frac{\partial u}{\partial x_i} \operatorname{div} \phi \, dx &= -\sum_{j=1}^{n} \int_{\Omega'} u \frac{\partial^2 \phi^j}{\partial x_i \partial x_j} \, dx \\ &= -\sum_{j=1}^{n} \int_{\Omega'} \phi^j \, d\mu^{ij} \\ &\leqslant \sum_{j=1}^{n} |\mu^{ij}|(\Omega') < \infty ,\end{aligned}$$

where $|\mu^{ij}|$ is the total variation of the Radon measure μ^{ij}. This proves the theorem. □

3. Twice differentiability

Let u be a k-convex function, $k \geqslant 2$. Then by the Theorem 2.4, we have $D^2 u = (\mu^{ij})_{i,j}$, where μ^{ij} are Radon measures. By Lebesgue's Decomposition Theorem, we may write

$$\mu^{ij} = \mu^{ij}_{\mathrm{ac}} + \mu^{ij}_{\mathrm{s}} \quad \text{for } i, j = 1, \dots, n ,$$

where μ^{ij}_{ac} is absolutely continuous with respect to $\mathcal{L}^n$ and μ^{ij}_{s} is supported on a set with Lebesgue measure zero. Let u_{ij} be the density of the absolutely continuous part, that is, $d\mu^{ij}_{\mathrm{ac}} = u_{ij} \, dx$, $u_{ij} \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Set $u_{ij} := \partial_{ij} u$, $\nabla^2 u := \partial_{ij} u = (u_{ij})_{i,j} \in L^1_{\mathrm{loc}}(\mathbb{R}^n; \mathbb{R}^{n \times n})$ and $[D^2 u]_s := (\mu^{ij}_{\mathrm{s}})_{i,j}$. Thus the vector valued Radon measure $D^2 u$ can be decomposed as $D^2 u = [D^2 u]_{\mathrm{ac}} + [D^2 u]_s$, where $d[D^2 u]_{\mathrm{ac}} = \nabla^2 u \, dx$. Now we are in a position prove theorem 1.1. To carry out the proof, we use a similar approach to Evans and Gariepy, see [**3**, Section 6.4].

Proof of Theorem 1.1: Let $n \geqslant 2$ and u be a k-convex function on $\mathbb{R}^n$, $k > n/2$. Then by Theorem 2.4, and Theorem 2.5, we have for $\mathcal{L}^n$ almost everywhere x

$$\lim_{r \to 0} ⨍_{B(x,r)} \left|\nabla u(y) - \nabla u(x)\right| dy = 0 , \tag{3.1}$$

$$\lim_{r \to 0} ⨍_{B(x,r)} \left|\nabla^2 u(y) - \nabla^2 u(x)\right| dy = 0 \tag{3.2}$$

and

$$\lim_{r \to 0} \frac{|[D^2 u]_s|(B(x,r))}{r^n} = 0 . \tag{3.3}$$

where $⨍_E f \, dx$ we denote the mean value $(\mathcal{L}^n(E))^{-1} \int_E f \, dx$. Fix a point x for which (3.2)–(3.3) holds. Without loss generality we may assume $x = 0$. Then following similar calculations as in the proof of [**3**, Theorem 1, Section 6.4], we obtain,

$$⨍_{B(r)} \left|u(y) - u(0) - \langle \nabla u(0), y \rangle - \frac{1}{2} \langle \nabla^2 u(0) y, y \rangle \right| dy = o(r^2) , \tag{3.4}$$

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as $r \to 0$. In order to establish

$$\sup_{B(r/2)} \left| u(y) - u(0) - \langle \nabla u(0), y \rangle - \frac{1}{2} \langle \nabla^2 u(0) y, y \rangle \right| = o(r^2) \quad \text{as } r \to 0, \tag{3.5}$$

we need the following lemma.

LEMMA 3.1. *Let $h(y) := u(y) - u(0) - \langle \nabla u(0), y \rangle - \langle \nabla^2 u(0) y, y \rangle / 2$. Then there exists a constant $C > 0$ depending only on n, k and $|\nabla^2 u(0)|$, such that for any $0 < r < 1$*

$$\sup_{\substack{y,z \in B(r) \\ y \neq z}} \frac{|h(y) - h(z)|}{|y - z|^\alpha} \leqslant \frac{C}{r^\alpha} \fint_{B(2r)} |h(y)| \, dy + C r^{2-\alpha}, \tag{3.6}$$

where $\alpha := (2 - n/k)$.

PROOF: Let $\Lambda := |\nabla^2 u(0)|$ and define $g(y) := h(y) + \Lambda |y|^2/2$. Since $\Lambda |y|^2/2 - u(0) - \langle \nabla u(0), y \rangle - \langle \nabla^2 u(0) y, y \rangle / 2$ is convex and the sum of two k-convex functions are k-convex (follows from (1.4)), we conclude that g is k-convex. Applying the Hölder estimate in (2.1) for g with $\Omega' = B(2r)$, there exists $C := C(n,k) > 0$, such that

$$\begin{aligned}
r^{n+\alpha} \sup_{\substack{y,z \in B(r) \\ y \neq z}} \frac{|g(y) - g(z)|}{|y - z|^\alpha} &= \operatorname{dist}\big(B(r), \partial B(2r)\big)^{n+\alpha} \sup_{\substack{y,z \in B(r) \\ y \neq z}} \frac{|g(y) - g(z)|}{|y - z|^\alpha} \\
&\leqslant \sup_{\substack{y,z \in B(2r) \\ y \neq z}} d_{y,z}^{n+\alpha} \frac{|g(y) - g(z)|}{|y - z|^\alpha} \\
&\leqslant C \int_{B(2r)} |g(y)| \, dy \\
&\leqslant C \int_{B(2r)} |h(y)| \, dy + C r^{n+2}, \qquad (3.7)
\end{aligned}$$

where $d_{y,z} := \min\Big\{\operatorname{dist}\big(y, \partial B(2r)\big), \operatorname{dist}\big(z, \partial B(2r)\big)\Big\}$. Therefore the estimate (3.6) for h follows from the estimate (3.7) and the definition of g. ∎

PROOF OF THEOREM 1.1. To prove (3.5), take $0 < \varepsilon$, $\delta < 1$, such that $\delta^{1/n} \leqslant 1/2$. Then there exists r_0 depending on ε and δ, sufficiently small, such that, for $0 < r < r_0$

$$\begin{aligned}
\mathcal{L}^n \Big\{ z \in B(r) : |h(z)| \geqslant \varepsilon r^2 \Big\} &\leqslant \frac{1}{\varepsilon r^2} \int_{B(r)} |h(z)| \, dz \\
&= o(r^n) \qquad \text{by} \quad (3.4) \\
&< \delta \mathcal{L}^n\big(B(r)\big) \qquad (3.8)
\end{aligned}$$

Set $\sigma := \delta^{1/n} r$. Then for each $y \in B(r/2)$ there exists $z \in B(r)$ such that

$$|h(z)| \leqslant \varepsilon r^2 \quad \text{and} \quad |y - z| \leqslant \sigma.$$

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Hence for each $y \in B(r/2)$, we obtain by (3.4) and (3.6),

$$\begin{aligned}|h(y)| &\leqslant |h(z)| + |h(y) - h(z)| \\ &\leqslant \varepsilon r^2 + C|y-z|^\alpha \left(\frac{1}{r^\alpha} \fint_{B(2r)} |h(y)|\, dy + r^{2-\alpha}\right) \\ &\leqslant \varepsilon r^2 + C\delta^{\alpha/n} r^\alpha \left(\frac{1}{r^\alpha} \fint_{B(2r)} |h(y)|\, dy + r^{2-\alpha}\right) \\ &\leqslant \varepsilon r^2 + C\delta^{\alpha/n} \left(\fint_{B(2r)} |h(y)|\, dy + r^2\right) \\ &= r^2(\varepsilon + C\delta^{\alpha/n}) + o(r^2) \qquad \text{as} \quad r \to 0\end{aligned}$$

By choosing δ such that, $C\delta^{\alpha/n} = \varepsilon$, we have for sufficiently small $\varepsilon > 0$ and $0 < r < r_0$,

$$\sup_{B(r/2)} |h(y)| \leqslant 2\varepsilon r^2 + o(r^2).$$

Hence

$$\sup_{B(r/2)} \left|u(y) - u(0) - \langle \nabla u(0), y\rangle - \frac{1}{2}\langle \nabla^2 u(0) y, y\rangle\right| dy = o(r^2) \quad \text{as } r \to 0.$$

This proves (1.7) for $x = 0$ and hence u is twice differentiable at $x = 0$. Therefore u is twice differentiable at almost every x and satisfies (1.7), for which (3.2)–(3.3) holds. This proves the theorem. ∎

Let u be a k-convex function and $\mu_k[u]$ be the associated k-Hessian measure. Then $\mu_k[u]$ can be decomposed as the sum of a regular part $\mu_k^{ac}[u]$ and a singular part $\mu_k^{s}[u]$. As an application of the Theorem 1.1, we prove the following theorem.

THEOREM 3.2. *Let $\Omega \subset \mathbb{R}^n$ be an open set and $u \in \Phi^k(\Omega)$, $k > n/2$. Then the absolutely continuous part of $\mu_k[u]$ is represented by the k-Hessian operator $F_k[u]$. That is*

$$\mu_k^{ac}[u] = F_k[u]\, dx. \tag{3.9}$$

PROOF: Let u be a k-convex function, $k > n/2$ and u_ε be the mollification of u. Then by (1.5) and the properties of mollification (see for example [**3**, Theorem 1, Section 4.2]) it follows that $u_\varepsilon \in \Phi^k(\Omega) \cap C^\infty(\Omega)$. Since u is twice differentiable almost everywhere (by Theorem 1.1) and $u \in W^{2,1}_{\mathrm{loc}}(\Omega)$ (by Theorem 2.5), we conclude that $\nabla^2 u_\varepsilon \to \nabla^2 u$ in L^1_{loc}. Let $\mu_k[u_\varepsilon]$ and $\mu_k[u]$ be the Hessian measures associated to the functions u_ε and u respectively. Then by weak continuity Theorem 2.3 ([**13**, Theorem 1.1]), $\mu_k[u_\varepsilon]$ converges to $\mu_k[u]$ in measure and $\mu_k[u_\varepsilon] = F_k[u_\varepsilon]\, dx$. It follows that for any compact set $E \subset \Omega$,

$$\mu_k[u](E) \geqslant \limsup_{\varepsilon\to 0} \mu_k[u_\varepsilon](E) = \limsup_{\varepsilon \to 0} \int_E F_k[u_\varepsilon]. \tag{3.10}$$

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Since $F_k[u_\varepsilon] \geqslant 0$ and $F_k[u_\varepsilon](x) \to F_k[u](x)$ almost everywhere, by Fatou's Lemma, for every relatively compact measurable subset E of Ω, we have

$$\int_E F_k[u] \leqslant \liminf_{\varepsilon\to 0} \int_E F_k[u_\varepsilon]. \tag{3.11}$$

Therefore by Theorem 3.1, [**13**], it follows that $F_k[u] \in L^1_{\mathrm{loc}}(\Omega)$. Let $\mu_k[u] = \mu_k^{\mathrm{ac}}[u] + \mu_k^{\mathrm{s}}[u]$, where $\mu_k^{\mathrm{ac}}[u] = h\,dx$, $h \in L^1_{\mathrm{loc}}(\Omega)$ and $\mu_k^{\mathrm{s}}[u]$ is the singular part supported on a set of Lebesgue measure zero. We would like to prove that $h(x) = F_k[u](x)$ $\mathcal{L}^n$ almost everywhere x. By taking $E := \overline{B}(x,r)$, from (3.10) and (3.11), we obtain

$$⨍_{\overline{B}(x,r)} F_k[u]\,dy \leqslant \frac{\mu_k[u](\overline{B}(x,r))}{\mathcal{L}^n(B(x,r))} = ⨍_{\overline{B}(x,r)} h\,dy + \frac{\mu_k^{\mathrm{s}}[u](\overline{B}(x,r))}{\mathcal{L}^n(B(x,r))}. \tag{3.12}$$

Hence by letting $\varepsilon \to 0$, we obtain

$$F_k[u](x) \leqslant h(x) \quad \mathcal{L}^n \text{ almost everywhere } x. \tag{3.13}$$

To prove the reverse inequality, let us recall that h is the density of the absolutely continuous part of the measure $\mu_k[u]$, that is for $\mathcal{L}^n$ almost everywhere x

$$h(x) = \lim_{r\to 0} \frac{\mu_k^{\mathrm{ac}}[u](\overline{B}(x,r))}{\mathcal{L}^n(B(x,r))} = \lim_{r\to 0} \frac{\mu_k[u](\overline{B}(x,r))}{\mathcal{L}^n(B(x,r))}. \tag{3.14}$$

Since $\mu_k^{\mathrm{s}}[u]$ is supported on a set of Lebesgue measure zero,

$$\mu_k^{\mathrm{s}}[u]\big(\partial B(x,r)\big) = 0, \quad \mathcal{L}^1 \text{ almost everywhere } r > 0.$$

Therefore by the weak continuity of $\mu_k[u_\varepsilon]$ (see for example Theorem 1, [**3**, Theorem 1]), we conclude that

$$\lim_{\varepsilon\to 0} \mu_k[u_\varepsilon]\big(B(x,r)\big) = \mu_k[u]\big(B(x,r)\big), \quad \mathcal{L}^1 \text{ almost everywhere } r > 0. \tag{3.15}$$

Let $\delta > 0$. Then for $\varepsilon < \varepsilon' = \varepsilon(\delta)$ and for $\mathcal{L}^1$ almost everywhere $r > 0$, $\mathcal{L}^n$ almost everywhere x

$$\begin{aligned} h(x) &\leqslant \lim_{r\to 0} \frac{(1+\delta)\mu_k[u_\varepsilon](B(x,r))}{\mathcal{L}^n(B(x,r))} \\ &= (1+\delta) \lim_{r\to 0} ⨍_{B(x,r)} F_k[u_\varepsilon]\,dy \\ &= (1+\delta) F_k[u_\varepsilon](x) \end{aligned} \tag{3.16}$$

By letting $\varepsilon \to 0$ and finally $\delta \to 0$, we obtain

$$h(x) \leqslant F_k[u](x), \mathcal{L}^n \text{ almost everywhere } x.$$

This proves the theorem. $\square$

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