A Note on Convergence Factors

By C. E. Walsh

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Suppose that $u_n \geqslant u_{n+1} > 0$, for $n = 1, 2, \ldots$ and that $e_1, e_2, \ldots$ are factors which make $\Sigma e_n u_n$ convergent.

Let $\sum_{r=1}^{n} a_r = A_n$, where all the a's are positive, Σa_n diverges, and $A_{n+1} = O(A_n)$. Define E_n by the equation

$$A_n E_n = \sum_1^n e_r.$$

For the case where $a_n = 1$ it has been shewn by Fuchs [1] and Karamata [1] that, under various conditions, $E_n = o(1)$. The object of this note is to extend some of their results.

Let $S_n = \sum_1^{n-1} e_r u_r = S + \theta_n k_n$, where $k_n > 0$, $S_1 = 0$.

With the above notation

$$e_n u_n = S_{n+1} - S_n,$$

and so

$$E_n = \frac{1}{A_n} \sum_1^{n-1} \left(\frac{1}{u_r} - \frac{1}{u_{r+1}}\right) k_{r+1} \theta_{r+1} + \frac{k_{n+1}\theta_{n+1}}{A_n u_n}.$$

Hence by the Toeplitz-Schur theorem we have

Let $k_n > 0$ $(n = 1, 2, \ldots)$. Then in order that $E_n = o(1)$ for every sequence (e_n) for which $\theta_n = o(1)$ [or $O(1)$] it is necessary and sufficient that

$$\frac{1}{A_n} \sum_1^{n-1} \left(\frac{1}{u_{r+1}} - \frac{1}{u_r}\right) k_{r+1} + \frac{k_{n+1}}{A_n u_n} = O(1) \quad [\text{or } o(1)]. \tag{1}$$

It follows immediately from this that if

$$\gamma_n = \frac{k_n}{A_n}\left(\frac{1}{u_n} - \frac{1}{u_{n-1}}\right) \leqslant B_n - \frac{A_{n-1}}{A_n} B_{n-1} \tag{2}$$

[1] W. H. J. Fuchs, *Proc. Edinburgh Math. Soc.* (2), 7 (1942), 27–30; J. Karamata *Journal London Math. Soc.*, 21 (1946), 162–166.

where $0 < B_n < \infty$, *and if in addition*

$$\frac{k_{n+1}}{A_n u_n} = O(1), \tag{3}$$

then $E_n = o(1)$. *Since* $A_{n+1} = O(A_n)$, (3) *is true if*

$$\frac{k_n}{A_n u_n} = O(1). \tag{4}$$

We now enumerate three choices of k_n which will satisfy (2) and (4), and for which, accordingly, $E_n = o(1)$.

[A]. *Assume that* k_n *satisfies* (4) *and also*

$$1 - \frac{u_n}{u_{n-1}} \leqslant \lambda \left(1 - \frac{k_{n-1}}{k_n} \frac{u_n}{u_{n-1}}\right), \tag{5}$$

λ *being a positive constant.*

Then $$\gamma_n \leqslant \lambda \left(\frac{k_n}{A_n u_n} - \frac{A_{n-1}}{A_n} \frac{k_{n-1}}{A_{n-1} u_{n-1}}\right),$$

and so (2) is also satisfied.

A case of this kind occurs when $k_n = v_n u^{\alpha}{}_n$ where $v_n \leqslant v_{n+1}$ for $n = 1, 2, \ldots, 0 \leqslant \alpha < 1$ (the inequality (5) being satisfied with $\lambda = 1/(1-\alpha)$), and $v_n u_n^{\alpha-1}/(A_n) = O(1)$.

The special case where $v_n = (n-1)^{\beta}$, $0 \leqslant \beta < 1$, was proved by Karamata.

[B]. *Let us assume now that* $k_n/(A_n u_n) < K$, *a constant, and that*

$$\frac{k_n}{A_{n-1} A_n \cdot u_{n-1} u_n}$$

steadily decreases.

This gives

$$\gamma_n < \frac{k_n}{A_n u_n} - \frac{A_{n-1}}{A_n} \frac{k_{n+1}}{A_{n+1} u_{n+1}}$$

$$< \left(2K - \frac{k_{n+1}}{A_{n+1} u_{n+1}}\right) - \frac{A_{n-1}}{A_n} \left(2K - \frac{k_n}{A_n u_n}\right)$$

and so (2), also, holds good.

The conditions required here are fulfilled if, *e.g.*,

$$k_n = A_{n-1} A_n u_{n-1} u_n \Big/ \sum_{1}^{n-1} a_r u_r.$$

Karamata proved this with $a_n = 1$.

[C]. *Let* $k_n = A_{n-1} u_{n-1}$, *where* $u_{n-1}/u_n < K$, *a constant.*

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Then (4) is satisfied, and

$$\gamma_n = \frac{A_{n-1}}{A_n}\left(\frac{u_{n-1}}{u_n} - 1\right).$$

Using an inequality employed by Karamata we have

$$\frac{u_{n-1}}{u_n} - 1 < \frac{u_{n-1}}{u_n} \log \frac{u_{n-1}}{u_n} < K \log \frac{u_{n-1}}{u_n}.$$

Consequently

$$\gamma_n < K \frac{A_{n-1}}{A_n} \log \frac{u_{n-1}}{u_n} = K\left(\frac{\log \alpha_n}{A_n} - \frac{A_{n-1}}{A_n} \frac{\log \alpha_{n-1}}{A_{n-1}}\right)$$

where $\alpha_1 = 1$, $\alpha_n = u_1^{a_1} u_2^{a_2} \dots u_{n-1}^{a_{n-1}} u_n^{-A_{n-1}}$, $n > 1$.

Accordingly (2), *also, will be satisfied if*

$$\alpha_n^{1/A_{n-1}} = O(1).$$

When $a_n = 1$ this is so if $u_n = O(u_{2n})$ [when, *a fortiori*, $u_{n-1} = O(u_n)$, as is easily shewn]. This case was given by Karamata.

Finally, let $k_n = a_n u_n$, *where* $u_n > 0$ *still, but no longer necessarily decreases.*

Let $\theta_n = O(1)$ *and*

$$1 - u_n/(u_{n-1}) = b_n \epsilon_n$$

where $b_n > 0$ $(n = 2, 3, \dots)$ *and* $\epsilon_n \to 0$, $b_n \epsilon_n = O(1)$.

Assume now that $A_n = \sum_1^n a_r b_r \to \infty$, *and that*

$$A_{n+1} = O(A_n), \quad a_{n+1} = o(A_n). \tag{6}$$

Then
$$\gamma_n = \frac{a_n b_n}{A_n} \epsilon_n.$$

In addition

$$\frac{k_{n+1}}{A_n u_n} = \frac{a_{n+1}}{A_n}(1 - b_{n+1}\epsilon_{n+1}) = o(1) \text{ from (6)}$$

since $b_{n+1}\epsilon_{n+1}$ is bounded.

The conditions of (1), for the case where $\theta_n = O(1)$, are now easily seen to be fulfilled, and so $E_n = o(1)$.

If $u_n \geqslant u_{n+1}$ always, the restriction $A_{n+1} = O(A_n)$ can be dispensed with here.

Writing $a_n = b_n = 1$ we obtain a result given by Fuchs.

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TRINITY COLLEGE,
DUBLIN.