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EXISTENCE OF NONINNER AUTOMORPHISMS OF ORDER p IN SOME FINITE p-GROUPS

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Abstract

Let G be a nonabelian finite p-group of order p^m. A long-standing conjecture asserts that G admits a noninner automorphism of order p. In this paper we prove the validity of the conjecture if $\exp(G) = p^{m-2}$. We also show that if G is a finite p-group of maximal class, then G has at least $p(p-1)$ noninner automorphisms of order p which fix $\Phi(G)$ elementwise.

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1. Introduction

Let G be a nonabelian finite p-group. A long-standing conjecture asserts that G admits a noninner automorphism of order p (see also [9, Problem 4.13]). Liebeck [8] has shown that finite p-groups of class 2 with $p > 2$ must have a noninner automorphism of order p fixing $\Phi(G)$ elementwise. For $p = 2$, Liebeck produced an example of a 2-group G of class 2 and order 2^7 with the property that all automorphisms of order two fixing $\Phi(G)$ are inner. Deaconescu and Silberberg [5] reduced the verification of the conjecture to the case where $C_G(Z(\Phi(G))) = \Phi(G)$. Abdollahi [1–3] proved that if G is a finite p-group of class 2, 3 or $G/Z(G)$ is powerful, then G has a noninner automorphism of order p leaving either $\Phi(G)$ or $\Omega_1(Z(G))$ fixed elementwise. Jamali and Viseh [7] proved that every nonabelian finite 2-group with a cyclic commutator subgroup has a noninner automorphism of order two fixing either $\Phi(G)$ or $Z(G)$ elementwise. In [11] we showed the validity of the conjecture when G satisfies one of the following conditions:

(1) $\operatorname{rank}(G' \cap Z(G)) \neq \operatorname{rank}(Z(G))$;
(2) $Z_2(G)/Z(G)$ is cyclic;
(3) $C_G(Z(\Phi(G))) = \Phi(G)$ and $(Z_2(G) \cap Z(\Phi(G)))/Z(G)$ is not elementary abelian of rank rs, where $r = d(G)$ and $s = \operatorname{rank}(Z(G))$.

Here we show the validity of the conjecture for some finite p-groups. In fact we prove the following theorem.

THEOREM A. *Let G be a nonabelian finite p-group of order p^m satisfying one of the following conditions:*

(1) $\Phi(G)$ *is cyclic;*
(2) $\exp(G) = p^{m-2}$*;*
(3) $s = \mathrm{rank}(Z(G)) \geq (m-1)/2$*;*
(4) $s = \mathrm{rank}(Z(G)) \geq 2$ *and* $[G : Z(G)] \leq p^4$.

Then G has a noninner automorphism of order p leaving either $\Phi(G)$ or $\Omega_1(Z(G))$ fixed elementwise.

A p-group G of order p^n with $n \geq 3$ and nilpotency class $n-1$ is said to be of maximal class. The cornerstone in the theory of p-groups of maximal class is the paper by Blackburn [4] (see also Huppert [6, III.14]). If G is a p-group of maximal class, then G has a noninner automorphism of order p which fixes $\Phi(G)$ elementwise (see [11, Corollary 2.7]). In this paper, we prove a stronger version of [11, Corollary 2.7].

THEOREM B. *If G is a group of order p^n $(n \geq 4)$ and of maximal class, then G has at least $p(p-1)$ noninner automorphisms of order p which fix $\Phi(G)$ elementwise.*

2. Proofs

PROOF OF THEOREM A. If $C_G(Z(\Phi(G))) \neq \Phi(G)$, then by [5], G has a noninner automorphism of order p which fixes $\Phi(G)$ elementwise. Hence we need only consider the case where $C_G(Z(\Phi(G))) = \Phi(G)$. Also by [1, Theorem] and [11, Corollary 2.7] we can assume that $\mathrm{cl}(G) \neq 2, m-1$.

(1) Since $\Phi(G)$ is cyclic,

$$\frac{Z_2(G) \cap Z(\Phi(G))}{Z(G)} = \frac{Z_2(G) \cap \Phi(G)}{Z(G)} \leq \frac{\Phi(G)}{Z(G)}$$

is cyclic. Hence by [11, Theorem], G has a noninner automorphism of order p which fixes $\Phi(G)$ elementwise.

(2) Let a be the element of G of order p^{m-2}. Assume that $C_G(a) \neq \langle a \rangle$. Choose $x \in C_G(a) \setminus \langle a \rangle$. Hence $M = \langle x, a \rangle$ is an abelian maximal subgroup of G. Therefore

$$M \leq C_G(M) \leq C_G(\Phi(G)) = C_G(Z(\Phi(G))) = \Phi(G),$$

a contradiction. Thus $C_G(a) = \langle a \rangle$. Suppose first that p is odd. Hence by [10, Proposition 5], G is isomorphic, for $m \geq 4$, to
$G_7 = \langle a, b, c \mid a^{p^{m-2}} = 1, b^p = 1, c^p = 1, b^{-1}ab = a^{1+p^{m-3}}, c^{-1}ac = ab, bc = cb \rangle$;

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and, for $m \geq 5$, to
$G_8 = \langle a, b \mid a^{p^{m-2}} = 1, b^{p^2} = 1, b^{-1}ab = a^{1+p^{m-4}} \rangle$,
$G_{10} = \langle a, b \mid a^{p^{m-2}} = 1, a^{p^{m-3}} = b^{p^2}, a^{-1}ba = b^{1-p} \rangle$.

If G is isomorphic to G_7, then $M = \langle a, b \mid a^{p^{m-2}} = 1, b^p = 1, b^{-1}ab = a^{1+p^{m-3}} \rangle$ is a maximal subgroup of G and $Z(G) = Z(M) = \langle a^p \rangle$. Now the map ϕ defined by $\phi(a) = a$, $\phi(b) = b$ and $\phi(c) = a^{p^{m-3}}c$ is a noninner automorphism of order p which fixes $\Phi(G)$ elementwise. Now let G be isomorphic to G_8. Thus $G' = \langle a^{p^{m-4}} \rangle$ and $Z(G) = \langle a^{p^2} \rangle$. If $m = 5$, then $\Phi(G) = \langle a^p, b^p \rangle$, whence $Z(\Phi(G)) = Z(G) = \langle a^{p^2} \rangle$. Thus

$$C_G(Z(\Phi(G))) = C_G(Z(G)) = G \neq \Phi(G),$$

a contradiction. If $m \geq 6$, then G is of class 2, a contradiction. Finally let G be isomorphic to G_{10}. Thus $Z(G) = \langle a^{p^2} \rangle$ and $G' = \langle b^p \rangle$. Set $\bar{a} = aZ(G)$ and $\bar{b} = bZ(G)$. Hence

$$\bar{G} = G/Z(G) = \langle \bar{a}, \bar{b} \mid \bar{a}^{p^2} = \bar{b}^{p^2} = \bar{1}, [\bar{a}, \bar{b}] = \bar{b}^p \rangle.$$

Therefore $\bar{G}' = \langle \bar{b}^p \rangle \leq \bar{G}^p$. Thus $G/Z(G)$ is a powerful p-group and so by [2, Theorem 2.6], G has a noninner automorphism of order p which fixes $\Phi(G)$ elementwise.

Now let $p = 2$. By [10, Proposition 7], G is isomorphic, for $m \geq 5$, to
$G_{15} = \langle a, b, c \mid a^{2^{m-2}} = 1, b^2 = 1, c^2 = 1, b^{-1}ab = a^{1+2^{m-3}}, c^{-1}ac = a^{-1+2^{m-3}}, bc = cb \rangle$,
$G_{16} = \langle a, b, c \mid a^{2^{m-2}} = 1, b^2 = 1, c^2 = 1, b^{-1}ab = a^{1+2^{m-3}}, c^{-1}ac = a^{-1+2^{m-3}}$,
$c^{-1}bc = a^{2^{m-3}}b \rangle$,
$G_{17} = \langle a, b, c \mid a^{2^{m-2}} = 1, b^2 = 1, c^2 = 1, b^{-1}ab = a^{1+2^{m-3}}, c^{-1}ac = ab, bc = cb \rangle$,
$G_{18} = \langle a, b, c \mid a^{2^{m-2}} = 1, b^2 = 1, c^2 = b, b^{-1}ab = a^{1+2^{m-3}}, c^{-1}ac = a^{-1}b \rangle$;
for $m \geq 6$, to
$G_{20} = \langle a, b \mid a^{2^{m-2}} = 1, b^4 = 1, b^{-1}ab = a^{-1+2^{m-4}} \rangle$,
$G_{21} = \langle a, b \mid a^{2^{m-2}} = 1, a^{2^{m-3}} = b^4, a^{-1}ba = b^{-1} \rangle$,
$G_{24} = \langle a, b, c \mid a^{2^{m-2}} = 1, b^2 = 1, c^2 = 1, b^{-1}ab = a^{1+2^{m-3}}, c^{-1}ac = a^{-1+2^{m-4}}b, bc = cb \rangle$,
$G_{25} = \langle a, b, c \mid a^{2^{m-2}} = 1, b^2 = 1, c^2 = a^{2^{m-3}}, b^{-1}ab = a^{1+2^{m-3}}$,
$c^{-1}ac = a^{-1+2^{m-4}}b, bc = cb \rangle$;
and for $m = 5$ to
$G_{26} = \langle a, b, c \mid a^8 = 1, b^2 = 1, c^2 = a^4, b^{-1}ab = a^5, c^{-1}ac = ab, bc = cb \rangle$.

If G is one of the groups G_{15} or G_{16}, then $G' = \langle a^2b \rangle \cong C_{2^{m-3}}$ and $Z(G) = \langle a^{2^{m-3}} \rangle \cong C_2$. Hence the map ϕ defined by $\phi(a) = a^{-1}$, $\phi(b) = b$ and $\phi(c) = c$ is a noninner automorphism of order two which fixes $Z(G)$ elementwise. If G is the group G_{17}, then

$$G' = \langle a^{2^{m-3}}, b \rangle \cong C_2 \times C_2 \quad \text{and} \quad Z(G) = \langle a^4 \rangle \cong C_{2^{m-4}}.$$

Hence the map ϕ defined by $\phi(a) = ac$, $\phi(b) = b$ and $\phi(c) = c$ is a noninner automorphism of order two which fixes $Z(G)$ elementwise. If G is the group G_{18}, then the map ϕ defined by $\phi(a) = ab$, $\phi(b) = b$ and $\phi(c) = c^{-1}$ is a noninner automorphism of order two which fixes $\Omega_1(Z(G))$ elementwise. If G is the group G_{20}, then $G' = \langle a^2 \rangle \cong C_{2^{m-3}}$ and $Z(G) = \langle a^{2^{m-3}} \rangle \cong C_2$. Hence the map ϕ defined by $\phi(a) = a^{-1}$ and $\phi(b) = b$ is a

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noninner automorphism of order two which fixes $Z(G)$ elementwise. If G is the group G_{21}, then $G' = \langle b^2\rangle \cong C_4$ and $Z(G) = \langle a^2\rangle \cong C_{2^{m-3}}$. Since $|G/Z(G)| = 8$ and $G/Z(G)$ is not abelian, we have $Z_2(G)/Z(G) \cong C_2$ and hence by [11, Theorem], G has a noninner automorphism of order two which fixes $\Phi(G)$ elementwise. Let G be one of the groups G_{24} or G_{25}. Then

$$G' = \langle a^2 b\rangle \cong C_{2^{m-3}} \quad \text{and} \quad Z(G) = \langle a^{2^{m-3}}\rangle \cong C_2.$$

Hence the map ϕ defined by $\phi(a) = ab$, $\phi(b) = b$ and $\phi(c) = bc$ is a noninner automorphism of order two which fixes $Z(G)$ elementwise. Finally, let G be the group G_{26}. Then $G' = \langle a^4, b\rangle \cong C_2 \times C_2$ and $Z(G) = \langle a^4\rangle \cong C_2$. Hence the map ϕ defined by $\phi(a) = ac$, $\phi(b) = b$ and $\phi(c) = c^{-1}$ is a noninner automorphism of order two which fixes $Z(G)$ elementwise.

(3) If $s = 1$, then $m = 3$ and so $\Phi(G) = Z(G) \cong C_p$. Hence $C_G(Z(\Phi(G))) = G \neq \Phi(G)$, which is a contradiction. Therefore $s \geq 2$. We claim that

$$\frac{Z_2(G) \cap C_G(\Phi(G))}{Z(G)} \ncong \mathrm{Hom}\Big(\frac{G}{\Phi(G)}, Z(G)\Big).$$

Assume to the contrary that

$$\frac{Z_2(G) \cap C_G(\Phi(G))}{Z(G)} \cong \mathrm{Hom}\Big(\frac{G}{\Phi(G)}, Z(G)\Big).$$

Since G is nonabelian and $s = \mathrm{rank}(Z(G)) \geq (m-1)/2$,

$$\Big|\frac{G}{Z(G)}\Big| \geq \Big|\frac{Z_2(G) \cap C_G(\Phi(G))}{Z(G)}\Big| = \Big|\mathrm{Hom}\Big(\frac{G}{\Phi(G)}, Z(G)\Big)\Big| \geq p^{2s} \geq p^{m-1}.$$

This is a contradiction, since $s \geq 2$. Hence by [11, Proposition 2.5], G has a noninner automorphism of order p which fixes $\Phi(G)$ elementwise.

(4) Since $G/Z(G)$ is nonabelian, $|G/Z(G)| = p^3$ or p^4. If $|G/Z(G)| = p^3$, then $|Z_2(G)/Z(G)| = p$. Hence by [11, Theorem], G has a noninner automorphism of order p which fixes $\Phi(G)$ elementwise. Now let $|G/Z(G)| = p^4$. By [11, Theorem], we can assume that $Z_2(G)/Z(G)$ is not cyclic. Therefore $Z_2(G)/Z(G) \cong C_p \times C_p$. It follows from $s \geq 2$ and $d(G) \geq 2$ that

$$\frac{Z_2(G) \cap C_G(\Phi(G))}{Z(G)} \ncong \mathrm{Hom}\Big(\frac{G}{\Phi(G)}, Z(G)\Big).$$

Hence by [11, Proposition 2.5], G has a noninner automorphism of order p which fixes $\Phi(G)$ elementwise. □

Proof of Theorem B. Since G is of maximal class, $|Z_2(G)| = p^2$. Hence by [12, Step 1], $C_G(Z_2(G))$ is a maximal subgroup of G, M_0 say. Since G is of maximal class, by [4, p. 53] G has just $p+1$ maximal subgroups. Let $M_1, \ldots, M_p$ denote the maximal subgroups different from M_0.

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We now divide the proof into the following three steps.

Step 1. G is not the union of $M_1, \ldots, M_p$.

For $i = 2, \ldots, p$, $|M_i \cap M_1| = p^{n-2}$, and so $|M_i \setminus M_1| = p^{n-2}(p-1)$. Hence

$$\left|\left(\bigcup_{i=1}^{p} M_i\right) \setminus M_1\right| \le \sum_{i=2}^{p} |M_i \setminus M_1| = p^{n-2}(p-1)^2 < p^n - p^{n-1} = |G \setminus M_1|.$$

Step 2. $Z(M_i) = Z(G) \cong C_p$ for $i = 1, \ldots, p$.

Suppose that $|Z(M)| > p$ for some $M = M_i$. Since by [4, Lemma 2.2], $Z_2(G)$ is the only normal subgroup of G of order p^2, and $Z(M)$ is normal in G, we have $Z_2(G) \le Z(M)$, and therefore

$$M \le C_G(Z(M)) \le C_G(Z_2(G)) = M_0,$$

a contradiction.

Step 3. G has at least $p(p-1)$ noninner automorphisms of order p which fix $\Phi(G)$ elementwise.

By Step 1, we can pick $x \in G \setminus (M_1 \cup \cdots \cup M_p)$. Thus

$$G = \langle x\rangle M_1 = \langle x\rangle M_2 = \cdots = \langle x\rangle M_p.$$

By Step 2, $Z(M_j) = Z(G) \cong C_p$ for all $1 \le j \le p$. Let $Z(G) = \langle z\rangle$ and $1 \le j \le p$. It follows from

$$Z(G) \le \Phi(G) \quad \text{and} \quad Z(G) = Z(M_j) = C_G(M_j)$$

that the map α_j defined on G by $\alpha_j(x^i m_j) = x^i m_j z^i$ for every $m_j \in M_j$ and every $i \in \{0, 1, \ldots, p-1\}$ is a noninner automorphism of order p which fixes $\Phi(G)$ elementwise. Let $\alpha_j = \alpha_k$ for some $1 \le j \ne k \le p$. Pick any $x_0 \in M_j \setminus M_k$. Since $G = \langle x\rangle M_k$, we have $x_0 = x^u m_k$ for some $0 < u < p$ and some $m_k \in M_k$. Then

$$x_0 = \alpha_j(x_0) = \alpha_k(x_0) = x^u m_k z^u = x_0 z^u.$$

Therefore $z^u = 1$ and so p must divide u, a contradiction. It can be verified that if α_j is one of the above automorphisms, then $\alpha_j^2, \ldots, \alpha_j^{p-1}$ are noninner automorphisms of order p which fix $\Phi(G)$ elementwise. By imitating the proof of the above we get $\alpha_j^s \ne \alpha_j^t$ for all $1 \le j \ne k \le p$ and $1 \le s, t \le p-1$. Therefore G has at least $p(p-1)$ noninner automorphisms of order p which fix $\Phi(G)$ elementwise. □

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