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A NOTE ON TIGHTNESS

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Abstract. The purpose of this note is to prove a results of Jain and López-Permouth under a weaker conditions replacing R-weak injectivity by R-tightness and even getting a simpler proof.

1. Introduction. Throughout this paper all rings are associative with identity and all modules are unitary right modules. We denote the category of all right R-modules by Mod-R. Given a module M_R the injective hull of M in Mod-R is denoted by $E(M)$. The purpose of this paper is to further the study of the concept of tightness [**1**], [**4**]. Following Jain and López-Permouth, given two modules M and N $\in$Mod-R, a module M is N-*tight* if every quotient of N which is embeddable in $E(M)$ is embeddable in M. A module is *tight* if it is tight relative to all finitely generated submodules of $E(M)$.

A ring R is called right *CEP*-ring if every cyclic right R-module is essentially embeddable in a projective module. In this paper we assume all modules are unital right R-modules unless otherwise indicated.

We start first with some basic results that will be needed in this note.

Lemma 1. *Let* R *be an artinian ring, and let* N, M *be finitely generated modules. If* M *is* N-*tight and* N *is* M-*tight and* Soc(M) $\simeq$ Soc(N) *then* M $\simeq$ N.

Proof. Let σ: $N \to E(M)$ be the monomorphism induced by the isomorphism between Soc*(M)* and Soc*(N)*. Since M is N-tight, N is embeddable in M. Similarly, M is embeddable in N. Since M and N are finitely generated over artinian ring, $M \simeq N$.

Lemma 2. [**2**, **3**, **4**] *A right* CEP-*ring is right artinian. All projective indecomposable right modules over a right* CEP-*ring are uniform.*

Lemma 3. *Let* R *be a right artinian ring such that all indecomposable projective right* R-*modules are uniform and* R-*tight. Then the following holds:*

(i) *every simple right* R-*module is isomorphic to the socle of an indecomposable projective module,*
(ii) *every simple right* R-*module is embeddable in* Soc(R),
(iii) *if* P *and* Q *are projective right modules with* Soc(P) $\simeq$ Soc(Q) *then* P $\simeq$ Q.

Proof. The proof follows from Lemma 1 and [**3**, Lemma 5. 1].

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The next Theorem was proved by Jain and López-Permouth in [**3**] with the assumption that every indecomposable projective right module is weakly R-injective. We show that the theorem is true under a weaker condition. In fact all we need is to have every indecomposable projective right module is R-tight.

THEOREM. *A ring* R *is a right* CEP-*ring if and only if the following holds:*

(i) R *is right artinian,*
(ii) *every indecomposable projective right module is uniform and R-tight.*

Proof. Let R be a CEP-ring. By Lemma 2, R is right artinian. Let P be an indecomposable projective right module. Once more Lemma 2 implies that $Soc(P)$ is simple. Let $\sigma : R/I \to E(P)$ be a monomorphism. Then $Soc(R/I) \simeq Soc(E(P)) = Soc(P)$. Since R is a CEP-ring, R/I embeds essentially in some projective module, say, Q. Thus $Soc(P) \simeq Soc(Q)$. Hence by Lemma 3, $P \simeq Q$, and thus R/I embeds in P, proving that P is R-tight. Conversely, assume R satisfies the two conditions. Write $R = \oplus \sum_{i=1}^{n} e_i R$ as a direct sum of indecomposable right ideals. Let I be a right ideal of R. By Lemma 3, $Soc(R/I) \simeq \oplus \sum_{i=1}^{k} Soc(e_i R)^{n_i}$. Let $P = \oplus \sum_{i=1}^{k} (e_i R)^{n_i}$. Since $Soc(P) = Soc(E(P))$, the above isomorphism between $Soc(R/I)$ and $\oplus \sum_{i=1}^{k} Soc(e_i R)^{n_i} = Soc(P)$ may be looked upon as an essential embedding φ : $Soc(R/I) \to E(P)$. This extends into an essential embedding $\hat{\varphi}$: $R/I \to E(P)$. By tightness, there exists an embedding $\sigma : R/I \to P$ which is essential in P, proving that R is a CEP-ring.

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