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A NOTE ON ENDOMORPHISM SEMIGROUPS

BY
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If $\mathfrak{A}$ is a universal algebra, the set of endomorphisms of $\mathfrak{A}$ forms a monoid (i.e., semigroup with identity) under composition. We denote it by End $(\mathfrak{A})$. For definitions and notations, see [1]. It is well known (e.g., [1], Theorem 12.3) that for any monoid M there is a unary algebra $\mathfrak{A}$ with $M \cong \text{End}(\mathfrak{A})$. E. Mendelsohn and Z. Hedrlín [3] have proved that the monoid of a subalgebra of an algebra $\mathfrak{A}$ is independent of the monoid of $\mathfrak{A}$. In [2], Hedrlín proves the same for the monoid of a homomorphic image of $\mathfrak{A}$. The proofs of these depend heavily on graph-theoretical and category-theoretical considerations. In this note considerably shorter direct algebraic proofs are given of these results.

THEOREM 1. *Let M_1 and M_2 be monoids. There exist an algebra $\mathfrak{A}$ with* End $(\mathfrak{A}) \cong M_1$ *and a subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ with* End $(\mathfrak{B}) \cong M_2$.

Proof. Let $\mathfrak{A}_i = \langle A_i; F_i \rangle$ be a unary algebra with End $(\mathfrak{A}_i) \cong M_i$ for $i \in \{1, 2\}$. Assume $A_1 \cap A_2 = \varnothing$, and choose distinct objects a, b, c not in $A_1 \cup A_2$. We define an algebra on the set $A = A_1 \cup A_2 \cup \{a, b, c\}$ as follows: for $i \in \{1, 2\}, f \in F_i$, we define $\bar{f}$ by $\bar{f}(x) = f(x)$ if $x \in A_i$ and $\bar{f}(x) = x$ if $x \in A - A_i$; define $\alpha(x) = a$ if $x \in A_1 \cup \{a\}$, $\alpha(x) = b$ if $x \in A_2 \cup \{c\}$, and $\alpha(b) = c$; define $\beta(x) = x$ if $x \in A_1 \cup A_2$, $\beta(a) = \beta(c) = b$, and $\beta(b) = c$; define $\gamma(x) = b$ for all $x \in A$; for each $y \in A_2$, define unary δ_y by $\delta_y(a) = y$ and $\delta_y(x) = x$ if $x \in A - \{a\}$. Let $\mathfrak{A}$ be the resulting algebra. The subset $A_2 \cup \{b, c\}$ determines a subalgebra, which we denote by $\mathfrak{B}$. If $\varphi \in \text{End}(\mathfrak{A}_1)$, extend φ to φ^*: $A \to A$ by $\varphi^*(x) = x$ for all $x \notin A_1$. Then it is easily checked that $\varphi^* \in \text{End}(\mathfrak{A})$. On the other hand, if $\psi \in \text{End}(\mathfrak{A})$, then $\psi(a) = a$ and $\psi(b) = b$ since these are the only fixed points of α and γ, respectively. Then $\psi(c) = \psi(\alpha(b)) = \alpha(\psi(b)) = \alpha(b) = c$. If $y \in A_2$, then $\psi(y) = \psi(\delta_y(a)) = \delta_y(\psi(a)) = \delta_y(a) = y$. If $x \in A_1$, then $\alpha(\psi(x)) = \psi(\alpha(x)) = \psi(a) = a$, hence $\psi(x) \in A_1 \cup \{a\}$. Since $\beta(\psi(x)) = \psi(\beta(x)) = \psi(x)$, but $\beta(a) \neq a$, we must have $\psi(x) \in A_1$. If φ is the restriction of ψ to A_1, then by the definition of the operations $\bar{f}$ for $f \in F_1$, we have $\varphi \in \text{End}(\mathfrak{A}_1)$. Clearly $\psi = \varphi^*$, so the correspondence $\varphi \to \varphi^*$ is a bijection between End $(\mathfrak{A}_1)$ and End $(\mathfrak{A})$. Since this clearly preserves composition, we have End $(\mathfrak{A}) \cong \text{End}(\mathfrak{A}_1) \cong M_1$. In a similar way we can show End $(\mathfrak{B}) \cong \text{End}(\mathfrak{A}_2)$, since endomorphisms of $\mathfrak{B}$ are just the extensions of endomorphisms of $\mathfrak{A}_2$ that fix b and c. The details are omitted, and this completes the proof.

THEOREM 2. *Let M_1 and M_2 be monoids. There exist algebras $\mathfrak{A}$ and $\mathfrak{B}$ with* End $(\mathfrak{A}) \cong M_1$, End $(\mathfrak{B}) \cong M_2$, *and $\mathfrak{B}$ a homomorphic image of $\mathfrak{A}$.*

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Proof. Let $\mathfrak{A}_i = \langle A_i; F_i \rangle$ be a unary algebra with End $(\mathfrak{A}_i) \cong M_i$ for $i \in \{1, 2\}$, and assume $A_1 \cap A_2 = \varnothing$. Take a disjoint copy A_2' of A_2, and four new elements a, b, c, and d. Let $\eta: A_2 \to A_2'$ be a bijection. We define an algebra on the set $A = A_1 \cup A_2 \cup A_2' \cup \{a, b, c, d\}$ as follows: for $i \in \{1, 2\}$ and $f \in F_i$, define $\bar{f}$ by $\bar{f}(x) = f(x)$ for $x \in A_i$ and $\bar{f}(x) = x$ for $x \in A - A_i$; define α by $\alpha(x) = x$ if $x \in A_1$, $\alpha(x) = \eta(x)$ if $x \in A_2$, $\alpha(x) = a$ if $x \in A_2' \cup \{b, c, d\}$, and $\alpha(a) = b$; for each $y \in A_2' \cup \{a, b, c, d\}$ define β_y by $\beta_y(x) = y$ for all $x \in A$; finally, define γ by $\gamma(x) = x$ if $x \in A_2$, $\gamma(x) = c$ if $x \in A_1 \cup A_2' \cup \{a, b, d\}$, and $\gamma(c) = d$. Let $\mathfrak{A}$ be the resulting algebra. We claim that End $(\mathfrak{A}) \cong$ End $(\mathfrak{A}_1)$. Let $\varphi \in$ End $(\mathfrak{A})$. Then $\varphi(x) = x$ for $x \in A_2' \cup \{a, b, c, d\}$ because of the operations β_x. If $x \in A_2$, then $\varphi(x) \in A_2$ because it is a fixed point of γ. Then $\alpha(x) \in A_2'$, so $\alpha(\varphi(x)) = \varphi(\alpha(x)) = \alpha(x)$, and since α is 1-1 on A_2, $\varphi(x) = x$. If $x \in A_1$, then $\varphi(x) \in A_1$ since it is fixed under α. Thus the restriction of φ to A_1 sends A_1 into itself; and because of the definitions of $\bar{f}$ for $f \in F_1$, it is an endomorphism of $\mathfrak{A}_1$. As in the previous proof, it is easily checked that this correspondence is an isomorphism between End $(\mathfrak{A})$ and End $(\mathfrak{A}_1)$.

Now define $\Theta = (A_1 \cup A_2' \cup \{a, b\})^2 \cup \{(x, x) : x \in A\}$. Then Θ is easily seen to be a congruence relation on $\mathfrak{A}$. The factor algebra $\mathfrak{B} = \mathfrak{A}/\Theta$ consists of a copy of $A_2 \cup \{a, c, d\}$ with which we identify it. If $\psi \in$ End $(\mathfrak{B})$, then because of β_a, β_b, and β_c, we have $\psi(a) = a$, $\psi(b) = b$, and $\psi(c) = c$. Also if $x \in A_2$, then $\psi(x) \in A_2$ because it is fixed under γ. As before, restriction to A_2 establishes an isomorphism from End $(\mathfrak{B})$ to End $(\mathfrak{A}_2)$; hence $\mathfrak{B}$ is the required homomorphic image.

Remark. It should be mentioned that in [2] and [3] the authors prove results stronger than our Theorems 1 and 2. Namely, they require that the algebras $\mathfrak{A}$ have only one binary operation or two unary operations. These stronger forms follow immediately from Theorems 1 and 2 using the following result, which is implicit in [4]: Given any algebra $\mathfrak{A}$ there is an algebra $\mathfrak{A}^*$ having only one binary operation (respectively, two unary operations), with End $(\mathfrak{A}) \cong$ End $(\mathfrak{A}^*)$, and if $\mathfrak{A}$ is a subalgebra or homomorphic image of $\mathfrak{B}$, then $\mathfrak{A}^*$ is a subalgebra or homomorphic image, respectively, of $\mathfrak{B}^*$.

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