Bull. Aust. Math. Soc. **95** (2017), 250–259
doi:10.1017/S0004972716000782

THE HILBERT–SCHMIDT NORM OF A COMPOSITION OPERATOR ON THE BERGMAN SPACE

CHENG YUAN and ZE-HUA ZHOU✉

(Received 3 May 2016; accepted 25 June 2016; first published online 2 November 2016)

Abstract

We use a generalised Nevanlinna counting function to compute the Hilbert–Schmidt norm of a composition operator on the Bergman space $L_a^2(\mathbb{D})$ and weighted Bergman spaces $L_a^1(dA_\alpha)$ when α is a nonnegative integer.

2010 *Mathematics subject classification*: primary 30H20; secondary 32A36, 47B10, 47B33.

Keywords and phrases: Bergman space, composition operators, counting functions.

1. Introduction

1.1. Background. Let $\mathbb{D}$ denote the unit disc in the complex plane $\mathbb{C}$ and let φ be a holomorphic function on $\mathbb{D}$ with $\varphi(\mathbb{D}) \subset \mathbb{D}$. For every function f analytic in $\mathbb{D}$, the composition operator C_φ is a linear operator defined by $C_\varphi(f) = f \circ \varphi$.

Properties of composition operators on various analytic function spaces have been widely investigated (see, for example, [1, 5, 8, 9]). One of the classical spaces is the Hardy space H^2, the space consisting of the analytic functions f on $\mathbb{D}$ such that

$$\|f\|_{H^2}^2 = \sup_{0<r<1} \int_0^{2\pi} |f(re^{i\theta})|^2 \frac{d\theta}{2\pi} < \infty.$$

Another is the Bergman space $L_a^2(\mathbb{D})$, which is the space consisting of those holomorphic functions f on $\mathbb{D}$ satisfying

$$\|f\|_{L_a^2(\mathbb{D})}^2 = \int_{\mathbb{D}} |f(z)|^2 \, dA(z) < \infty,$$

where

$$dA(z) = \frac{1}{\pi} \, dx \, dy = \frac{r}{\pi} \, dr \, d\theta$$

is the normalised area measure on $\mathbb{D}$. It is well known that C_φ is always bounded on both H^2 and $L_a^2(\mathbb{D})$.

Cheng Yuan is supported by the National Natural Science Foundation of China (Grant No. 11501415); Ze-Hua Zhou is supported by the National Natural Science Foundation of China (Grant No. 11371276).

In [7], Shapiro computed the essential norm of C_φ acting on H^2 in terms of the Nevanlinna counting function of φ. The essential norm of a bounded operator T on a Banach space X, denoted by $\|T\|_{e,X}$, is the distance from T to the subspace of all compact operators acting on X in the operator norm. Also, for a self-map φ on $\mathbb{D}$, the Nevanlinna counting function N_φ is defined on $\mathbb{D} \setminus \{\varphi(0)\}$ and given by

$$N_\varphi(w) = \sum_{\varphi(z)=w} \log \frac{1}{|z|},$$

where multiplicities are counted and $N_\varphi(w)$ is taken to be zero if w is not in the range of φ. The fundamental work of Shapiro [7, Theorem 2.3] asserts that

$$\|C_\varphi\|_{e,H^2} = \limsup_{|w|\to 1} \frac{N_\varphi(w)}{\log(1/|w|)}.$$

Later, in [4], Luecking and Zhu proved that for $0 < p < \infty$, C_φ is in the Schatten class $\mathcal{S}_p$ of H^2 if and only if

$$\int_{\mathbb{D}} \left(\frac{N_\varphi(z)}{\log(1/|z|)} \right)^{p/2} d\lambda(z) < \infty,$$

where $d\lambda(z) = dA(z)/(1 - |z|^2)^2$ is the Möbius invariant measure on $\mathbb{D}$.

On the Bergman space $L^2_a(\mathbb{D})$, Poggi-Corradini verified in [6] that

$$\|C_\varphi\|_{e,L^2_a(\mathbb{D})} = \limsup_{|w|\to 1} \frac{N_{\varphi,2}(w)}{(\log(1/|w|))^2},$$

where

$$N_{\varphi,2}(w) = \sum_{\varphi(z)=w} \left(\log \frac{1}{|z|} \right)^2, \quad w \in \mathbb{D} \setminus \{\varphi(0)\}.$$

Moreover, it is shown in [4] that for $0 < p < \infty$, C_φ is in the Schatten class $\mathcal{S}_p$ of $L^2_a(\mathbb{D})$ if and only if

$$\int_{\mathbb{D}} \left(\frac{N_{\varphi,2}(z)}{(\log(1/|z|))^2} \right)^{p/2} d\lambda(z) < \infty.$$

1.2. Overview. From the above remarks, we know that the Schatten p-class membership of composition operators is closely related to the Nevanlinna counting functions. The Schatten 1-class $\mathcal{S}_1$ is usually called the trace class and $\mathcal{S}_2$ is usually called the Hilbert–Schmidt class.

For any $T \in \mathcal{S}_1$ on a separable Hilbert space H, the trace of T is given by

$$\mathrm{tr}(T) = \sum_{k=0}^{\infty} \langle Te_k, e_k \rangle,$$

where $\{e_k\}$ is any orthonormal basis of H. It is known that the sum is independent of the choice of the orthonormal basis. The Hilbert–Schmidt norm of T is defined by

$$\|T\|^2_{HS} = \mathrm{tr}(T^*T).$$

In this paper, we will compute the Hilbert–Schmidt norm of a composition operator on $L^2_a(\mathbb{D})$. The following theorem is established.

https://doi.org/10.1017/S0004972716000782 Published online by Cambridge University Press

Theorem 1.1. *For an analytic self-map φ of $\mathbb{D}$, let*

$$\tilde{N}_\varphi(w) = 2N_\varphi(w) - \sum_{\varphi(z)=w} (1-|z|^2)$$

be the general counting function of φ.

(i) *For $f, g \in L_a^2(\mathbb{D})$,*

$$\int_{\mathbb{D}} f(\varphi(z))\overline{g(\varphi(z))}\, dA(z) = f(\varphi(0))\overline{g(\varphi(0))} + \int_{\mathbb{D}} f'(z)\overline{g'(z)}\tilde{N}_\varphi(z)\, dA(z).$$

(ii) *If C_φ is in the Hilbert–Schmidt class of $L_a^2(\mathbb{D})$, then*

$$\|C_\varphi\|_{HS}^2 = 1 + \frac{|\varphi(0)|^2(2-|\varphi(0)|^2)}{(1-|\varphi(0)|^2)^2} + \int_{\mathbb{D}} \frac{\tilde{N}_\varphi(z)(2+4|z|^2)}{(1-|z|^2)^4}\, dA(z).$$

2. Proof of Theorem 1.1(i)

The argument is inspired by [3]. For $f, g \in L_a^2(\mathbb{D})$, we can use the Littlewood–Paley formula [2, page 228] to deduce that

$$\begin{aligned}
&\int_{\mathbb{D}} f(\varphi(z))\overline{g(\varphi(z))}\, dA(z)\\
&= \int_0^1 \Big(\frac{1}{\pi}\int_0^{2\pi} f(\varphi(re^{i\theta}))\overline{g(\varphi(re^{i\theta}))} d\theta\Big) r\, dr\\
&= \int_0^1 2\Big(f(\varphi(0))\overline{g(\varphi(0))} + r^2 \int_{\mathbb{D}} f'(\varphi(rw))\overline{g'(\varphi(rw))}|\varphi'(rw)|^2 \log\frac{1}{|w|^2}\, dA(w)\Big) r\, dr\\
&= f(\varphi(0))\overline{g(\varphi(0))} + 2\int_0^1 \Big(\int_{\mathbb{D}} f'(\varphi(rw))\overline{g'(\varphi(rw))}|\varphi'(rw)|^2 \log\frac{1}{|w|^2}\, dA(w)\Big) r^3\, dr.
\end{aligned}$$

Put $u = rw$ in the inner integral. Then

$$\begin{aligned}
&\int_{\mathbb{D}} f'(\varphi(rw))\overline{g'(\varphi(rw))}|\varphi'(rw)|^2 \log\frac{1}{|w|^2}\, dA(w)\\
&\quad = \frac{2}{r^2}\int_{r\mathbb{D}} f'(\varphi(u))\overline{g'(\varphi(u))}|\varphi'(u)|^2 \log\frac{r}{|u|}\, dA(u)\\
&\quad = \frac{2}{\pi r^2}\int_0^{2\pi}\int_0^r f'(\varphi(se^{it}))\overline{g'(\varphi(se^{it}))}|\varphi'(se^{it})|^2 \log\frac{r}{s}\, s\, ds\, dt.
\end{aligned}$$

Fubini's theorem implies that

$$\begin{aligned}
&2\int_0^1 \Big(\int_{\mathbb{D}} f'(\varphi(rw))\overline{g'(\varphi(rw))}|\varphi'(rw)|^2 \log\frac{1}{|w|^2}\, dA(w)\Big) r^3\, dr\\
&= \frac{4}{\pi}\int_0^{2\pi}\int_0^1\int_0^r f'(\varphi(se^{it}))\overline{g'(\varphi(se^{it}))}|\varphi'(se^{it})|^2 \log\frac{r}{s}\, s\, ds\, r\, dr\, dt\\
&= \frac{4}{\pi}\int_0^{2\pi}\int_0^1 f'(\varphi(se^{it}))\overline{g'(\varphi(se^{it}))}|\varphi'(se^{it})|^2 \int_s^1 r\log\frac{r}{s}\, dr s\, ds\, dt.
\end{aligned}$$

https://doi.org/10.1017/S0004972716000782 Published online by Cambridge University Press

Using the identity

$$\int_s^1 r\log\frac{r}{s}\,dr=\frac{1}{2}\log\frac{1}{s}-\frac{1}{4}(1-s^2) \tag{2.1}$$

in the inner integral yields

$$\begin{aligned}
&\frac{4}{\pi}\int_0^{2\pi}\int_0^1 f'(\varphi(se^{it}))\overline{g'(\varphi(se^{it}))}|\varphi'(se^{it})|^2\Big(\frac{1}{2}\log\frac{1}{s}-\frac{1}{4}(1-s^2)\Big)s\,ds\,dt\\
&=\frac{1}{\pi}\int_0^{2\pi}\int_0^1 f'(\varphi(se^{it}))\overline{g'(\varphi(se^{it}))}|\varphi'(se^{it})|^2\Big(\log\frac{1}{s^2}-(1-s^2)\Big)s\,ds\,dt\\
&=\int_{\mathbb{D}} f'(\varphi(w))\overline{g'(\varphi(w))}|\varphi'(w)|^2\Big(\log\frac{1}{|w|^2}-(1-|w|^2)\Big)\,dA(w)\\
&=\int_{\mathbb{D}} f'(z)\overline{g'(z)}\tilde{N}_\varphi(z)\,dA(z).
\end{aligned}$$

This completes the proof.

Taking $\varphi(z)=z$, the identity on $\mathbb{D}$, in Theorem 4.2(i) gives the following corollary.

Corollary 2.1. *If $f\in L_a^2(\mathbb{D})$, then*

$$\|f\|^2_{L_a^2(\mathbb{D})}=|f(0)|^2+\int_{\mathbb{D}}|f'(z)|^2\Big(\log\frac{1}{|z|^2}-(1-|z|^2)\Big)\,dA(z).$$

3. The Hilbert–Schmidt norm

In this section, we prove Theorem 1.1(ii). It is well known that

$$e_n(z)=\sqrt{n+1}\,z^n,\quad n\ge 0,$$

is an orthonormal basis for $L_a^2(\mathbb{D})$. Thus,

$$\begin{aligned}
\|C_\varphi\|^2_{HS}&=\mathrm{tr}(C_\varphi^*C_\varphi)=\sum_{k=0}^\infty\langle C_\varphi^*C_\varphi e_k,e_k\rangle=\sum_{k=0}^\infty\langle C_\varphi e_k,C_\varphi e_k\rangle\\
&=\sum_{k=0}^\infty\langle\sqrt{k+1}\varphi(z)^k,\sqrt{k+1}\varphi(z)^k\rangle=\sum_{k=0}^\infty(k+1)\langle\varphi(z)^k,\varphi(z)^k\rangle\\
&=1+\sum_{k=1}^\infty(k+1)\int_{\mathbb{D}}\varphi(z)^k\overline{\varphi(z)^k}\,dA(z).
\end{aligned}$$

Now we can use Theorem 1.1(i) to deduce that

$$\begin{aligned}
\|C_\varphi\|^2_{HS}&=1+\sum_{k=1}^\infty(k+1)\Big(|\varphi(0)|^{2k}+\int_{\mathbb{D}}k^2|z|^{2k-2}\tilde{N}_\varphi(z)\,dA(z)\Big)\\
&=1+\frac{|\varphi(0)|^2(2-|\varphi(0)|^2)}{(1-|\varphi(0)|^2)^2}+\int_{\mathbb{D}}\sum_{k=1}^\infty(k+1)k^2|z|^{2k-2}\tilde{N}_\varphi(z)\,dA(z)\\
&=1+\frac{|\varphi(0)|^2(2-|\varphi(0)|^2)}{(1-|\varphi(0)|^2)^2}+\int_{\mathbb{D}}\tilde{N}_\varphi(z)\frac{(2+4|z|^2)}{(1-|z|^2)^4}\,dA(z).
\end{aligned}$$

This completes the proof.

4. Composition operators on the weighted Bergman space

For $\alpha > -1$, the weighted Bergman space $L_a^2(dA_\alpha)$ is the space of analytic functions in $\mathbb{D}$ satisfying

$$\|f\|^2_{L_a^2(dA_\alpha)} = \int_{\mathbb{D}} |f(z)|^2 \, dA_\alpha(z) < \infty,$$

where $dA_\alpha(z) = (\alpha+1)(1-|z|^2)^\alpha \, dA(z)$. In some sense, H^2 can be treated as $L_a^2(dA_{-1})$. We have the following corollary.

Corollary 4.1. *If C_φ is in the Hilbert–Schmidt class of H^2, then*

$$\|C_\varphi\|^2_{HS,H^2} = 1 + \frac{|\varphi(0)|^2}{1-|\varphi(0)|^2} + \int_{\mathbb{D}} \frac{1+|z|^2}{(1-|z|^2)^3} N_\varphi(z)\, dA(z).$$

Proof. An orthonormal basis for H^2 can be given as

$$e_n(z) = z^n, \quad n \geq 0.$$

Thus,

$$\begin{aligned}\|C_\varphi\|^2_{HS,H^2} &= 1 + \sum_{n=1}^{\infty}\Big(|\varphi(0)|^{2n} + \int_{\mathbb{D}} n^2|z|^{2n-2} N_\varphi(z)\, dA(z)\Big)\\ &= 1 + \frac{|\varphi(0)|^2}{1-|\varphi(0)|^2} + \int_{\mathbb{D}} \sum_{n=1}^{\infty} n^2|z|^{2n-2} N_\varphi(z)\, dA(z)\\ &= 1 + \frac{|\varphi(0)|^2}{1-|\varphi(0)|^2} + \int_{\mathbb{D}} \frac{1+|z|^2}{(1-|z|^2)^3} N_\varphi(z)\, dA(z).\end{aligned}$$

□

When α is an arbitrary nonnegative integer, we can extend the results of Theorem 1.1 to the weighted Bergman space case. In the rest of this section, we discuss the cases when $\alpha = 1$ and 2.

Theorem 4.2. *For an analytic self-map φ of $\mathbb{D}$, let*

$$\tilde{N}^1_\varphi(w) = 2N_\varphi(w) - \frac{1}{2}\sum_{\varphi(z)=w}(3 - 4|z|^2 + |z|^4)$$

be the general 1-order counting function of φ.

(i) *For $f, g \in L_a^2(dA_1)$,*

$$\int_{\mathbb{D}} f(\varphi(z))\overline{g(\varphi(z))}\, dA_1(z) = f(\varphi(0))\overline{g(\varphi(0))} + \int_{\mathbb{D}} f'(z)\overline{g'(z)}\tilde{N}^1_\varphi(z)\, dA(z).$$

(ii) *If C_φ is in the Hilbert–Schmidt class of $L_a^2(dA_1)$, then*

$$\|C_\varphi\|^2_{HS,L_a^2(dA_1)} = 1 + \frac{|\varphi(0)|^2(3-3|\varphi(0)|^2+|\varphi(0)|^4)}{(1-|\varphi(0)|^2)^3} + \int_{\mathbb{D}} \frac{3(3|z|^2+1)}{(1-|z|^2)^5}\tilde{N}^1_\varphi(z)\, dA(z).$$

https://doi.org/10.1017/S0004972716000782 Published online by Cambridge University Press

PROOF.

(i) The argument is parallel to the proof of Theorem 1.1, with equation (2.1) replaced by

$$\int_s^1 r(1-r^2)\log\frac{r}{s}\,dr = \frac{1}{4}\log\frac{1}{s} - \frac{1}{16}(3-4s^2+s^4).$$

(ii) According to [9, page 78], an orthonormal basis for $L_a^2(dA_1)$ is given by

$$e_n(z) = \sqrt{\frac{(n+1)(n+2)}{2}}z^n, \quad n \geq 0.$$

Thus,

$$\begin{aligned}
\|C_\varphi\|^2_{HS,L_a^2(dA_1)} &= \sum_{n=0}^{\infty}\Big\langle \sqrt{\frac{(n+1)(n+2)}{2}}(\varphi(z))^n, \sqrt{\frac{(n+1)(n+2)}{2}}(\varphi(z))^n\Big\rangle_{L_a^2(dA_1)} \\
&= 1 + \sum_{n=1}^{\infty}\frac{(n+1)(n+2)}{2}\int_{\mathbb{D}}\varphi(z)^n\overline{\varphi(z)^n}\,dA_1(z) \\
&= 1 + \sum_{n=1}^{\infty}\frac{(n+1)(n+2)}{2}\Big(|\varphi(0)|^{2n} + \int_{\mathbb{D}} n^2|z|^{2n-2}\tilde{N}_\varphi^1(z)\,dA(z)\Big) \\
&= 1 + \frac{|\varphi(0)|^2(3-3|\varphi(0)|^2+|\varphi(0)|^4)}{(1-|\varphi(0)|^2)^3} \\
&\quad + \int_{\mathbb{D}}\sum_{n=1}^{\infty}\frac{(n+1)(n+2)}{2}n^2|z|^{2n-2}\tilde{N}_\varphi^1(z)\,dA(z) \\
&= 1 + \frac{|\varphi(0)|^2(3-3|\varphi(0)|^2+|\varphi(0)|^4)}{(1-|\varphi(0)|^2)^3} + \int_{\mathbb{D}}\frac{3(3|z|^2+1)}{(1-|z|^2)^5}\tilde{N}_\varphi^1(z)\,dA(z).
\end{aligned}$$

□

COROLLARY 4.3. *If $f \in L_a^2(dA_1)$, then*

$$\|f\|^2_{L_a^2(dA_1)} = |f(0)|^2 + \int_{\mathbb{D}}|f'(z)|^2\Big(\log\frac{1}{|z|^2} - \frac{1}{2}(3-4|z|^2+|z|^4)\Big)\,dA(z).$$

THEOREM 4.4. *For an analytic self-map φ of $\mathbb{D}$, let*

$$\tilde{N}_\varphi^2(w) = 2N_\varphi(w) - \frac{1}{6}\sum_{\varphi(z)=w}(11-18|z|^2+9|z|^4-2|z|^6)$$

be the general 2-order counting function of φ.

(i) *For $f, g \in L_a^2(dA_2)$,*

$$\int_{\mathbb{D}} f(\varphi(z))\overline{g(\varphi(z))}\,dA_2(z) = f(\varphi(0))\overline{g(\varphi(0))} + \int_{\mathbb{D}} f'(z)\overline{g'(z)}\tilde{N}_\varphi^2(z)\,dA(z).$$

https://doi.org/10.1017/S0004972716000782 Published online by Cambridge University Press

(ii) *If C_φ is in the Hilbert–Schmidt class of $L_a^2(dA_2)$, then*

$$\|C_\varphi\|^2_{HS,L_a^2(dA_2)} = 1 + \frac{(2-|\varphi(0)|^2)(2-2|\varphi(0)|^2+|\varphi(0)|^4)}{|\varphi(0)|^{-2}(1-|\varphi(0)|^2)^4} + \int_{\mathbb{D}} \frac{6(6|z|^2+1)}{(1-|z|^2)^6}\tilde{N}_\varphi^2(z)\,dA(z).$$

PROOF.

(i) In this case, equation (2.1) should be replaced by

$$\int_s^1 r(1-r^2)^2 \log\frac{r}{s}\,dr = \frac{1}{6}\log\frac{1}{s} - \frac{1}{72}(11-18s^2+9s^4-2s^6).$$

(ii) An orthonormal basis for $L_a^2(dA_2)$ is given by

$$e_n(z) = \sqrt{\frac{(n+1)(n+2)(n+3)}{6}}\,z^n, \quad n \ge 0.$$

Thus,

$$\begin{aligned}
\|C_\varphi\|^2_{HS,L_a^2(dA_2)} &= 1 + \sum_{n=1}^\infty \frac{(n+1)(n+2)(n+3)}{6}\int_{\mathbb{D}} \varphi(z)^n\overline{\varphi(z)^n}\,dA_2(z)\\
&= 1 + \sum_{n=1}^\infty \frac{(n+1)(n+2)(n+3)}{6}\Big(|\varphi(0)|^{2n} + \int_{\mathbb{D}} n^2|z|^{2n-2}\tilde{N}_\varphi^2(z)\,dA(z)\Big)\\
&= 1 + \frac{|\varphi(0)|^2(2-|\varphi(0)|^2)(2-2|\varphi(0)|^2+|\varphi(0)|^4)}{(1-|\varphi(0)|^2)^4}\\
&\quad + \int_{\mathbb{D}}\sum_{n=1}^\infty \frac{(n+1)(n+2)(n+3)}{6} n^2|z|^{2n-2}\tilde{N}_\varphi^2(z)\,dA(z)\\
&= 1 + \frac{|\varphi(0)|^2(2-|\varphi(0)|^2)(2-2|\varphi(0)|^2+|\varphi(0)|^4)}{(1-|\varphi(0)|^2)^4}\\
&\quad + \int_{\mathbb{D}} \frac{6(6|z|^2+1)}{(1-|z|^2)^6}\tilde{N}_\varphi^2(z)\,dA(z).
\end{aligned}$$

□

COROLLARY 4.5. *If $f \in L_a^2(dA_2)$, then*

$$\|f\|^2_{L_a^2(dA_2)} = |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2\Big(\log\frac{1}{|z|^2} - \frac{1}{6}(11-18|z|^2+9|z|^4-2|z|^6)\Big)\,dA(z).$$

REMARK 4.6. We can use `Maple` to compute that

$$\int r(1-r^2)^n \log\frac{r}{s}\,dr = -\frac{n}{16}r^4\,{}_3F_2([2,2,-n+1];[3,3];r^2) + \frac{r^2}{4}\Big(\Big(2\log\frac{r}{s}-1\Big)\,{}_2F_1([1,-n];[2];r^2)\Big),$$

https://doi.org/10.1017/S0004972716000782 Published online by Cambridge University Press

where ${}_mF_l([\alpha_1,\dots,\alpha_m];[\beta_1,\dots,\beta_l];x)$ is the hypergeometric function given by

$$ {}_mF_l([\alpha_1,\dots,\alpha_m];[\beta_1,\dots,\beta_l];x)=\sum_{k=0}^{\infty}\frac{x^k\prod_{j=1}^{m}(\alpha_j)_k}{k!\prod_{j=1}^{l}(\beta_j)_k} $$

and $(\alpha)_k$ is the Pochhammer symbol defined by

$$ (\alpha)_k=\frac{\Gamma(\alpha+k)}{\Gamma(\alpha)}. $$

It is easy to check that if α_j is a negative integer for some $j\in\{1,\dots,m\}$, then the hypergeometric function ${}_mF_l([\alpha_1,\dots,\alpha_m];[\beta_1,\dots,\beta_l];x)$ is a polynomial. In particular, if $n\ge 2$ is a positive integer, ${}_3F_2([2,2,-n+1];[3,3];x)$ and ${}_2F_1([1,-n];[2];x)$ are polynomials. Thus, if n is a positive integer with $n\ge 2$,

$$ \begin{aligned} &\int_s^1 r(1-r^2)^n\log\frac{r}{s}\,dr\\ &\quad=-\frac{n}{16}\,{}_3F_2([2,2,-n+1];[3,3];1)+\frac{1}{4}\Big(2\log\frac{1}{s}-1\Big)\,{}_2F_1([1,-n];[2];1)\\ &\qquad+\frac{n}{16}s^4\,{}_3F_2([2,2,-n+1];[3,3];s^2)+\frac{s^2}{4}\,{}_2F_1([1,-n];[2];s^2). \end{aligned} $$

The corresponding result similar to Theorem 1.1 can then be obtained.

5. The Hilbert–Schmidt norm of C_φ^*

It is well known that T is in the Schatten-p class $\mathcal{S}_p$ on a Hilbert space H if and only if T^* is in $\mathcal{S}_p$. Moreover, $\|T^*\|_{\mathcal{S}_p}=\|T\|_{\mathcal{S}_p}$.

From [9, Theorem 6.4], the trace of a positive operator T on $L_a^2(dA_\alpha)$ can be expressed as

$$ \operatorname{tr}(T)=(\alpha+1)\int_{\mathbb{D}}\widetilde{T}(z)\,d\lambda(z), $$

where

$$ \widetilde{T}(z)=\langle Tk_z,k_z\rangle_{L_a^2(dA_\alpha)},\qquad z\in\mathbb{D}, $$

is the Berezin transform of T and

$$ k_z(w)=\frac{(1-|z|^2)^{(2+\alpha)/2}}{(1-w\bar{z})^{2+\alpha}} $$

is the normalised reproducing kernel of $L_a^2(dA_\alpha)$. The reproducing kernel of $L_a^2(dA_\alpha)$ is given by

$$ K_\alpha(w,z)=\frac{1}{(1-w\bar{z})^{2+\alpha}},\qquad z,w\in\mathbb{D}. $$

Obviously,

$$ k_z(w)=K_\alpha(w,z)/\sqrt{K_\alpha(z,z)}. $$

https://doi.org/10.1017/S0004972716000782 Published online by Cambridge University Press

For the composition operator C_φ on $L^2_a(dA_\alpha)$, it is easy to check that

$$C_\varphi^* K_\alpha(w,z) = K_\alpha(w,\varphi(z))$$

and

$$\widetilde{C_\varphi C_\varphi^*}(z) = \|C_\varphi^* k_z\|^2 = \left(\frac{1-|z|^2}{1-|\varphi(z)|^2}\right)^{2+\alpha}.$$

Combining these facts together yields

$$\|C_\varphi^*\|^2_{HS,L^2_a(dA_\alpha)} = \operatorname{tr}(C_\varphi C_\varphi^*) = (\alpha+1)\int_{\mathbb{D}}\left(\frac{1-|z|^2}{1-|\varphi(z)|^2}\right)^{2+\alpha} d\lambda(z).$$

In particular, for $\alpha = 0, 1, 2$, respectively, we have the following results.

Corollary 5.1. *Let φ be an analytic self-map of $\mathbb{D}$. Then*

$$\int_{\mathbb{D}}\left(\frac{1-|z|^2}{1-|\varphi(z)|^2}\right)^2 d\lambda(z) = 1 + \frac{|\varphi(0)|^2(2-|\varphi(0)|^2)}{(1-|\varphi(0)|^2)^2} + \int_{\mathbb{D}} \frac{\tilde{N}_\varphi(z)(2+4|z|^2)}{(1-|z|^2)^4}\, dA(z),$$

$$2\int_{\mathbb{D}}\left(\frac{1-|z|^2}{1-|\varphi(z)|^2}\right)^3 d\lambda(z) = 1 + \frac{|\varphi(0)|^2(3-3|\varphi(0)|^2+|\varphi(0)|^4)}{(1-|\varphi(0)|^2)^3} + \int_{\mathbb{D}} \frac{3(3|z|^2+1)}{(1-|z|^2)^5}\tilde{N}^1_\varphi(z)\, dA(z),$$

$$3\int_{\mathbb{D}}\left(\frac{1-|z|^2}{1-|\varphi(z)|^2}\right)^4 d\lambda(z) = 1 + \frac{(2-|\varphi(0)|^2)(2-2|\varphi(0)|^2+|\varphi(0)|^4)}{|\varphi(0)|^{-2}(1-|\varphi(0)|^2)^4} + \int_{\mathbb{D}} \frac{6(6|z|^2+1)}{(1-|z|^2)^6}\tilde{N}^2_\varphi(z)\, dA(z).$$

References

[1] C. C. Cowen and B. D. MacCluer, *Composition Operators on Spaces of Analytic Functions*, Studies in Advanced Mathematics (CRC Press, Boca Raton, FL, 1995).

[2] J. B. Garnet, *Bounded Analytic Functions*, Graduate Texts in Mathematics, 236 (Springer, New York, 2007), revised first version.

[3] K. Guo and D. Zheng, ‘Rudin orthogonality problem on the Bergman space’, *J. Funct. Anal.* **261** (2011), 51–68.

[4] D. Luecking and K. Zhu, ‘Composition operators belonging to the Schatten ideals’, *Amer. J. Math.* **114** (1992), 1127–1145.

[5] E. Nordgren, ‘Composition operators’, *Canad. J. Math.* **20** (1968), 442–449.

[6] P. Poggi-Corradini, ‘The essential norm of composition operators revisited’, *Contemp. Math.* **213** (1998), 167–173.

[7] J. H. Shapiro, ‘The essential norm of a composition operator’, *Ann. of Math. (2)* **125** (1987), 375–404.

[8] J. H. Shapiro, *Composition Operators and Classical Function Theory* (Springer, New York, 1993).

[9] K. Zhu, *Operator Theory in Function Spaces*, 2nd edn (American Mathematical Society, Providence, RI, 2007).

CHENG YUAN, Institute of Mathematics, School of Science,
Tianjin University of Technology and Education, Tianjin 300222,
PR China
e-mail: yuancheng1984@163.com

ZE-HUA ZHOU, Department of Mathematics, Tianjin University,
Tianjin 300354, PR China
e-mail: zehuazhoumath@aliyun.com, zhzhou@tju.edu.cn