On Determinants whose elements are Determinants.

By Professor E. T. WHITTAKER.

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§ 1. *Object of paper.*

The present paper is concerned with determinants whose elements are themselves determinants. The best-known determinants of this kind are those whose elements are minors of a given determinant; these are called the "adjugate" and the "compounds" of the given determinant. Determinants whose elements are themselves determinants also occur frequently in Muir's theory of Extensionals.

The determinants considered in the present paper are of a somewhat more general type than adjugates, compounds, and extensionals; the principal result obtained is "Theorem A" (at the end of § 2), which relates to determinants whose elements are formed from any number of arrays. It is shown in § 3 that many other formulae, both new and old, may be obtained by specialising the arrays in "Theorem A."

§ 2. *A General theorem.*

Let (P_1), (P_2), (P_3), ..., (P_n) denote any arrays, each of $(n-1)$ rows and n columns; let (0) denote an array of $(n-1)$ rows and n columns whose elements are all zeros; and let (1) denote an array of n rows and n columns whose elements in the leading diagonal are each unity, and whose other elements are each zero. Consider the determinant of order n^2

$$\begin{vmatrix} (P_1) & (0) & (0) & (0) & \cdots & (0) \\ (0) & (P_2) & (0) & (0) & \cdots & (0) \\ (0) & (0) & (P_3) & (0) & \cdots & (0) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ (0) & (0) & (0) & (0) & \cdots & (P_n) \\ (1) & (1) & (1) & (1) & \cdots & (1) \end{vmatrix} \qquad \ldots\ldots\ldots\ldots\ldots (1)$$

It will frequently be convenient to take the case $n=4$ as an illustrative example; in this case the determinant may be written

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ b_1 & b_2 & b_3 & b_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ c_1 & c_2 & c_3 & c_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & d_1 & d_2 & d_3 & d_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e_1 & e_2 & e_3 & e_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & f_1 & f_2 & f_3 & f_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & g_1 & g_2 & g_3 & g_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & h_1 & h_2 & h_3 & h_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & k_1 & k_2 & k_3 & k_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & l_1 & l_2 & l_3 & l_4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & m_1 & m_2 & m_3 & m_4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & n_1 & n_2 & n_3 & n_4 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{vmatrix} \quad \ldots(1\text{A})$$

By an obvious transformation the determinant (1) may be replaced by

$$-\begin{vmatrix} (P_1) & (0) & (0) & (0) & \ldots\ldots & (0) \\ (0) & (P_2) & (0) & (0) & \ldots\ldots & (0) \\ (0) & (0) & (P_3) & (0) & \ldots\ldots & (0) \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ (0) & (0) & (0) & (0) & \ldots & (P_{n-1}) \\ (P_n) & (P_n) & (P_n) & (P_n) & \ldots & (P_n) \end{vmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

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which in the case $n=4$ may be written

$$-\begin{vmatrix} a_1 & a_2 & a_3 & a_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ b_1 & b_2 & b_3 & b_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ c_1 & c_2 & c_3 & c_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & d_1 & d_2 & d_3 & d_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e_1 & e_2 & e_3 & e_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & f_1 & f_2 & f_3 & f_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & g_1 & g_2 & g_3 & g_4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & h_1 & h_2 & h_3 & h_4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & k_1 & k_2 & k_3 & k_4 \\ l_1 & l_2 & l_3 & l_4 & l_1 & l_2 & l_3 & l_4 & l_1 & l_2 & l_3 & l_4 \\ m_1 & m_2 & m_3 & m_4 & m_1 & m_2 & m_3 & m_4 & m_1 & m_2 & m_3 & m_4 \\ n_1 & n_2 & n_3 & n_4 & n_1 & n_2 & n_3 & n_4 & n_1 & n_2 & n_3 & n_4 \end{vmatrix} \quad \ldots(2\text{A})$$

Now expand the determinant (2A) by Laplace's theorem as a sum of products of minors, each product containing one minor formed from the first three rows of (2A), one minor formed from the second three rows, and so on. It is readily seen that the expansion may be written in the determinental form

$$\begin{vmatrix} \begin{vmatrix} a_2 & a_3 & a_4 \\ b_2 & b_3 & b_4 \\ c_2 & c_3 & c_4 \end{vmatrix} & \begin{vmatrix} a_1 & a_3 & a_4 \\ b_1 & b_3 & b_4 \\ c_1 & c_3 & c_4 \end{vmatrix} & \begin{vmatrix} a_1 & a_2 & a_4 \\ b_1 & b_2 & b_4 \\ c_1 & c_2 & c_4 \end{vmatrix} & \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \\ \begin{vmatrix} d_2 & d_3 & d_4 \\ e_2 & e_3 & e_4 \\ f_2 & f_3 & f_4 \end{vmatrix} & \begin{vmatrix} d_1 & d_3 & d_4 \\ e_1 & e_3 & e_4 \\ f_1 & f_3 & f_4 \end{vmatrix} & \begin{vmatrix} d_1 & d_2 & d_4 \\ e_1 & e_2 & e_4 \\ f_1 & f_2 & f_4 \end{vmatrix} & \begin{vmatrix} d_1 & d_2 & d_3 \\ e_1 & e_2 & e_3 \\ f_1 & f_2 & f_3 \end{vmatrix} \\ \begin{vmatrix} g_2 & g_3 & g_4 \\ h_2 & h_3 & h_4 \\ k_2 & k_3 & k_4 \end{vmatrix} & \begin{vmatrix} g_1 & g_3 & g_4 \\ h_1 & h_3 & h_4 \\ k_1 & k_3 & k_4 \end{vmatrix} & \begin{vmatrix} g_1 & g_2 & g_4 \\ h_1 & h_2 & h_4 \\ k_1 & k_2 & k_4 \end{vmatrix} & \begin{vmatrix} g_1 & g_2 & g_3 \\ h_1 & h_2 & h_3 \\ k_1 & k_2 & k_3 \end{vmatrix} \\ \begin{vmatrix} l_2 & l_3 & l_4 \\ m_2 & m_3 & m_4 \\ n_2 & n_3 & n_4 \end{vmatrix} & \begin{vmatrix} l_1 & l_3 & l_4 \\ m_1 & m_3 & m_4 \\ n_1 & n_3 & n_4 \end{vmatrix} & \begin{vmatrix} l_1 & l_2 & l_4 \\ m_1 & m_2 & m_4 \\ n_1 & n_2 & n_4 \end{vmatrix} & \begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{vmatrix} \end{vmatrix}, \quad (3\text{A})$$

and similarly in general, if P_n denotes the determinant of order

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$(n-1)$ which is formed from the array (P_t) by omitting its t^{th} column,* the determinant (1) or (2) is equal to the determinant

$$\begin{vmatrix} P_{11} & P_{12} & \cdots\cdots & P_{1n} \\ P_{21} & P_{22} & \cdots\cdots & P_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ P_{n1} & P_{n2} & \cdots\cdots & P_{nn} \end{vmatrix} \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots (3)$$

Now expand the determinant (2A) by Laplace's theorem in a different fashion, namely, as a sum of products of minors, each product containing one minor formed from the first four columns of (2A), one from the second four columns, and one from the third four columns. It is readily seen that the expansion may be written in the determinantal form

$$\begin{vmatrix} \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ l_1 & l_2 & l_3 & l_4 \end{vmatrix} & \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ m_1 & m_2 & m_3 & m_4 \end{vmatrix} & \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ n_1 & n_2 & n_3 & n_4 \end{vmatrix} \\ \begin{vmatrix} d_1 & d_2 & d_3 & d_4 \\ e_1 & e_2 & e_3 & e_4 \\ f_1 & f_2 & f_3 & f_4 \\ l_1 & l_2 & l_3 & l_4 \end{vmatrix} & \begin{vmatrix} d_1 & d_2 & d_3 & d_4 \\ e_1 & e_2 & e_3 & e_4 \\ f_1 & f_2 & f_3 & f_4 \\ m_1 & m_2 & m_3 & m_4 \end{vmatrix} & \begin{vmatrix} d_1 & d_2 & d_3 & d_4 \\ e_1 & e_2 & e_3 & e_4 \\ f_1 & f_2 & f_3 & f_4 \\ n_1 & n_2 & n_3 & n_4 \end{vmatrix} \\ \begin{vmatrix} g_1 & g_2 & g_3 & g_4 \\ h_1 & h_2 & h_3 & h_4 \\ k_1 & k_2 & k_3 & k_4 \\ l_1 & l_2 & l_3 & l_4 \end{vmatrix} & \begin{vmatrix} g_1 & g_2 & g_3 & g_4 \\ h_1 & h_2 & h_3 & h_4 \\ k_1 & k_2 & k_3 & k_4 \\ m_1 & m_2 & m_3 & m_4 \end{vmatrix} & \begin{vmatrix} g_1 & g_2 & g_3 & g_4 \\ h_1 & h_2 & h_3 & h_4 \\ k_1 & k_2 & k_3 & k_4 \\ n_1 & n_2 & n_3 & n_4 \end{vmatrix} \end{vmatrix} . \quad (4\text{A})$$

The general determinant (2) may be expanded in a similar fashion, being thus transformed into the determinant

$$\begin{vmatrix} D_{11} & D_{12} & \cdots & D_{1,\,n-1} \\ D_{21} & D_{22} & \cdots & D_{2,\,n-1} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ D_{n-1,\,1} & D_{n-1,\,2} & \cdots & D_{n-1,\,n-1} \end{vmatrix} \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots (4)$$

*** It is best to assign to P_{tt} a definite sign, namely, the sign which it would have as an algebraic complement in a determinant formed by superadding a new row to the array P_t.**

where D_{st} denotes the determinant of the array formed by adjoining to the array (P_s) the t^{th} row of the array (P_n).

Now the array (P_n) occupies an eccentric position in the determinant (2), and any other one of the arrays (P_1), (P_2), ... (P_n) may be brought into this eccentric position instead of (P_n), the function being essentially symmetrical with respect to the n arrays. Thus we can obtain n different determinants of the type (4), differing in the choice of the array which occupies the eccentric position.

We have now obtained a general theorem, which may be stated thus:

Theorem A.—The determinant (1), *the determinant* (3), *and the n determinants of the type* (4), *are all equal to each other.*

Even this does not exhaust the forms in which this function of the n arrays ca be represented. For example, it may be shown without difficulty that ar other form is the determinant of order $(2n-1)$

$$\begin{vmatrix} (0) & (P_n) \\ (\pi) & (1) \end{vmatrix} \qquad \text{.......... ....} \quad (5)$$

where (0) denotes a square array of order $(n-1)$ whose elements are all zero, (P_n) and (1) have the same meanings as before, and (π) denotes the array

$$\begin{pmatrix} P_{11} & P_{21} & \cdots & P_{n-1,1} \\ P_{12} & P_{22} & \cdots & P_{n-2,2} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ P_{1n} & P_{2n} & \cdots & P_{n-1,n} \end{pmatrix}$$

§3. *Applications of Theorem A.*

The import of Theorem A will be most clearly seen by considering some special cases of it.

We shall first take the arrays (P_1), (P_2), ... (P_p) to be the arrays obtained by omitting the first, second, ..., p^{th} row respectively in a determinant

$$A = \begin{vmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nn} \end{vmatrix},$$

and we shall take the remaining arrays (P_{p+1}), (P_{p+2}), ..., (P_n) to be the arrays obtained by omitting the $(p+1)^{\text{th}}$, $(p+2)^{\text{th}}$, ... n^{th} row respectively in another determinant,

$$B = \begin{vmatrix} \beta_{11} & \beta_{12} & \cdots & \beta_{1n} \\ \beta_{21} & \beta_{22} & \cdots & \beta_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \beta_{n1} & \beta_{n2} & \cdots & \beta_{nn} \end{vmatrix},$$

then if we denote the co-factor of α_{rs} in A by A_{rs}, and the co-factor of β_{rs} in B by B_{rs}, the determinant (3) becomes

$$\begin{vmatrix} A_{11} & A_{12} & \cdots\cdots & A_{1n} \\ A_{21} & A_{22} & \cdots\cdots & A_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ A_{p1} & A_{p2} & \cdots\cdots & A_{pn} \\ B_{p+1,1} & B_{p+1,2} & \cdots\cdots & B_{p+1,n} \\ \cdots & \cdots & \cdots & \cdots \\ B_{n1} & B_{n2} & \cdots\cdots & B_{nn} \end{vmatrix},$$

that is to say, its first p rows are the same as the first p rows in the adjugate of A, while its last $(n-p)$ rows are the same as the last $(n-p)$ rows in the adjugate of B.

Now let $|A_s, B_t|$ denote the determinant formed from A by replacing its s^{th} row by the t^{th} row of B. Then the determinant (4) becomes

$$\begin{vmatrix} |A_1, B_1| & |A_1, B_2| & \cdots & |A_1, B_p| & |A_1, B_{p+1}| & \cdots & |A_1, B_{n-1}| \\ |A_2, B_1| & |A_2, B_2| & \cdots & |A_2, B_p| & |A_2, B_{p+1}| & \cdots & |A_2, B_{n-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ |A_p, B_1| & |A_p, B_2| & \cdots & |A_p, B_p| & |A_p, B_{p+1}| & \cdots & |A_p, B_{n-1}| \\ 0 & 0 & \cdots & 0 & B & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & B \end{vmatrix}$$

or

$$B^{n-p-1} \begin{vmatrix} |A_1, B_1| & |A_2, B_2| & \cdots & |A_1, B_p| \\ |A_2, B_1| & |A_2, B_2| & \cdots & |A_2, B_p| \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ |A_p, B_1| & |A_p, B_2| & \cdots & |A_p, B_p| \end{vmatrix},$$

We have thus obtained, as a special case of Theorem A, the following result:

Theorem B.—The determinant formed by taking the first p rows of the adjugate of a determinant A of order n, and the last (n − p) rows of the adjugate of a determinant B, also of order n, is equal to

$$B^{n-p-1}\begin{vmatrix} |A_1, B_1| & |A_1, B_2| & \ldots & |A_1, B_p| \\ |A_2, B_1| & |A_2, B_2| & \ldots & |A_2, B_p| \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ |A_p, B_1| & |A_p, B_2| & \ldots & |A_p, B_p| \end{vmatrix}$$

where $|A_s, B_t|$ denotes the determinant formed from A by replacing its s^{th} row by the t^{th} row of B.

As a particular case of this, take the determinant A to be the determinant of the unit matrix

$$\begin{pmatrix} 1 & 0 & 0 \ldots \\ 0 & 1 & 0 \ldots \\ 0 & 0 & 1 \ldots \\ \ldots & \ldots & \ldots \end{pmatrix}.$$

Then $|A_r, B_s|$ becomes β_{sr}, and therefore Theorem B becomes

$$\begin{vmatrix} B_{p+1,p+1} & B_{p+1,p+2} & \ldots & B_{p+1,n} \\ B_{p+2,p+1} & B_{p+2,p+2} & \ldots & B_{p+2,n} \\ \ldots & \ldots & \ldots & \ldots \\ B_{n,p+1} & B_{n,p+2} & \ldots & B_{nn} \end{vmatrix} = B^{n-p-1}\begin{vmatrix} \beta_{11} & \beta_{21} & \ldots & \beta_{p1} \\ \beta_{12} & \beta_{22} & \ldots & \beta_{p2} \\ \ldots & \ldots & \ldots & \ldots \\ \beta_{1p} & \beta_{2p} & \ldots & \beta_{pp} \end{vmatrix}$$

and this is no other than Jacobi's well-known theorem that *any minor of order r of the adjugate of any determinant B is equal to B^{r-1} multiplied by the algebraic complement of the corresponding minor of B.* Thus Theorem B, which is itself a special case of Theorem A, includes Jacobi's theorem as a special case of itself. It may be remarked that this method of deriving Jacobi's theorem frees it (and all kindred theorems which may be derived from Theorem A) from all dependence on the multiplication theorem, which has not been used in the present paper.

Next, we note that by interchanging the parts played by the determinants A and B in Theorem B, we obtain a form in which A^{p-1} appears as a factor. Therefore (since A and B have no common factors in the general case) the determinant

$$\begin{vmatrix} |A_1, B_1| & |A_1, B_2| & \dots & |A_1, B_p| \\ |A_2, B_1| & |A_2, B_2| & \dots & |A_2, B_p| \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ |A_p, B_1| & |A_p, B_2| & \dots & |A_p, B_p| \end{vmatrix}$$

contains A^{p-1} as a factor. By considering dimensions, we see that the other factor must be of degree $(n-p)$ in the letters α combined and of degree p in the letters β combined; and in fact it must be linear and homogeneous with respect to the elements of each of the first p rows of B and the last $(n-p)$ rows of A, and must vanish when any two of these rows become identical. It must therefore be a mere numerical multiple of the determinant formed of these numbers: and thus we obtain the formula

$$\begin{vmatrix} |A_1, B_1| & |A_1, B_2| & \dots & |A_1, B_p| \\ |A_2, B_1| & |A_2, B_2| & \dots & |A_2, B_p| \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ |A_p, B_1| & |A_p, B_2| & \dots & |A_p, B_p| \end{vmatrix} = A^{p-1} \begin{vmatrix} \beta_{11} & \dots & \beta_{1n} \\ \beta_{21} & \dots & \beta_{2n} \\ \dots & \dots & \dots \\ \beta_{p1} & \dots & \beta_{pn} \\ \alpha_{p+1,1} & \dots & \alpha_{p+1,n} \\ \dots & \dots & \dots \\ \alpha_{n1} & \dots & \alpha_{nn} \end{vmatrix}.$$

This is an extension (given in substance, though not in form, by Bazin himself) of a theorem given by Bazin* in 1854, namely, that, *Given two determinants A and B, each of order n, if in one of them A we substitute in all possible ways, in place of one of its rows, one of the rows of B, so that we obtain n^2 determinants, the determinant formed of these as elements has the value $A^{n-1}B$.* The above formula bears to Bazin's theorem the same relation that Jacobi's theorem on the minors of the adjugate bears to Cauchy's theorem that the adjugate of A has the value A^{n-1}.

Another application of Theorem A is the following:

If in equation (3A) we take the d's and g's equal to the l's respectively, the h's and a's equal to the m's, and the f's and b's equal to the n's, the formula becomes

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* *Liouville's Journal* **16** (1854), p. 145. It was rediscovered by Lloyd Tanner, *Educ. Times Rep.* **28** (1877), p. 41.

$$\begin{vmatrix}
\begin{vmatrix} m_2 & m_3 & m_4 \\ n_2 & n_3 & n_4 \\ c_2 & c_3 & c_4 \end{vmatrix} & \begin{vmatrix} m_1 & m_3 & m_4 \\ n_1 & n_3 & n_4 \\ c_1 & c_3 & c_4 \end{vmatrix} & \begin{vmatrix} m_1 & m_2 & m_4 \\ n_1 & n_2 & n_4 \\ c_1 & c_2 & c_4 \end{vmatrix} & \begin{vmatrix} m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \\
\begin{vmatrix} l_2 & l_3 & l_4 \\ e_2 & e_3 & e_4 \\ n_2 & n_3 & n_4 \end{vmatrix} & \begin{vmatrix} l_1 & l_3 & l_4 \\ e_1 & e_3 & e_4 \\ n_1 & n_3 & n_4 \end{vmatrix} & \begin{vmatrix} l_1 & l_2 & l_4 \\ e_1 & e_2 & e_4 \\ n_1 & n_2 & n_4 \end{vmatrix} & \begin{vmatrix} l_1 & l_2 & l_3 \\ e_1 & e_2 & e_3 \\ n_1 & n_2 & n_3 \end{vmatrix} \\
\begin{vmatrix} l_2 & l_3 & l_4 \\ m_2 & m_3 & m_4 \\ k_2 & k_3 & k_4 \end{vmatrix} & \begin{vmatrix} l_1 & l_3 & l_4 \\ m_1 & m_3 & m_4 \\ k_1 & k_3 & k_4 \end{vmatrix} & \begin{vmatrix} l_1 & l_2 & l_4 \\ m_1 & m_2 & m_4 \\ k_1 & k_2 & k_4 \end{vmatrix} & \begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ k_1 & k_2 & k_3 \end{vmatrix} \\
\begin{vmatrix} l_2 & l_3 & l_4 \\ m_2 & m_3 & m_4 \\ n_2 & n_3 & n_4 \end{vmatrix} & \begin{vmatrix} l_1 & l_3 & l_4 \\ m_1 & m_3 & m_4 \\ n_1 & n_3 & n_4 \end{vmatrix} & \begin{vmatrix} l_1 & l_2 & l_4 \\ m_1 & m_2 & m_4 \\ n_1 & n_2 & n_4 \end{vmatrix} & \begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{vmatrix}
\end{vmatrix}$$

$$= - \begin{vmatrix} l_1 & l_2 & l_3 & l_4 \\ m_1 & m_2 & m_3 & m_4 \\ n_1 & n_2 & n_3 & n_4 \\ c_1 & c_2 & c_3 & c_4 \end{vmatrix} \begin{vmatrix} l_1 & l_2 & l_3 & l_4 \\ m_1 & m_2 & m_3 & m_4 \\ n_1 & n_2 & n_3 & n_4 \\ e_1 & e_2 & e_3 & e_4 \end{vmatrix} \begin{vmatrix} l_1 & l_2 & l_3 & l_4 \\ m_1 & m_2 & m_3 & m_4 \\ n_1 & n_2 & n_3 & n_4 \\ k_1 & k_2 & k_3 & k_4 \end{vmatrix}$$

This result may evidently be extended to determinants of any order : the general formula may be stated thus :

Theorem C.—Let $\Delta_1, \Delta_2, \Delta_3, \ldots \Delta_n$ *be any determinants of order* n *which have their first* $(n-1)$ *rows in common ; and let* E *denote a determinant whose* p^{th} *row (for* $p = 1, 2, \ldots n$*) is the same as the* p^{th} *row of the adjugate of* Δ_p. *Then*

$$E = \Delta_1 \Delta_2 \ldots \Delta_{n-1}.$$

The well-known theorem regarding the adjugate (namely, that the adjugate of a determinant of order n is the $(n-1)^{\text{th}}$ power of the determinant itself) is a particular case of this theorem ; for if we specialise our result by taking all the determinants $\Delta_1, \Delta_2, \ldots \Delta_n$ to be the same determinant Δ, the above formula becomes

$$E = \Delta^{n-1}$$

when E is now the adjugate of Δ.

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