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Abstract

In this paper, we determine the group of contact transformations modulo contact isotopies for Legendrian circle bundles over closed surfaces of non-positive Euler characteristic. These results extend and correct those presented by the first author in a former work. The main ingredient we use is connectedness of certain spaces of embeddings of surfaces into contact 3-manifolds. This connectedness question is also studied for itself with a number of (hopefully instructive) examples.

Introduction

In this paper, we study contact transformations of 3-manifolds which are circle bundles equipped with contact structures tangent to the fibers. The main example of such a manifold is the unit cotangent bundle $V = T_1^*S$ of a surface S, endowed with its canonical contact structure ξ: this contact manifold is also called the manifold of cooriented contact elements over S. Other examples are obtained as follows: for any positive integer d dividing $|2g-2|$ where g is the genus of S, the manifold V admits a d-fold fibered cyclic cover V_d and the pullback ξ_d of ξ on V_d is a contact structure tangent to the fibers of V_d over S. It is a nice and easy observation that all Legendrian circle bundles are of this form (see [Lut83, p. 179]).

Our goal here is to determine the contact mapping class group of (V_d, ξ_d), namely the group $\pi_0\mathcal{D}(V_d; \xi_d)$, where $\mathcal{D}(V_d; \xi_d)$ denotes the group of contact transformations of (V_d, ξ_d) (diffeomorphisms preserving the contact structure with its coorientation). This group has an obvious homomorphism to the usual (smooth) mapping class group $\pi_0\mathcal{D}(V_d)$ (where $\mathcal{D}(V_d)$ consists of all diffeomorphisms of V_d) which has been computed in [Wal67]. By standard fibration results (see § 1), the kernel of this homomorphism is tightly related to the fundamental group of the isotopy class of ξ_d, i.e. the connected component of ξ_d in the space $\mathcal{CS}(V_d)$ of all contact structures on V_d.

Our main result is the following theorem, in which V_d is endowed with any principal circle bundle structure inherited from one on $V = T_1^*S$.

Theorem (Theorem 2.5, Corollary 2.6, and Theorem 2.9). *Let S be a closed, connected, orientable surface of genus $g \geqslant 1$ and d a positive integer dividing $2g-2$. Denote by $R_t\colon V_d \to V_d$ the action of $2\pi t \in \mathbb{R}/2\pi\mathbb{Z}$ by rotation along the fibers. Then:*

– *the fundamental group $\pi_1(\mathcal{CS}(V_d), \xi_d)$ is infinite cyclic and generated by the loop $(R_t)_*\xi_d$, $t \in [0, 1/d]$;*

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– *the kernel of the natural homomorphism*

$$\pi_0\mathcal{D}(V_d;\xi_d) \to \pi_0\mathcal{D}(V_d)$$

is the cyclic group $\mathbb{Z}/d\mathbb{Z}$ spanned by the contact mapping classes of the deck transformations of V_d over T_1^*S.

In the torus case, $g = 1$, Geiges and Gonzalo Pérez proved in [GG04, Theorem 2.1], also using topological methods, that the loop appearing in the previous theorem generates an infinite cyclic group in $\pi_1(\mathcal{CS},\xi_d)$ (but they did not prove that this cyclic subgroup is the full group). Bourgeois reproved this using contact homology in [Bou06, Proposition 2]. Then Geiges and Klukas in [GK14] proved the theorem when $g = 1$ and $d = 1$.

As a direct consequence of the theorem above, we obtain the following result.

Corollary (Corollary 2.8). *Let S be a closed orientable surface of genus $g \geqslant 2$. Then the natural homomorphism*

$$\pi_0\mathcal{D}(S) \to \pi_0\mathcal{D}(T_1^*S;\xi)$$

induced by the differential is an isomorphism.

This corollary is stated as [Gir01b, Theorem 1] but the 'proof' given there contains a mistake. See § 3.1 for a detailed erratum and § 3.2 for several related examples.

In the case $g = 1$, each manifold V_d is diffeomorphic to $\mathbb{T}^3 = (\mathbb{R}/\mathbb{Z})^3$ fibering over $S = \mathbb{T}^2$ by the projection $(x, y, z) \mapsto (x, y)$, and its contact structure ξ_d can be defined by

$$\cos(2d\pi z)\,dx - \sin(2d\pi z)\,dy = 0, \quad x, y, z \in \mathbb{R}/\mathbb{Z}.$$

Then the results of [Gir94, Gir99] about the so-called 'pre-Lagrangian tori' readily imply that the image of the obvious homomorphism

$$\pi_0\mathcal{D}(\mathbb{T}^3,\xi_d) \to \pi_0\mathcal{D}(\mathbb{T}^3) = \mathrm{SL}_3(\mathbb{Z})$$

is the subgroup Π of transformations preserving $\mathbb{Z}^2 \times \{0\} \subset \mathbb{Z}^3$ (see also [EP94]). Therefore, we have the following result.

Corollary (Theorem 2.9). *The induced homomorphism $\pi_0\mathcal{D}(\mathbb{T}^3,\xi_1) \to \Pi$ is an isomorphism.*

Finally, for $g = 0$, an unpublished result of Fraser shows that the contact transformation group of the standard projective 3-space (namely, the unit cotangent bundle of the 2-sphere) is connected. This completes the list of contact mapping class groups for unit cotangent bundles of closed orientable surfaces.

1. Natural fibrations in contact topology

For any compact manifold V with (possibly empty) boundary, we denote by $\mathcal{D}(V,\partial V)$ the group of diffeomorphisms of V relative to a neighborhood of the boundary. When the boundary of V is empty, we sometimes drop ∂V from our notation.

Lemma 1.1. *Let (V,ξ) be a compact contact manifold. The natural map*

$$\mathcal{D}(V,\partial V) \to \mathcal{D}(V,\partial V)\cdot\xi, \quad \phi \mapsto \phi_*\xi,$$

is a locally trivial fibration whose fiber is the contact transformation group $\mathcal{D}(V,\partial V;\xi)$.

Proof. By the classical Cerf–Palais fibration criterion (see [Cer91, Lemma 2, p. 240, § 0.4.4] or [Pal60, Theorem A]), it suffices to show that the above map admits a continuous local section near every point $\xi_0 \in \mathcal{D}(V, \partial V) \cdot \xi$. Choose a vector field ν transverse to ξ_0 and observe that ξ_0 has a convex open neighborhood $\mathcal{U}$ in $\mathcal{D}(V, \partial V) \cdot \xi$ which consists of contact structures transverse to ν. Then Gray's theorem associates to any contact structure $\xi_1 \in \mathcal{U}$ an isotopy $\phi_t \in \mathcal{D}(V, \partial V)$, $t \in [0, 1]$, such that $\phi_0 = \mathrm{Id}$ and $\phi_{t*}\xi_0 = (1-t)\xi_0 + t\xi_1$ for all $t \in [0, 1]$. Moreover, one can easily arrange that this isotopy varies continuously with ξ_1. Indeed, it is uniquely determined by a smooth path σ_t of sections $V \to TV/\xi_t$ and, if this path depends continuously on ξ_1 (if σ_t is identically 0, for instance), then so does the corresponding isotopy, and the map $\xi_1 \mapsto \phi_1$ gives the required continuous section. □

Assume from now on that the contact manifold (V, ξ) has dimension three, and let F be a compact orientable surface properly embedded in V. Recall that the characteristic foliation ξF of F in (V, ξ) is the (singular) foliation spanned by the line field $\xi \cap TF$ (the singularities are the points where $\xi = TF$). We denote by:

- $\mathcal{P}(F, V)$ the space of proper embeddings $F \to V$ which coincide with the inclusion $\iota \colon F \to V$ near ∂F;
- $\mathcal{P}_o(F, V) \subset \mathcal{P}(F, V)$ the connected component of the inclusion ι;
- $\mathcal{P}(F, V; \xi) \subset \mathcal{P}(F, V)$ the subspace of embeddings ψ which induce the same characteristic foliation as the inclusion, i.e. satisfy $\xi\, \psi(F) = \psi_*(\xi F)$;
- $\mathcal{P}_o(F, V; \xi)$ the intersection $\mathcal{P}_o(F, V) \cap \mathcal{P}(F, V; \xi)$.

The same standard tools as in the proof of Lemma 1.1 give the following result.

Lemma 1.2. *Let (V, ξ) be a compact contact manifold of dimension three. For every properly embedded surface $F \subset V$, the restriction map*

$$\mathcal{D}(V, \partial V; \xi) \to \mathcal{P}(F, V; \xi), \quad \phi \mapsto \phi|_F,$$

is a locally trivial fibration over its image.

Proof. Each embedding $\psi_0 \in \mathcal{P}(F, V; \xi)$ which lies in the image is also in the image of the restriction map

$$\mathcal{D}(V, \partial V) \to \mathcal{P}(F, V), \quad \phi \mapsto \phi|_F,$$

which is a locally trivial fibration by the Cerf–Palais fibration theorem. As a result, there exists a neighborhood $\mathcal{V}$ of ψ_0 in $\mathcal{P}(F, V; \xi)$, and a continuous extension map $\mathcal{V} \to \mathcal{D}(V, \partial V)$ which associates to every embedding $\psi_1 \in \mathcal{V}$ a diffeomorphism $\phi_1 \in \mathcal{D}(V, \partial V)$ such that $\psi_1 = \phi_1|_F$. Using Gray's theorem, and the fact that embeddings in $\mathcal{V}$ induce the same characteristic foliation, it is easy to correct this extension map so that it takes values in $\mathcal{D}(V, \partial V; \xi)$. We conclude applying again the Cerf–Palais fibration criterion. □

Remark 1.3. The above lemma is a typical result where it is useful to work relatively to a neighborhood of the boundary and not just to the boundary itself. Indeed, any diffeomorphism relative to both ∂V and a properly embedded surface F is tangent to the identity along ∂F, and so the fibration property fails in this case. However, since the inclusion of $\mathcal{D}(V, \partial V; \xi)$ into the group of contact transformations relative to the boundary is a homotopy equivalence, this does not matter.

We now recall how the theory of ξ-convex surfaces can be used to study the homotopy type of $\mathcal{P}(F, V; \xi)$ (see [Gir91, Gir01b]). Let F be a compact orientable surface properly embedded in (V, ξ) with (possibly empty) Legendrian boundary; F is *ξ-convex* if it admits a *homogeneous neighborhood*, namely a product neighborhood

$$U := F \times \mathbb{R} \supset F = F \times \{0\} \quad \text{with } \partial U = \partial F \times \mathbb{R} \subset \partial V,$$

in which the vector field ∂_t, $t \in \mathbb{R}$, preserves ξ. The points $p \in F$ where $\partial_t(p) \in \xi$ then form a multi-curve Γ called the *dividing set* of F associated with U. This curve depends on U and its product structure, but its isotopy class does not, and is uniquely determined by the foliation ξF: specifically, Γ is the unique multi-curve (up to isotopy) which avoids the singularities of ξF, is transverse to ξF and divides F into regions where the dynamics of ξF is alternatively expanding and contracting (see [Gir91, Gir01b] for more details). It follows that the curves dividing a given singular foliation on a surface, in the above sense, form a contractible space. Among them, the dividing sets associated with all possible homogeneous neighborhoods U of F are those intersecting ∂F at the points where ξ is tangent to ∂V. Moreover, we have the following proposition (see [Gir01b, Lemmas 6 and 7]).

Proposition 1.4. *Let F be a ξ-convex surface, U a homogeneous neighborhood, and Γ the associated dividing set.*

(a) *The space $\mathscr{F}(F; \Gamma)$ of singular foliations on F which are tangent to ∂F and admit Γ_U as a dividing set is an open contractible neighborhood of ξF in the space of all singular foliations on F.*

(b) *There exists a continuous map $\mathscr{F}(F; \Gamma) \to \mathcal{P}(F, V)$, $\sigma \mapsto \psi_\sigma$, with the following properties:*

- (i) *$\psi_{\xi F}$ is the inclusion $F \to V$;*
- (ii) *$\psi_\sigma(F)$ is contained in $U = F \times \mathbb{R}$ and transverse to the contact vector field ∂_t for all $\sigma \in \mathscr{F}(F; \Gamma)$;*
- (iii) *$\xi\, \psi_\sigma(F) = \psi_\sigma(\sigma)$ for all $\sigma \in \mathscr{F}(F; \Gamma)$.*

(c) *Let $\mathcal{P}(F, V; \Gamma)$ denote the space of embeddings $\psi \in \mathcal{P}(F, V)$ such that $\psi(F)$ is ξ-convex with dividing set $\psi(\Gamma)$. Then the inclusion $\mathcal{P}(F, V; \xi) \to \mathcal{P}(F, V; \Gamma)$ is a homotopy equivalence.*

We will also need the following result which shows that the homotopy type of $\mathcal{D}(V, \partial V; \xi)$ is locally constant when ∂V is ξ-convex (see [Gir01b, Proposition 8].

Proposition 1.5. *Let V be a compact 3-manifold, Δ a multi-curve on ∂V, and $\mathcal{CS}(V, \Delta)$ the space of contact structures ξ on V for which ∂V is ξ-convex with dividing set Δ. For $\xi \in \mathcal{CS}(V, \Delta)$, the homotopy type of $\mathcal{D}(V, \partial V; \xi)$ depends only on the connected component of $\mathcal{CS}(V; \Delta)$ containing ξ.*

2. Legendrian circle bundles over surfaces

2.1 The general case

In this section, we consider a compact oriented surface S which is neither a sphere nor a torus. The torus case will be treated in § 2.2. Actually, using results of [Mas08], the following discussion can be carried over to orbifolds.

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As in the introduction, (V_d, ξ_d) denotes the d-fold fibered cyclic cover of the unit cotangent bundle $V := V_1 = T_1^*S$, equipped with the pullback of the canonical contact structure ξ of V.

In any trivialization of $\pi : V_d \to S$ over a subsurface $R \subset S$ with non-empty boundary, the contact structure ξ_d can be described as follows: if J denotes the complex structure on S induced by our choice of a principal bundle structure on T_1^*S, then the restriction of ξ_d to $V_d|_R \simeq R \times \mathbb{S}^1$ has the form

$$\xi_{\lambda,d} = \ker(\cos(d\theta)\lambda + \sin(d\theta)\lambda \circ J), \quad \theta \in \mathbb{S}^1,$$

where λ is a non-singular 1-form on R. In practice, we will take R equal to S if S has non-empty boundary, and to S with an open disk removed if S is closed.

Now observe that the preimage $F := \pi^{-1}(\gamma)$ of any properly embedded curve γ in S is a ξ_d-convex surface in V_d. Indeed, any vector field X in S transverse to γ (and tangent to ∂S) lifts to a contact vector field $\bar{X}$ transverse to F (and tangent to ∂V_d). The dividing set of $\xi_d F$ associated with $\bar{X}$ is the set of points in F where ξ_d projects down (by the differential of π) to the line spanned by X.

If γ is contained in the subsurface R, the trivialization of $V_d|_R$ induces a diffeomorphism $F = \pi^{-1}(\gamma) \simeq \gamma \times \mathbb{S}^1$. Then the dividing set, provided all its components are consistently oriented (as parallel curves), represents the homology class $2(d, x-1) \in \mathbb{Z}^2 = H_1(\gamma \times \mathbb{S}^1)$, where x denotes the index of λ along γ.

The following proposition can be proved using [Gir00, Lemma 4.7] (a special case of the semi-local Bennequin inequality proved later as [Gir01a, Proposition 4.10]) exactly as in [Gir01a, Lemma 3.9] which dealt with circle bundles without boundary.

Proposition 2.1. *Let F be a torus fibered over a homotopically essential circle in S, and Γ a dividing set for $\xi_d F$. For any isotopy φ such that $\varphi_1(F)$ is also ξ_d-convex, the foliation $\xi_d \varphi_1(F)$ is divided by a collection of curves isotopic to the components of $\varphi_1(\Gamma)$.*

We now turn to spaces of embeddings of surfaces. The following lemma is useful to prove the existence of contact transformations which are smoothly but not contact isotopic to the identity.

Lemma 2.2. *Let T be a fibered torus over a homotopically non-trivial circle C in S, and $i\colon T \to V_d$ the inclusion map. Let R_t be the action of $e^{2i\pi t}$ on V_d. For any non-zero integer k in $\mathbb{Z}$, the path $\gamma_k\colon [0,1] \to \mathcal{P}_o(T, V_d)$ defined by $\gamma_k(t) = R_{kt/d} \circ i$ is non-trivial in $\pi_1(\mathcal{P}_o(T, V_d), \mathcal{P}_o(T, V_d; \xi_d))$.*

For any integer k between 1 and $d-1$, the action of $R_{k/d}$ on $\pi_0(\mathcal{P}_o(T, V_d; \xi_d))$ is non-trivial.

The above lemma will be reduced to the following statement.

Proposition 2.3 [Ghi06, Proposition 7.1]. *In $(\mathbb{T}^2 \times \mathbb{R}, \ker(\cos(2n\pi z)\, dx - \sin(2n\pi z)\, dy))$, the Legendrian circles $\{0\} \times \mathbb{S}^1 \times \{0\}$ and $\{0\} \times \mathbb{S}^1 \times \{k\}$ are not contact isotopic for any $k \neq 0$.*

Alternatively, one could use the stronger result due to Eliashberg *et al.* [EHS95] saying that, in $(\mathbb{T}^3, \ker(\cos(2n\pi z)\, dx - \sin(2n\pi z)\, dy))$, the Legendrian circle $\{0\} \times \mathbb{S}^1 \times \{0\}$ cannot be displaced from the pre-Lagrangian torus $\mathbb{S}^1 \times \mathbb{S}^1 \times \{0\}$ by a contact isotopy. However, this result uses holomorphic curves in symplectizations so it has a different flavor from the techniques we use in this paper.

Proof of Lemma 2.2. We first explain how to reduce the first statement to a statement in a thickened torus. Suppose for contradiction that there is a path j in $\mathcal{P}_o(T, V_d; \xi_d)$ from the inclusion to $\gamma_k(1)$. The path lifting property of the fibration $\mathcal{D}(V_d; \xi_d) \to \mathcal{P}(T, V_d; \xi_d)$ of Lemma 1.2 gives a contact isotopy φ_t such that $\gamma_k(t) = \varphi_t \circ i$. Let $p\colon \hat{S} \to S$ be the covering map associated to the subgroup generated by C in $\pi_1(S)$, and let $\hat{V}$ be the induced circle bundle over $\hat{S}$. We denote by $\hat{T}$ the compact component of $p^{-1}(T)$. The contact isotopy φ lifts to a contact isotopy $\hat{\varphi}$ for the induced contact structure $p^*\xi_d$. The interior of $\hat{S}$ is an open annulus, and contains a closed sub-annulus A such that $\hat{\varphi}_t(\hat{T})$ stays above A for all t. One can then cut off $\hat{\varphi}_t$, using Libermann's theorem from [Lib59], to get a contact isotopy with support in $A \times \mathbb{S}^1$. In $H_1(\hat{T}, \mathbb{Z})$, we consider a $\mathbb{Z}$-basis (S, F), where F is the homology class of fibers. The torus $\hat{T}$ has circles of singularities which can be oriented consistently to get a total homology class $2(dS + mF)$, where m is an unknown integer. After pulling back everything under a d-fold covering map from A to itself, we can assume there are exactly $2d$ circles of singularities. The circle bundle over A then embeds into $\mathbb{T}^3$ equipped with $\ker(\cos(2d\pi z)\, dx - \sin(2d\pi z)\, dy)$, so it is sufficient to get a contradiction there. Note that we do not make any claim concerning how the circle bundle structure embeds inside $\mathbb{T}^3$ and we will not use it.

We now consider the covering map $\mathbb{T}^2 \times \mathbb{R} \to \mathbb{T}^3$ sending (x, y, s) to $(x, y, s \mod \mathbb{Z})$. The contact isotopy $\hat{\varphi}$ lifts to a contact isotopy contradicting Proposition 2.3.

The statement about $\pi_0(\mathcal{P}_o(T, V_d; \xi_d))$ follows immediately from what we proved and the long exact sequence of the pair $(\mathcal{P}_o(T, V_d), \mathcal{P}_o(T, V_d; \xi_d))$, since the path corresponding to k between 1 and $d-1$ does not come from a loop in $\mathcal{P}_o(T, V_d)$. □

Proposition 2.4. *If T is a fibered torus over a non-separating embedded circle in S, then the group of deck transformations of $V_d \to V$ acts freely and transitively on $\pi_0(\mathcal{P}_o(T, V_d; \xi_d))$. If A is a fibered annulus over a non-separating properly embedded arc in S, then $\mathcal{P}_o(A, V_d; \xi_d)$ is connected.*

The following proof will need one more technical ingredient from the study of contact structures on circle bundles: the twisting number. Fibers in V_d have a canonical framing coming from vector fields along the fiber which project to some constant vector on the base. If L is any Legendrian curve isotopic to a fiber, we call twisting number of L the number $t(L)$ of turns made by ξ_d along L compared to the canonical framing transported by isotopy from some fiber, see [Gir01a, p. 227] for further discussion. As explained in [Gir01a, Lemma 3.6], it follows from Bennequin's inequality in $\mathbb{R}^3$ that $t(L) \leqslant -d$ for all L.

Proof. We first prove connectedness of $\mathcal{P}_o(A, V_d; \xi_d)$. Let $(j_t)_{t \in [0,1]}$ be an isotopy of embeddings of A which coincides with the inclusion map on a neighborhood of ∂A. Let Γ be a dividing set on A associated to some homogeneous neighborhood. According to Proposition 1.4(c), we only need to prove that j is homotopic to a path in $\mathcal{P}_o(A, V_d; \Gamma)$.

We use Colin's discretization technique [Col97], which relies on the following observation. We can find times $t_0 = 0 < t_1 < \cdots < t_k = 1$ such that, for t in $[t_i, t_{i+1}]$, the annuli $j_t(A)$ are all contained in some pinched product: $A_i \times [0,1]$ with $\{x\} \times [0,1]$ collapsed to a point for x in a neighborhood of ∂A_i. The sub-path j_t, $t \in [t_i, t_{i+1}]$ is then homotopic, with fixed end points, to the concatenation of two paths whose common extremity has image $A_i \times \{0\}$. In addition, genericity of ξ_d-convex surfaces allows us to assume $A_i \times \{0\}$ is ξ_d-convex. So we have replaced j by the concatenation of $2k$ paths of embeddings sweeping out pinched products. We can assume there is only one such path, and the general case follows by induction.

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The classification of tight contact structures on solid tori [Gir00, Hon00] guarantees that, in this situation, j is homotopic to an isotopy in $\mathcal{P}_o(A, V_d; \Gamma)$ as soon as $A' = j_1(A)$ is divided by a curve Γ' isotopic to $j_1(\Gamma)$. So we prove that fact.

We first remark that both Γ and Γ' are made of $2d$ traversing curves, because otherwise we could use the realization lemma (Proposition 1.4(b)) to produce a Legendrian circle L isotopic to the fibers with twisting number $t(L) > -d$. We now need two separate arguments, depending on the value of d.

Suppose first that d is greater than one. Let A'' and A''' be annuli isotopic to A' through ξ_d-convex surfaces and such that the annuli A, A', A'', A''' pairwise bound pinched products, and there is an arc going from A to A''' in the pinched product they bound and meeting A' and then A'' in its interior. Near their boundary, all of these annuli are fibered over arcs in S. Let $T \times [0,1]$ be a thickened torus with ξ_d-convex boundary $T_0 \sqcup T_1$, such that T_0 is a smoothing of $A \cup A'''$, T_1 is a smoothing of $A' \cup A''$, and those tori are fibered in the smoothing region. We identify the first homology groups of T_0 and T_1 using the product $T \times [0,1]$, and fix an integer basis (S, F) where F is the class coming from fibers of V.

If Γ' is not isotopic to $j_1(\Gamma)$, then, after orienting all dividing curves of T_0 and T_1 in the same way, their total homology class is $2(d, x_0)$ on T_0 and $2(d, x_1)$ on T_1 with $x_1 \neq x_0$. Pick's formula, proved in [Pic99], ensures that the triangle with vertices $(0,0)$, (d, x_0) and (d, x_1) in $H_1(T; \mathbb{R})$ contains integer points outside its vertical edge. If (a, b) is such a point, then $a < d$. The classification of tight contact structures on thickened tori then gives a ξ_d-convex torus in $T \times [0,1]$ divided by a collection of parallel curves which can be oriented all in the same way to have total homology class $2(a, b)$. The realization lemma gives again a Legendrian curve with twisting number $t = -a > -d$, hence a contradiction.

We now handle the case $d = 1$. In particular (V_d, ξ_d) is isomorphic to (V, ξ). Adding 1-handles, one can easily embed S into a surface S' with connected boundary so that ξ extends to a contact structure tangent to the fibers of $S' \times \mathbb{S}^1$ and, of course, the projection of A stays non-separating in S'. So we assume S has connected boundary. Let S'' be the surface obtained by gluing a disk along the boundary of S. The description of the contact structure ξ in terms of a 1-form λ on S allows us to understand the characteristic foliation $\xi \partial V$ in terms of the index of λ along ∂S. The latter is given by the Poincaré–Hopf theorem, so it is fixed. Using this information, we can embed (V, ξ) into the space V'' of contact elements of S'', with its canonical contact structure. In V'', one can extend A and A' to isotopic non-separating tori, which coincide outside V. Both tori are divided by two curves, and Proposition 2.1 guarantees those curves are isotopic. This implies that Γ' is isotopic to $j_1(\Gamma)$.

We now turn to the case of a torus T fibered over a non-separating circle C in S. We fix a dividing set Γ of $\xi_d T$. Let j be any isotopy of embeddings of T in V_d such that $T' = j_1(T)$ is ξ_d-convex. Proposition 2.1 ensures that any component of any dividing set of $\xi_d T'$ is isotopic in T' to a component of $j_1(\Gamma)$. However, there always exist isotopies of T which change the number of dividing curves and, as explained in § 3, there is no general result allowing us to get rid of them. This is where we need C to be non-separating, and not only homotopically non-trivial. The idea, which was born in [Ghi05, Proposition 5.4] and further developed in [Mas08, Lemma 8.5], is to consider a fibered torus F intersecting T along one fiber. Then for any isotopy φ_t, one can discretize the movement of F while constructing an isotopy of T through ξ_d-convex surfaces. There is no boundary in [Mas08], but one can check that it does not change anything here. This trick constructs a contact isotopy φ'_t such that $\varphi'_1(T) = \varphi_1(T)$. Of course parametrizations do not match in general: $(\varphi'_1)^{-1} \circ \varphi_1$ induces a self-diffeomorphism of T which may fail to be isotopic to the identity among diffeomorphisms preserving $\xi_d T$. However, after composing φ'_1 by a deck transformation, we can assume that each circle of singularities of T is globally preserved and, after an ultimate contact isotopy, we get a path in $\mathcal{P}_o(T, V_d; \xi_d)$. So the deck transformations group acts transitively on $\pi_0(\mathcal{P}_o(T, V_d; \xi_d))$. The last part of Lemma 2.2 states that this action is also free. □

Theorem 2.5. *If S is closed, then the kernel of the canonical homomorphism $\pi_0\mathcal{D}(V_d, \xi_d) \to \pi_0\mathcal{D}(V_d)$ is the cyclic group of deck transformations of V_d over V. If V has non-empty boundary, then $\pi_0\mathcal{D}(V_d, \partial V; \xi_d) \to \pi_0\mathcal{D}(V_d, \partial V_d)$ is injective.*

Proof. We first assume V has non-empty boundary and prove that the map $\pi_0\mathcal{D}(V_d, \partial V_d; \xi_d) \to \pi_0\mathcal{D}(V_d, \partial V_d)$ is injective. The proof proceeds by induction on

$$n(S) = -2\chi(S) - \beta(S) = \beta(S) + 4g(S) - 4,$$

where $\chi(S)$ and $g(S)$ are the Euler characteristic and genus of S and $\beta(S)$ is the number of connected components of ∂S. So $n(S) \geqslant -3$ with equality when S is a disk.

We first explain the induction step so we assume $n(S) > -3$. Let φ be a contactomorphism of V_d relative to some neighborhood U of ∂V_d, and smoothly isotopic to the identity relative to U. Let a be a properly embedded non-separating arc in S, and denote by A the annulus fibered over a, and $i\colon A \to V_d$ the inclusion map. According to Proposition 2.4, $\mathcal{P}_o(A, V_d, \xi_d)$ is connected. Hence, the path lifting property of the fibration $\mathcal{D}_o(V_d, \partial V_d; \xi_d) \to \mathcal{P}_o(A, V_d; \xi_d)$ from Lemma 1.2 implies that φ is contact isotopic to some φ' which is relative to A and U. Using Remark 1.3, we can assume φ' is relative to a neighborhood of $\partial V_d \cup A$ which is fibered over some neighborhood W of $a \cup \partial S$ in S. We cut S along a, and round the corners inside W to get a subsurface $S' \subset S$ with $n(S') < n(S)$. By induction hypothesis applied to $\pi^{-1}(S')$, φ' is contact isotopic to the identity so the induction step is complete.

The induction starts with the disk case, which is already explained with all details in [Gir01b, p. 345]. The idea is the same as for the induction step, but the cutting surface in the solid torus V_d is a meridian disk. There are no such disks with Legendrian boundary in V_d but one can use the realization lemma (Proposition 1.4) to deform ξ_d near ∂V_d until such a disk exists. This does not change the homotopy type of $\mathcal{D}(V_d, \partial V_d; \xi_d)$ according to Proposition 1.5. A variation on Colin's result about embedding of disks in [Col99, Theorem 3.1] then replaces Proposition 2.4, and the final isotopy is provided by Eliashberg's result in [Eli92] that $\pi_0\mathcal{D}(B^3, \partial B^3; \xi)$ is trivial for the standard ball.

We now turn to the case where V_d is closed. We first prove that the group of deck transformations injects into $\pi_0\mathcal{D}(V_d; \xi_d)$. Let C be a non-separating circle in S and T the fibered torus over C. Denote by i the inclusion of T in V_d. Proposition 2.4 guarantees that the action of a non-trivial deck transformation f on $\pi_0(\mathcal{P}_o(T, V_d; \xi_d))$ is non-trivial. Hence, f is non-trivial in $\pi_0\mathcal{D}(V_d; \xi_d)$.

We now prove surjectivity. Let φ be a contactomorphism of V_d which is smoothly isotopic to the identity. Proposition 2.4 gives a deck transformation f such that $f \circ \varphi \circ i$ is isotopic to i in $\mathcal{P}_o(T, V_d; \xi_d)$. As above, this implies that $f \circ \varphi$ is contact isotopic to a contactomorphism φ' which is relative to an open fibered neighborhood U of T. The circle bundle $V_d \backslash U$ has non-empty boundary hence we know that φ' is contact isotopic to identity. □

Corollary 2.6. *Assume that V_d has empty boundary and denote by $\mathcal{CS}$ the space $\mathcal{D}(V_d) \cdot \xi_d$ of contact structures isomorphic to ξ_d on V_d. Let R_t denote the action of $e^{2i\pi t} \in \mathbb{S}^1$ on V_d. The fundamental group $\pi_1(\mathcal{CS}, \xi_d)$ is an infinite cyclic group generated by the loop $t \mapsto (R_{t/d})_*\xi_d$.*

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Proof. The fibration $\mathcal{D}(V_d) \to \mathcal{CS}$ of Lemma 1.1 gives the exact sequence

$$\pi_1(\mathcal{D}(V_d), \mathrm{Id}) \to \pi_1(\mathcal{CS}, \xi_d) \to \pi_0(\mathcal{D}(V_d; \xi_d)) \to \pi_0(\mathcal{D}(V_d)).$$

We know from [Lau74, Hat76] that $\pi_1(\mathcal{D}(V_d), \mathrm{Id})$ is an infinite cyclic group generated by the loop $t \mapsto R_t$, $t \in [0, 1]$. Lemma 2.2 implies that this group injects into $\pi_0(\mathcal{D}_o(V_d), \mathcal{D}_o(V_d; \xi_d)) \simeq \pi_1(\mathcal{CS}, \xi_d)$. The result then follows from Theorem 2.5 describing the kernel of $\pi_0(\mathcal{D}(V_d; \xi_d)) \to \pi_0(\mathcal{D}(V_d))$. □

Before coming back to the special case of $V = V_1$, we note one more general corollary of Proposition 2.1.

Lemma 2.7. *A diffeomorphism of V_d which is fibered over the identity is isotopic to a contactomorphism only if it is isotopic to the identity.*

Proof. Let f be a diffeomorphism of V_d fibered over the identity of S. In order to guarantee that f is isotopic to the identity, it is enough to check that, for every torus T fibered over a homotopically essential circle, the restriction of f to T preserves an isotopy class of curves which is different from the class of fibers. Assume that f is isotopic to a contactomorphism. This condition means that $\xi'_d = f^*\xi_d$ is isotopic to ξ_d. Since f is fibered, the contact structure ξ'_d is also tangent to the fibers. Proposition 2.1 then implies that, for each torus T as above, f preserves the isotopy class of dividing curves. Those dividing curves are homotopically essential, and not isotopic to fibers. Hence, f is isotopic to the identity. □

Corollary 2.8. *The lifting map from $\pi_0\mathcal{D}(S)$ to $\pi_0\mathcal{D}(V, \xi)$ is an isomorphism.*

Proof. We denote by p the projection from V to S, and by $\mathcal{D}(S, \partial S)$ the group of diffeomorphisms of S relative to a neighborhood of ∂S. In the sequence of maps

$$\pi_0\mathcal{D}(S, \partial S) \to \pi_0\mathcal{D}(V, \partial V; \xi) \to \pi_0\mathcal{D}(V, \partial V)$$

the composite map is known to be injective (this follows from considerations of fundamental groups), so the first map is also injective. It remains to prove that it is surjective. Let φ be a contactomorphism. We want to prove that φ is contact isotopic to the lift of some diffeomorphism of S. According to Waldhausen [Wal67, Satz 10.1], φ is smoothly isotopic to a fibered diffeomorphism f: there exists an isotopy ψ, and a diffeomorphism $\bar{f}$ in $\mathcal{D}(S, \partial S)$, such that $f = \psi_1 \circ \varphi$ and $p \circ f = \bar{f} \circ p$. We will prove that φ is contact isotopic to the lift $D\bar{f}$. We first note that $f \circ D(\bar{f})^{-1}$ is fibered over the identity, and is smoothly isotopic to a contactomorphism (through the path $t \mapsto \psi_{1-t} \circ \varphi \circ D(\bar{f})^{-1}$). Lemma 2.7 then guarantees that $f \circ D(\bar{f})^{-1}$ is smoothly isotopic to the identity. Hence, φ is smoothly isotopic to $D\bar{f}$, hence contact isotopic to $D\bar{f}$ according to Theorem 2.5. □

2.2 The torus case

We now explain how the previous discussion can be modified to handle the case of a torus base. On $\mathbb{T}^3 = (\mathbb{R}/\mathbb{Z})^3$ with coordinates (x, y, z), we set

$$\xi_d = \ker(\cos(2d\pi z)\, dx - \sin(2d\pi z)\, dy).$$

The case $d = 1$ corresponds to the contact element bundle of $\mathbb{T}^2$ while higher values of d come from self-covering maps unwrapping the fibers. We denote by R_t the map $(x, y, z) \mapsto (x, y, z + 2\pi t)$.

Theorem 2.9. *On* $(\mathbb{T}^3, \xi_d)$*:*

(1) *a diffeomorphism is isotopic to a contactomorphism if and only its action on* $H_2(\mathbb{T}^3)$ *preserves the homology class, up to sign, of the pre-Lagrangian torus* $\{z = 0\}$*;*
(2) *the kernel of* $\pi_0\mathcal{D}(\mathbb{T}^3;\xi) \to \pi_0\mathcal{D}(\mathbb{T}^3)$ *is isomorphic to the cyclic group of order* $d-1$ *generated by* $(x, y, z) \mapsto (x, y, z + 1/d)$*;*
(3) *the fundamental group* $\pi_1(\mathcal{CS}(\mathbb{T}^3), \xi_d)$ *is an infinite cyclic group generated by the loop* $t \mapsto (R_{t/d})_*\xi_d$, $t \in [0, 1]$.

The first point comes directly from the classification of isotopy classes of tight contact structures on $\mathbb{T}^3$, that we now recall. The first result, proved in [Gir94], is that all incompressible pre-Lagrangian tori in $(\mathbb{T}^3, \xi_d)$ are isotopic to $\{z = 0\}$. In particular, they share a common homology class which is well defined up to sign in $H_2(\mathbb{T}^3)$. Next recall that the torsion of a contact manifold (V, ξ) was defined, in [Gir00, Definition 1.2], to be the supremum of all integers $n \geqslant 1$ such that there exist a contact embedding of

$$(T^2 \times [0, 1], \ker(\cos(2n\pi z)\, dx - \sin(2n\pi z)\, dy)), \quad (x, y, z) \in T^2 \times [0, 1]$$

into the interior of (V, ξ), or zero if no such integer n exists. It follows from [Gir00, Proposition 3.42] that the torsion of ξ_d on $\mathbb{T}^3$ is $d-1$. The classification of isotopy classes of tight contact structure on $\mathbb{T}^3$ established in [Gir00] is that any tight contact structure is isomorphic to some ξ_d, and two of them are isotopic if and only if they have the same torsion and their incompressible pre-Lagrangian tori are homologous. The first point of the above theorem follows from this classification, and the obvious observation that isomorphic contact structures have the same torsion.

The description of the kernel in the second point has exactly the same proof as in the preceding section.

The third point is slightly different because $\pi_1(\mathcal{D}(\mathbb{T}^3), \mathrm{Id})$ has rank three. It is generated by the three obvious circle actions on $(\mathbb{S}^1)^3$. However, two of these circle actions actually belong to $\mathcal{D}(\mathbb{T}^3; \xi_d)$ so that, in the exact sequence

$$\pi_1(\mathcal{D}(\mathbb{T}^3), \mathrm{Id}) \to \pi_1(\mathcal{CS}, \xi_d) \to \pi_0(\mathcal{D}(\mathbb{T}^3; \xi_d)) \to \pi_0(\mathcal{D}(\mathbb{T}^3))$$

considered in the proof of Corollary 2.6, the extra generators of $\pi_1(\mathcal{D}(\mathbb{T}^3, \mathrm{Id}))$ are mapped to trivial elements of $\pi_1(\mathcal{CS}, \xi_d)$, and the end result does not change.

3. Examples of disconnected spaces of embeddings

3.1 Erratum about the reference [Gir01b]

In the proof of [Gir01b, Proposition 10], the assertion following the words *Grâce au lemme 14* (namely, the claim that *il est possible de trouver des points* $s_0 = 0 < s_1 < \cdots < s_k = 1$ *tels que, pour* $0 \leqslant i \leqslant k-1$, *les surfaces* F_s, $s \in [s_i, s_{i+1}]$ *soient toutes incluses dans un voisinage rétractile* U_i *de* F_{s_i}) is wrong. In fact, the statement of Proposition 10 turns out to be wrong (Proposition 3.5 below provides a counter-example). As a consequence, Lemma 19 and the proofs of Theorems 1, 3 and 4 are also wrong. Other intermediate results (in particular, Lemma 15 which is sometimes useful as a complement to contact convexity theory) are not impacted.

To be more explicit, Lemma 14 allows us to assume that each surface F_s has a retractible neighborhood U_s. Each neighborhood U_s contains all surfaces sufficiently close to F_s; in other

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words, there exists a positive number ε_s such that $F_t \subset U_s$ as soon as $|s-t| < \varepsilon_s$. The intervals $J_s = (s-\varepsilon_s, s+\varepsilon_s)$ form an open covering of the segment $[0,1]$ so this covering admits a finite subcovering J_{s_i}, $0 \leqslant i \leqslant k$, and the points s_i can be chosen and indexed so that $s_0 = 0$, $s_k = 1$ and $s_{i-1} < s_i$ for $1 \leqslant i \leqslant k$. Thus, each $U_i = U_{s_i}$ is a retractible neighborhood of F_{s_i} such that $U_i \supset F_s$ for all s with $|s - s_i| < \varepsilon_{s_i}$. This does not imply that $U_i \supset F_s$ for all $s \in [s_i, s_{i+1}]$ and, in general, it is impossible to force U_i to contain $F_{s_{i+1}}$. Here is an example: assume that F_s is a convex torus for $s \notin \{1/3, 2/3\}$ and that $s = 1/3$ (respectively $s = 2/3$) corresponds to the death (respectively the birth) of a pair of parallel components in the dividing set; then $F_{1/3}$ cannot appear in any retractible neighborhood of any F_s with $s < 1/3$. Indeed [Gir00, Lemma 4.7] guarantees, in particular, that a homogeneous neighborhood of a torus divided by $2n$ essential curve contains no essential torus divided by $2k$ curves with $k < n$. In contrast, $F_{2/3}$ can appear in a retractible neighborhood of some F_s with $1/3 < s < 2/3$, see [Gir00, Lemma 3.31b].

3.2 Disconnected spaces of embeddings

In this section we describe examples of disconnected spaces consisting of smoothly isotopic embeddings inducing a fixed characteristic foliation. Those examples should be compared with the connectedness results which were crucial in § 2, and complement the erratum above. More specifically, we construct disconnected spaces of smoothly isotopic ξ-convex embeddings with a fixed dividing set, and the former spaces are deformation retracts of the latter by Proposition 1.4.

Proposition 3.1. *In $\mathbb{S}^3$ equipped with its standard contact structure, let D be an unknotted immersed disk with a single clasp self-intersection and such that the contact structure is tangent to D along its boundary (here unknotted means that D has a regular neighborhood which is an unknotted solid torus, see Figures 1 and 2). Let W be any unknotted solid torus which is a regular neighborhood of D. Assume that $T = \partial W$ is ξ-convex. The space $\mathcal{P}_o(T, \mathbb{S}^3; \xi)$ is not connected.*

Proof. We coorient T so that W is on the negative side of T and we denote by W' the solid torus which is the closure of $\mathbb{S}^3 \backslash W$. Since ξ orients $\mathbb{S}^3$, we also get an orientation on T. This orientation induces a cyclic ordering on $P(H_1(T;\mathbb{R}))$. We set $d = \ker(H_1(T) \to H_1(W))$ and $d' = \ker(H_1(T) \to H_1(W'))$ where maps are induced by inclusion. A direction in $H_1(T)$ distinct from d and d' will be called positive if it lies between d and d', and negative otherwise.

If T_1 is any cooriented unknotted torus, then we can repeat the above discussion and, for any isotopy sending T to T_1 (preserving orientations), positive directions will get identified with positive directions because such isotopies map meridian disks to meridian disks in each solid torus.

Claim. *Any unknotted ξ-convex torus in $\mathbb{S}^3$ is divided by a collection of essential closed curves whose direction is positive.*

Proof of claim. Because ξ is tight, we know from [Gir01a, Théorème 4.5] that T is divided by a collection of parallel homotopically essential circles. We fix a positive basis (μ, μ') of $H_1(T)$ such that μ is in d and μ' is in d' (using the notation above). Let Γ be a dividing set for T. For some choice of orientation, the components of Γ have homology class $p\mu + q\mu'$ with $q \geqslant 0$. The realization lemma (recalled as part of Proposition 1.4) allows to perturb T so that the characteristic foliation ξT has a circle of singularities parallel to Γ. Such a circle L is a Legendrian (p,q) torus knot along which ξ does not twist compared with T. The Seifert framing of L differs from the framing coming from T by pq, so that the Thurston–Bennequin invariant of L is $-pq$. Since the genus of L is $(|p|-1)(q-1)/2$, the Bennequin inequality gives $-pq \leqslant |p|q - |p| - q$. This condition is

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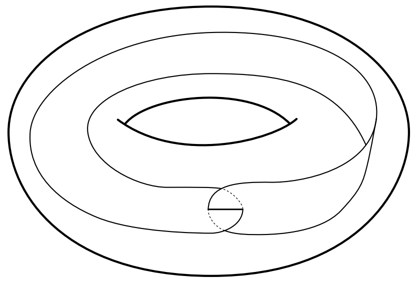

Figure 1. An unknotted immersed disk with a single clasp, sitting inside an unknotted solid torus.

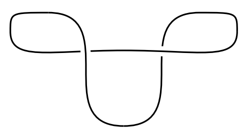

Figure 2. Lagrangian projection of the boundary of an immersed overtwisted disk in the standard contact $\mathbb{R}^3$.

equivalent to $p > 0$ and $(q-1)(p-1) \geqslant 1/4$ and, since p and q are integer, it is equivalent to $p \geqslant 1$ and $q \geqslant 1$. $\square$

We can see $\mathbb{S}^3$ as the union of two unknotted curves transverse to ξ and an open interval of pre-Lagrangian tori whose directions sweep out all positive directions. For each rational positive direction d and each positive integer n, we can perturb the corresponding pre-Lagrangian torus to a ξ-convex torus T' divided by $2n$ curves with direction d. So one of those T' has the same dividing set as T up to isotopy. After using once more the realization lemma, we can ensure that T' is the image of T under some embedding $j \in \mathcal{P}_o(T, \mathbb{S}^3; \xi)$. But the complement of T' is universally tight, whereas $\xi_{|W}$ becomes overtwisted in a two-fold cover. So j is not in the component of the inclusion in $\mathcal{P}_o(T, \mathbb{S}^3; \xi)$. $\square$

Remark 3.2. In order to get the weaker result that some solid torus W satisfies the conclusion of the above proposition, it is sufficient to observe that the complement of D contains an unknotted Legendrian knot L entwining D, and define W as the complement of a standard neighborhood of L. In that case, we already control the dividing set of ∂W by construction. Note also that the knot L is not isotopic to the canonical Legendrian unknot L_0, since the complement of the later is universally tight. The classification of Legendrian unknots in [EF98] guarantees that L is stabilization of L_0. So, in order to entwine D, one needs a somewhat tortuous Legendrian unknot.

Next we want to describe examples where we have explicit smooth isotopies among surfaces which are all ξ-convex except for a finite number of times, and exhibit various behaviors for those isotopies. We also want to highlight situations where persistent intersection phenomena occur, and situations where a contact isotopy exists in the ambient manifold, but not inside a smaller manifold (where a smooth isotopy still exists). For all this we need the following technical definition.

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Definition 3.3. A discretized isotopy of embeddings of an oriented surface S into a contact 3-manifold (V, ξ) is an isotopy of embeddings $j\colon S \times [0,1] \to V$ such that, for some (unique) integer n:

- the restriction of j to $S \times [i/n, (i+1)/n]$ is an embedding for each i from 0 to $n-1$;
- all surfaces $j_t(S)$ are ξ-convex except when $t = i/n + 1/(2n)$ for some integer i between 0 and $n-1$.

Each embedding of $S \times [i/n, (i+1)/n]$ is called a step of the discretized isotopy. It is called a forward or backward step depending on whether it is orientation preserving or reversing.

Colin's idea, described in our proof of Proposition 2.4, combines with [Gir01b, Lemme 15] to prove that any isotopy of embeddings which starts and ends at ξ-convex embeddings is homotopic, relative to its end points, to a discretized isotopy.

Any discretized isotopy j defines a sequence of isotopy classes of multi-curves $\Gamma_0, \dots, \Gamma_n$ such that the characteristic foliation of $j_{i/n}(S)$ is divided by $j_{i/n}(\Gamma_i)$.

Our examples below will use facts about $\mathbb{S}^1$-invariant contact structures on circle bundles, which we now recall. Let S be an oriented surface with non-empty boundary, and $V = S \times \mathbb{S}^1$ seen as a circle bundle over S. Let Γ be a properly embedded multi-curve on S such that components of the complement of Γ are labelled by plus or minus, so that adjacent components have different signs. Lutz proved, in [Lut77], that there is a cooriented $\mathbb{S}^1$-invariant positive contact structure ξ on V which is tangent to fibers exactly along $\Gamma \times \mathbb{S}^1$, and positively (respectively negatively) transverse to fibers over positive (respectively negative) components of $S \backslash \Gamma$. One says that Γ is the dividing set of ξ (it is indeed a dividing set for each surface $S \times \{\theta\}$). Lutz also proved that two invariant contact structures which agree near ∂V and have the same dividing set are isotopic relative to ∂V. Let ξ be such a contact structure. One can check that, for any properly embedded curve γ in S which intersects the dividing set Γ transversely (along a non-empty subset), the surface $\gamma \times \mathbb{S}^1$ is ξ-convex and divided by $(\gamma \cap \Gamma) \times \mathbb{S}^1$. Proposition 1.4 and Bennequin's theorem in [Ben83] can be used to prove that ξ is tight if and only if Γ has no homotopically trivial component or S is a disk and Γ is connected, see [Gir01a, Proposition 4.1b]. In addition, two tight $\mathbb{S}^1$-invariant contact structures on V are isotopic (relative to ∂V) if and only if their dividing sets are isotopic (relative to ∂S). This is stated only for closed surfaces in [Gir01a, Théorème 4.4b], but the proof is only easier if the boundary of S is not empty.

Recall from § 2.2 that the contact structures ξ_d on $\mathbb{T}^3$ with coordinates (x, y, z) are defined by

$$\xi_d = \ker(\cos(2d\pi z)\, dx - \sin(2d\pi z)\, dy)$$

and they are pairwise non-isomorphic.

Proposition 3.4. *In $(\mathbb{T}^3, \xi_d)$, let T be the torus $\{8d\, z = \cos x\}$. Denote by j_0 the inclusion of T into $\mathbb{T}^3$ and by j_1 the embedding obtained by restriction to T of the rotation $(x, y, z) \mapsto (x, y, z + 1/d)$. Those two embeddings are smoothly isotopic and:*

- *j_0 and j_1 induce the same characteristic foliation on T;*
- *j_0 is not isotopic to j_1 among ξ_d-convex embeddings;*
- *there is a discretized isotopy from j_0 to j_1 with only forward steps changing the direction of dividing curves;*
- *there is a discretized isotopy from j_0 to j_1 consisting of four forward steps which change the number of dividing curves without changing their direction.*

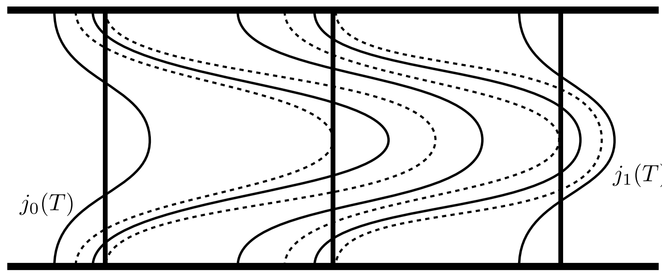

Figure 3. Discretized isotopy of curves lifting to tori in $(\mathbb{T}^3, \xi_d)$. Curves lifting to non-convex tori are dashed.

Proof. Since the rotation map is a contactomorphism, j_0 and j_1 induce the same characteristic foliation on T. Assume for contradiction that j_0 and j_1 are isotopic through ξ_d-convex surfaces. Proposition 1.4(c) and Lemma 1.2 then imply that there is a contact isotopy φ such that $j_1 = \varphi_1 \circ j_0$. We lift this isotopy to $\mathbb{T}^2 \times \mathbb{R}$ which covers $\mathbb{T}^3$ by $(x, y, s) \mapsto (x, y, s \mod 2\pi)$. We denote by φ' the lifted isotopy and by T' some (fixed) lift of T. We denote by τ_n the translation $(x, y, s) \mapsto (x, y, s+n)$ and by $T_{[a,b]}$ the compact manifold bounded by $\tau_a(T')$ and $\tau_b(T')$. Because T' is compact, and contact isotopies can be cut-off, we can assume that φ' is compactly supported. Then there is some N such that φ_1 sends $T_{[-N,0]}$ to $T_{[-N,1]}$. In particular, those submanifolds are contactomorphic. This contradicts the classification of tight contact structures on $\mathbb{T}^3$, since this contactomorphism could be used to build a contactomorphism from $(\mathbb{T}^3, \xi_{N+1})$ to $(\mathbb{T}^3, \xi_{N+2})$.

The existence of a discretized isotopy from j_0 to j_1 consisting of forward steps changing the direction of dividing curves follows from repeated uses of a small part of the classification of tight contact structures on thickened tori: if ξ is a tight contact structure on $\mathbb{T}^2 \times [0, 1]$ such that $\mathbb{T}^2 \times \{0\}$ and $\mathbb{T}^2 \times \{1\}$ are ξ-convex with two dividing curves γ_0, γ_0' and γ_1, γ_1' respectively, such γ_0 intersects γ_1 transversely at one point, then ξ is isotopic, relative to the boundary, to a contact structure ξ' such that all tori $\mathbb{T}^2 \times \{t\}$ are ξ'-convex except $\mathbb{T}^2 \times \{1/2\}$.

In order to construct a discretized isotopy where the direction of dividing curves is constant, we see ξ_d as an $\mathbb{S}^1$-invariant contact structure on $\mathbb{T}^3$ with $\mathbb{S}^1$ action given by rotation in the y direction. In order to describe an $\mathbb{S}^1$-equivariant isotopy of embeddings of T, it is enough to give an isotopy of curves in $\mathbb{T}^2$. Curves corresponding to ξ_d-convex tori are exactly those which are transverse to $\Gamma = \{x \in (\pi/d)\mathbb{Z}\}$. Figure 3 then finishes the proof. □

In our next example, the discretized isotopy oscillates, and there is persistent intersection.

Proposition 3.5. *Let V be the torus bundle over $\mathbb{S}^1$ with monodromy $B = \left(\begin{smallmatrix} 5 & 1 \\ 4 & 1 \end{smallmatrix}\right)$, i.e.*

$$V = (\mathbb{T}^2 \times \mathbb{R})/((Bx, t) \sim (x, t + 1)).$$

Let T be the image of $\mathbb{T}^2 \times \{1/2\}$ in V, and let j_0 be the inclusion map from T to V. There is a tight virtually overtwisted contact structure ξ on V, and an embedding $j_1 \in \mathcal{P}_o(T, V; \xi)$ such that:

- *j_0 is not isotopic to j_1 in $\mathcal{P}_o(T, V; \xi)$;*
- *any $j \in \mathcal{P}_o(T, V; \xi)$ such that $j(T)$ is disjoint from T is isotopic to j_0 in $\mathcal{P}_o(T, V; \xi)$ (in particular, $j_1(T)$ cannot be disjoined from T by contact isotopy);*
- *there is a discretized isotopy from j_0 to j_1 with one forward step and one backward step, both modifying the direction of dividing curves.*

Proof. We will use the theory of normal forms for tight contact structures on V established in [Gir00, § 3] (see also [Hon00]). With the notation used in [Gir00], $V = T^3_A$ where $A = B^{-1} = \begin{pmatrix} 1 & -1 \\ -4 & 5 \end{pmatrix}$. Other choices of monodromies are possible, we only want to explain one simple example.

We will describe the lifts of relevant contact structures on $\mathbb{T}^2 \times [0,1] \subset \mathbb{T}^2 \times \mathbb{R}$. In [Gir00], tight contact structures on thickened tori are described using two types of building blocks: rotation sequences and orbit flips, that we will briefly review.

Recall that a foliation σ on a torus T is called a suspension if there is a circle which transversely intersects all leaves. It then has an asymptotic direction $d(\sigma)$, which is a line through the origin in $H_1(T;\mathbb{R})$ spanned by limits of renormalized very long orbits of a directing vector field.

We fix a contact structure ξ on $T \times [0,1]$, and set $T_a = T \times \{a\}$. An interval $J \subset [0,1]$ is called a rotation sequence for ξ if all characteristic foliations ξT_t, $t \in J$ are suspensions. We say that J is minimally twisting if the directions $d(\xi T_t)$ do not sweep out the full projective line $P(H_1(T;\mathbb{R}))$. Theorem 3.3 from [Gir00] guarantees that two contact structures on $T \times J$ which agree along the boundary and have J as a minimally twisting rotation sequence are isotopic on $T \times J$ relative to boundary.

An interval $[a,b] \subset [0,1]$ is an orbit flip sequence for ξ, with homology class $d \in H_1(T;\mathbb{Z})$, if:

- ξT_a is a Morse–Smale suspension with two closed orbits whose homology classe is d;
- ξT_b is a Morse–Smale suspension with two closed orbits whose homology classe is $-d$;
- there is a multi-curve which divides all ξT_t, $t \in J$.

The uniqueness lemma [Gir00, Lemma 2.7] ensures that two contact structures on $T \times [a,b]$ which agree along the boundary, and admit $[a,b]$ as an orbit flip sequence, are isotopic relative to boundary. There is an explicit model in [Gir00, § 1.F] where all ξT_t are suspensions except one which has two circles of singularities instead of regular closed leaves.

Here we need two (isotopic) contact structures on $\mathbb{T}^2 \times [0,1]$. We fix a Morse–Smale suspension σ_0 on $\mathbb{T}^2$ with two closed orbits having homology class $(1,0)$, and we denote by σ_1 the image of σ_0 under A. We also fix a Morse–Smale suspension $\sigma_{1/2}$ with two closed orbits having homology class $(-1,1)$. Let ξ be a contact structure on $\mathbb{T}^2 \times [0,1]$ such that:

- ξ prints σ_t on $\mathbb{T}^2 \times \{t\}$ for $t \in \{0, 1/2, 1\}$;
- $[0,1]$ is a union of minimally twisting rotation sequences and two orbit flip sequences with homology classes $(1,0)$ and $(-1,1)$ respectively.

Let ξ' be a contact structure with the same properties except that orbit flip homology classes are $(1,-1)$ and $(-1,2)$. The explicit construction of [Gir00, Example 3.41] guarantees that ξ and ξ' are isotopic (relative to the boundary). More specifically, it builds a contact structure printing a non-generic movie of characteristic foliations where two saddle connections happen on the same torus, and such that the movies printed by ξ and ξ' are essentially obtained by choosing the order in which these connections appear. Alternatively, one can see the isotopy between ξ and ξ' as an application of the 'shuffling lemma' of Honda [Hon00, Lemma 4.14]. We will also denote by ξ and ξ' the induced contact structures on V. And we denote by T the image in V of $\mathbb{T}^2_{1/2}$.

Let φ be a smooth isotopy of V such that $\xi' = \varphi_1^* \xi$. Assume for contradiction that $j_0 : T \hookrightarrow V$ and $j_1 = \varphi_1 \circ j_0$ are in the same component of $\mathcal{P}_o(T, V; \xi)$. Using the path lifting property for the map $\mathcal{D}_o(V;\xi) \to \mathcal{P}_o(T,V;\xi)$ guaranteed by Lemma 1.2, we get a contact isotopy θ for ξ such that $j_1 = \theta_1 \circ j_0$. Then $\psi := \theta_1^{-1} \circ \varphi_1$ is a diffeomorphism relative to T, and pulls back ξ to ξ'. Thus, we can cut V along T to get a thickened torus Y, naturally identified with $\mathbb{T}^2 \times [1/2, 3/2]$.

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The diffeomorphism ψ induces a diffeomorphism of Y which is relative to the boundary, hence acts trivially on $H_1(Y)$. This is a contradiction because the restriction of ξ and ξ' to Y do not have the same relative Euler class in $H_1(Y)$. Recall that $e(Y;\xi)$ is the homology class of the vanishing locus of any generic section of ξ which spans $\xi\partial Y$ (with the correct orientation) along ∂Y. Here, contributions to this class come from orbit flips and we get $e(Y;\xi) = 2(-1,1)+2A(1,0) = 2(0,-3)$ while $e(Y;\xi') = 2(-1,2) + 2A(1,-1) = 2(1,-7)$. Note, for sanity check, that those two classes become the same in V, since $e(Y;\xi') - e(Y;\xi) = 2(1,-4) = (\mathrm{Id} - A)(0,1)$.

The second point of the proposition follows again from classification results. Let j be an embedding in $\mathcal{P}_0(T,V;\xi)$ such that $j(T)$ is disjoint from T. The classification of incompressible tori in the complement of T guarantees that T and $j(T)$ bound a thickened torus N in V. The classification of tight contact structures on thickened tori in [Gir00] or [Hon00] ensures that either j is isotopic to j_0 in $\mathcal{P}_0(T,V;\xi)$, or there exists a contact embedding of

$$(T^2 \times [0,1], \ker(\cos(\pi z)\, dx - \sin(\pi z)\, dy)), \quad (x,y,z) \in T^2 \times [0,1]$$

into the interior of N. But the existence of such an embedding is ruled out by the study of tight contact structure on V, specifically [Gir00, Proposition 1.8].

The announced discretized isotopy uses the image of $\mathbb{T}^2 \times \{0\}$ as a intermediate surface, and its existence is guaranteed by the classification result quoted in the proof of Proposition 3.4. □

Finally we describe an example on a manifold with boundary with the same situation as above, but things untangle inside a larger manifold.

Proposition 3.6. *Let V denote the manifold $\mathbb{T}^2 \times [0,1]$, and $V' = \mathbb{T}^2 \times [0,1/2]$. There is a universally tight contact structure ξ on V, and two smoothly isotopic ξ-convex embeddings $j_0, j_1 : \mathbb{T}^2 \to V'$ with images T_0 and T_1, such that:*

- *j_0 is isotopic to j_1 among ξ-convex embeddings in V;*
- *j_0 is not isotopic to j_1 among ξ-convex embeddings in V';*
- *T_0 cannot be disjoined from T_1 by an isotopy among ξ-convex surfaces in V';*
- *there is a discretized isotopy from j_0 to j_1 in V' with one forward step and one backward step, both modifying the direction of dividing curves;*
- *there is a discretized isotopy from j_0 to j_1 in V' with one forward step and one backward step, both modifying the number of dividing curves.*

Proof. The construction is pictured in Figure 4. Let S be the annulus $\{1 \leqslant |z| \leqslant 3\} \subset \mathbb{C}$ and $S' \subset S$ the subannulus $\{1 \leqslant |z| \leqslant 2\}$. We fix an identification between V and $S \times \mathbb{S}^1$ which identifies V' with $S' \times \mathbb{S}^1$. Let $\Gamma' = \Gamma'_1 \cup \Gamma'_2$ be a disjoint union of two properly embedded arcs in S' whose end points are on the circle $\{|z| = 2\}$. Let Γ be a smooth homotopically essential circle in S such that $\Gamma \cap S' = \Gamma'$. Let ξ be an $\mathbb{S}^1$-invariant contact structure on V with dividing set Γ, and denote by ξ' its restriction to V'. Let γ_0 and γ_1 be homotopically essential circles in S' such that γ_i intersects transversely Γ'_i in two points, and does not intersect the other component of Γ'. The tori we want are $T_0 = \gamma_0 \times \mathbb{S}^1$ and $T_1 = \gamma_1 \times \mathbb{S}^1$, parametrized by product maps.

There is an isotopy through ξ-convex surfaces in V because γ_0 and γ_1 are isotopic in S through curves transverse to Γ.

Assume for contradiction that there is such an isotopy in V'. We can arrange ξ so that j_0 and j_1 induce the same characteristic foliation on $\mathbb{T}^2$ and, using Proposition 1.4(c) and Lemma 1.2, our isotopy through convex surfaces can then be converted into a contact isotopy φ relative

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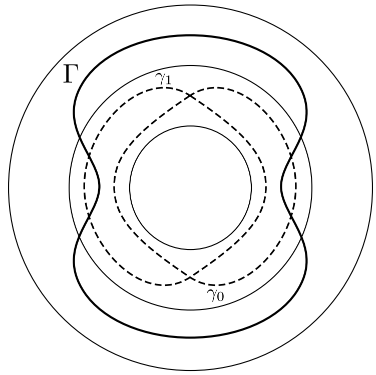

Figure 4. The example of Proposition 3.6. The dividing set Γ is the thick curve, γ_0 and γ_1 are dashed.

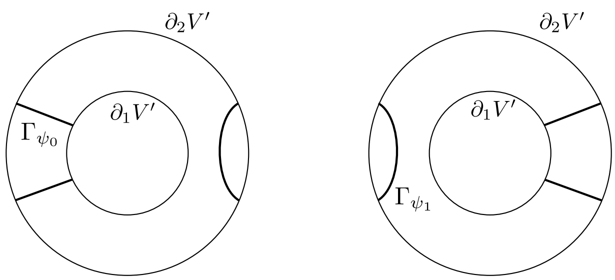

Figure 5. Dividing curves for the proof of Proposition 3.6. Thin curves are boundary components of S' and thick curves are the components of the dividing sets.

to the boundary. We denote by $\partial_1 V'$ and $\partial_2 V'$ the connected components $\{|z| = 1\} \times \mathbb{S}^1$ and $\{|z| = 2\} \times \mathbb{S}^1$ of $\partial V'$. Let ψ_0 and ψ_1 be smooth embeddings of V' into itself such that:

– each ψ_i is $\mathbb{S}^1$-equivariant;
– each ψ_i is the identity on $\partial_2 V'$;
– $\psi_i(\partial_1 V') = T_i$.

The contact structures $\psi_0^*\xi$ and $\psi_1^*\xi$ on V' are $\mathbb{S}^1$-invariant and the associated dividing sets Γ_{ψ_0} and Γ_{ψ_1} are shown on Figure 5. The contactomorphism φ_1 then induces a contactomorphism between $(V', \psi_0^*\xi)$ and $(V', \psi_1^*\xi)$ which is the identity on $\partial_2 V'$. However, the classification of $\mathbb{S}^1$-invariant contact structures forbids the existence of this contactomorphism. In this case, we can argue directly as follows. We denote by S'' the annulus $\{2 \leqslant |z| \leqslant 3\}$. Let ξ'' be a contact structure on $S'' \times \mathbb{S}^1$ which is $\mathbb{S}^1$-invariant and tangent to $\mathbb{S}^1$ along some Γ'' such that $\Gamma_{\psi_0} \cup \Gamma''$ has a homotopically trivial component but $\Gamma_{\psi_1} \cup \Gamma''$ has not. The contact structure $\xi'' \cup \psi_0^*\xi$ on V is overtwisted, whereas $\xi'' \cup \psi_1^*\xi$ is tight, so we have a contradiction.

So there is no contact isotopy φ in V' such that $j_1 = \varphi_1 \circ j_0$. Assume for contradiction that there is a contact isotopy φ in V' such that $T_0' = \varphi_1(T_0)$ is disjoint from T_1. The classification of incompressible surfaces in thickened tori ensures that $T_0' \cup T_1$ is the boundary of a thickened torus in the interior of V'. After some smooth deformation, we can assume that $T_0' = \{|z| = r_0\} \times \mathbb{S}^1$ and $T_1 = \{|z| = r_1\} \times \mathbb{S}^1$ and the contact structure is $\mathbb{S}^1$-invariant near T_0' and T_1 (note that we do not know the sign of $r_0 - r_1$). Those tori are both divided by vertical curves $\{*\} \times \mathbb{S}^1$, so the classification of universally tight contact structures on thickened tori guarantees that, after some further isotopy relative to T_0' and T_1, the contact structure is $\mathbb{S}^1$-invariant everywhere (see [Gir00, Théorème 4.4] or [Hon00]). The dividing set in the annulus $\{|z| \in [r_0, r_1]\}$ intersects each boundary component in two points, so it is either two boundary parallel arcs and some closed components, or two traversing arcs. The first possibility is ruled out by the classification of $\mathbb{S}^1$-invariant contact structures up to (non-necessarily invariant) isotopy since the full dividing set on S' would not be isotopic to Γ'. The second possibility is ruled out because T_0' and T_1 would then be isotopic among ξ'-convex surfaces, contradicting the previous point.

The construction of discretized isotopies is completely analogous to what we discussed for Proposition 3.4. □

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