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BOUNDEDNESS OF SINGULAR INTEGRAL OPERATORS OF CALDERÓN TYPE, VI

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§ 1. Introduction

In this paper, we improve an inequality given in [3]. Let L^p $(1 \leqq p \leqq \infty)$ and BMO denote the L^p space and the BMO space on the real line. Their norms are denoted by $\|\cdot\|_p$ and $\|\cdot\|_{BMO}$, respectively. For a locally integrable real-valued function $\phi(x)$, we define a kernel by

$$E[\phi](x, y) = \frac{1}{x-y} \exp\left\{i \frac{\Phi(x) - \Phi(y)}{x-y}\right\},$$

where $\Phi(x) = \int_0^x \phi(s)ds$. We denote by $E[\phi]$ the singular integral operator defined by this kernel and denote by $\|E[\phi]\|$ its norm as an operator from L^2 to itself. It is very important to estimate $\|E[\phi]\|$ and the author [3] showed that $\|E[\phi]\| \leqq \text{Const}\{1 + \|\phi\|_{BMO}^2\}$. The purpose of this paper is to show

THEOREM. $\|E[\phi]\| \leqq \text{Const}\{1 + \|\phi\|_{BMO}\}$.

Recently, Tchamitchian [5] showed that $\|C[\phi]\| \leqq \text{Const}\{1 + \|\phi\|_{BMO}\}$, where $C[\phi](x, y) = 1/\{(x-y) + i(\Phi(x) - \Phi(y))\}$. Our method also yields that $\|C[\phi]\| \leqq \text{Const}\{1 + \sqrt{\|\phi\|_{BMO}}\}$. The norms of these operators are likely dominated by $C_\varepsilon\{1 + \|\phi\|_{BMO}^\varepsilon\}$ for any $\varepsilon > 0$, where C_ε is a constant depending only on ε.

§ 2. Proof of Theorem

2.1. For $\phi \in L^\infty$ and an interval I, we put:

$$\tilde{\sigma}(\phi, I) = \int_I \left|\int_I E[\phi](x, y)dy\right|^2 dx$$

and

$$\tilde{\tau}(\alpha) = \sup\{\tilde{\sigma}(\phi, I)/|I|;\ \|\phi\|_\infty \leqq \alpha,\ I \text{ interval}\} \quad (\alpha \geqq 1),$$

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where $|I|$ denotes the length of I. Here is a main lemma in this paper.

LEMMA 1. *For $\phi, \psi \in L^\infty$ with $\|\phi\|_\infty \leqq \alpha$, $\|\psi\|_\infty \leqq \alpha$ and an interval I, we write*

$$\{x \in I; \Phi(x) \neq \Psi(x)\} = \bigcup_{k=1}^{\infty} I_k$$

with a sequence $\{I_k\}_{k=1}^\infty$ of mutually disjoint intervals, where $\Phi(x) = \int_a^x \phi(s)ds$ and $\Psi(x) = \int_a^x \psi(s)ds$ (a: the left endpoint of I). Suppose that $\psi(x)$ is a constant at each I_k. Then

$$\tilde{\sigma}(\phi, I) \leqq \tilde{\sigma}(\psi, I) + \sum_{k=1}^{\infty} \tilde{\sigma}(\phi, I_k) + \mathrm{Const}\, \nu(\alpha)|I| \quad (\alpha \geqq 1)\,, \tag{1}$$

where $\nu(\alpha) = \sqrt{\alpha\tilde{\tau}(\alpha)} + \alpha$.

Before the proof of this lemma, we show that our theorem is deduced from (1). In [3], [4], the author defined $\sigma(\phi, I) = \int_I \left|\int_I E[\phi](x, y)dy\right| dx$, the corresponding quantity $\tau(\alpha)$ to $\sigma(\cdot, \cdot)$ and showed that

$$\sigma(\phi, I) \leqq \sigma(\psi, I) + \sum_{k=1}^{\infty} \sigma(\phi, I_k) + \mathrm{Const}\, \alpha|I| \quad (\alpha \geqq 1)\,.$$

From this inequality and the rising sun lemma, we deduced

$$\tau(\alpha) \leqq \tau(\alpha/3) + 2\tau(2\alpha/3) + \mathrm{Const}\, \alpha \quad (\alpha \geqq 1)\,,$$

which yields $\tau(\alpha) \leqq \mathrm{Const}\, \alpha^2$. Using $\|E[\phi]\| \leqq \mathrm{Const}\, \{\tau(8\|\phi\|_{BMO}) + \|\phi\|_{BMO}\}$ (cf. [3, Lemma 6]), we obtained the previous inequality.

In the same manner, (1) and the rising sun lemma yield that

$$\tilde{\tau}(\alpha) \leqq \tilde{\tau}(\alpha/3) + 2\tilde{\tau}(2\alpha/3) + \mathrm{Const}\, \nu(\alpha) \quad (\alpha \geqq 1)\,. \tag{2}$$

Since $\sqrt{\tilde{\tau}(\alpha)} \leqq \mathrm{Const}\, \{\tau(\alpha) + \alpha\} \leqq \mathrm{Const}\, \alpha^2$ (cf. [1, pp. 49, 52], [4]), (2) is valid with $\nu(\alpha)$ replaced by $\alpha^{2.5}$, which yields $\sqrt{\tilde{\tau}(\alpha)} \leqq \mathrm{Const}\, \alpha^{1.3}$. Hence (2) is valid with $\nu(\alpha)$ replaced by $\alpha^{1.8}$. This gives $\sqrt{\tilde{\tau}(\alpha)} \leqq \mathrm{Const}\, \alpha$. Since $\tau(\alpha) \leqq \sqrt{\tilde{\tau}(\alpha)}$, we obtain the required inequality. Note that $\alpha^2 = (\alpha/3)^2 + 2(2\alpha/3)^2$. Hence our estimate by (2) is sharp.

2.2. Now we prove our lemma. For $\{I_k\}_{k=1}^\infty$, I in our lemma, we denote by x_k the midpoint of I_k and put $I_k^* = \tilde{I}_k \cap I$, where $\tilde{I}_k$ is an interval with midpoint x_k and of length $2|I_k|$. For a set E, $\chi_E(x)$ denotes its characteristic function. We write $\chi_k(x) = \chi_{I_k}(x)$. For a kernel $X(x, y)$ and

$f \in L^\infty$, we write simply $Xf(x) = \int_I X(x, y)f(y)dy$. For a set E and $f \in L^2$, we put $\|f\|_{2,E} = \left\{\int_E |f(x)|^2 dx\right\}^{1/2}$. Here are three lemmas necessary for the proof. The elementary calculus gives the first lemma. The second lemma immediately follows from the Carleson-Hunt inequality (cf. [2]).

LEMMA 2. *Let*

$$\mu_1^{(k)}(x) = \int_{I_k} 1/|x-y|dy, \qquad \mu_2^{(k)}(x) = \int_{I_k^*-I_k} 1/|x-y|dy \quad (k \geqq 1).$$

Then

$$\|\mu_1^{(k)}\|_{2,I_k^*-I_k} \leqq \text{Const}\sqrt{|I_k|}, \qquad \|\mu_2^{(k)}\|_{2,I_k} \leqq \text{Const}\sqrt{|I_k|} \quad (k \geqq 1).$$

LEMMA 3. *Let*

$$P_k(x) = \sqrt{|I_k|}/\{|x-x_k|^{3/2} + |I_k|^{3/2}\} \quad (k \geqq 1), \qquad \Delta(x) = \sum_{k=1}^\infty |I_k|P_k(x).$$

Then $\|\Delta\|_2 \leqq \text{Const}\sqrt{|I|}$.

LEMMA 4. *Let* $f \in L^\infty$ *satisfy* $\|f\|_\infty \leqq 1$. *Then*

$$\sum_{k=1}^\infty \|E[\theta]f\|_{2,I_k^*}^2 \leqq \text{Const}\{\tilde{\tau}(\alpha) + \alpha^2\}|I| \quad (\theta = \phi, \psi).$$

Proof. Note that

$$\|E[\theta]\| \leqq \text{Const}\{\sqrt{\tilde{\tau}(\alpha)} + \alpha\}, \qquad \|E[\theta]f\|_{BMO} \leqq \text{Const}\{\sqrt{\tilde{\tau}(\alpha)} + \alpha\}$$

(cf. [1, Chap. 4], [4]). Hence we have

$$|m_{I_k^*}(E[\theta]f)| \leqq |m_{I_k}(E[\theta]f)| + \text{Const}\{\sqrt{\tilde{\tau}(\alpha)} + \alpha\} \quad (k \geqq 1),$$

where $m_J g$ denotes the mean of $g(x)$ over J. By the John-Nirenberg theorem (cf. [1, p. 31]), we have

$$\begin{aligned}\|E[\theta]f\|_{2,I_k^*}^2 &\leqq 2\{\|E[\theta]f - m_{I_k^*}(E[\theta]f)\|_{2,I_k^*}^2 + |m_{I_k^*}(E[\theta]f)|^2|I_k^*|\} \\ &\leqq \text{Const}\{\|E[\theta]f\|_{BMO}^2|I_k^*| + |m_{I_k}(E[\theta]f)|^2|I_k| + (\tilde{\tau}(\alpha) + \alpha^2)|I_k|\} \\ &\leqq \text{Const}\{\|E[\theta]f\|_{2,I_k}^2 + (\tilde{\tau}(\alpha) + \alpha^2)|I_k|\}.\end{aligned}$$

Thus

$$\begin{aligned}\sum_{k=1}^\infty \|E[\theta]f\|_{2,I_k^*}^2 &\leqq \text{Const}\{\|E[\theta]f\|_{2,I}^2 + (\tilde{\tau}(\alpha) + \alpha^2)|I|\} \\ &\leqq \text{Const}\{\tilde{\tau}(\alpha) + \alpha^2\}|I|.\end{aligned}$$

Q.E.D.

2.3. Let $U = \bigcup_{k=1}^\infty I_k$, $V = I - U$ and

$$F(x, y) = E[\phi](x, y) - E[\psi](x, y) \quad (x, y \in I).$$

Then [3]:

(3) $$F(x, y) = -F(y, x).$$

(4) $$F(x, y) = 0 \quad (x, y \in V).$$

(5) $$|F(x, y)| \leqq \text{Const}\sqrt{\alpha}\, P_k(x) \quad (x \in I_k^{*c} \cap V,\ y \in I_k).$$

(6) $$|F(x, y)| \leqq \text{Const}\sqrt{\alpha}(\sqrt{|I_l|} + \sqrt{|I_k|})/|x - y|^{3/2} \quad (x \in I_l,\ y \in I_k).$$

We have

$$\begin{aligned} \tilde{\sigma}(\phi, I) &= \int_I |E[\psi]1(x) + F1(x)|^2 dx \\ &= \tilde{\sigma}(\psi, I) + 2\,\text{Re}\int_I F1(x)\overline{E[\psi]1(x)}dx + \int_I |F1(x)|^2 dx \\ &\qquad (= \tilde{\sigma}(\psi, I) + 2\,\text{Re}\, A + B, \text{ say}). \end{aligned} \tag{7}$$

We have, by (4),

$$\begin{aligned} A &= \int_I F\chi_U(x)\overline{E[\psi]1(x)}\,dx + \int_I F\chi_V(x)\overline{E[\psi]1(x)}\,dx \\ &= \int_U F\chi_U(x)\overline{E[\psi]1(x)}\,dx + \int_V F\chi_U(x)\overline{E[\psi]1(x)}\,dx \\ &\quad + \int_U F\chi_V(x)\overline{E[\psi]1(x)}\,dx \quad (= A^{(1)} + A^{(2)} + A^{(3)}, \text{ say}). \end{aligned} \tag{8}$$

In this paragraph, we estimate $A^{(3)}$ and $A^{(2)}$. We have, by (5) and Lemma 2,

$$\begin{aligned} |A^{(3)}| &\leqq \sum_{k=1}^{\infty}\int_{I_k} |F\chi_V(x)E[\psi]1(x)|\,dx \\ &\leqq \sum_{k=1}^{\infty}\int_{I_k} |F\chi_{V\cap I_k^*}(x)E[\psi]1(x)|\,dx + \sum_{k=1}^{\infty}\int_{I_k} |F\chi_{V\cap I_k^{*c}}(x)E[\psi]1(x)|\,dx \\ &\leqq 2\sum_{k=1}^{\infty}\int_{I_k} \mu_2^{(k)}(x)|E[\psi]1(x)|\,dx + \text{Const}\sqrt{\alpha}\sum_{k=1}^{\infty}\int_{I_k} \|P_k\|_1 |E[\psi]1(x)|\,dx \\ &\leqq 2\sum_{k=1}^{\infty} \|\mu_2^{(k)}\|_{2,I_k}\|E[\psi]1\|_{2,I_k} + \text{Const}\sqrt{\alpha}\,\|E[\psi]1\|_1 \\ &\leqq 2\left\{\sum_{k=1}^{\infty}\|\mu_2^{(k)}\|_{2,I_k}^2\right\}^{1/2}\|E[\psi]1\|_2 + \text{Const}\sqrt{\alpha}\sqrt{|I|}\,\|E[\psi]1\|_2 \\ &\leqq \text{Const}\,\nu(\alpha)|I|. \end{aligned}$$

We have, by (5) and Lemmas 2, 3, 4,

$$\begin{aligned} |A^{(2)}| &\leqq \sum_{k=1}^{\infty}\int_V |F\chi_k(x)E[\psi]1(x)|\,dx = \sum_{k=1}^{\infty}\int_{V\cap I_k^*} |F\chi_k(x)E[\psi]1(x)|\,dx \\ &\quad + \sum_{k=1}^{\infty}\int_{V\cap I_k^{*c}} |F\chi_k(x)E[\psi]1(x)|\,dx \leqq 2\sum_{k=1}^{\infty}\int_{I_k^*-I_k} \mu_1^{(k)}(x)|E[\psi]1(x)|\,dx \end{aligned}$$

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$$+ \text{Const}\sqrt{\alpha}\sum_{k=1}^{\infty}\int_I |I_k||P_k(x)|E[\psi]1(x)|dx$$

$$\leqq 2\sum_{k=1}^{\infty}\|\mu_1^{(k)}\|_{2,I_k^*-I_k}\|E[\psi]1\|_{2,I_k^*} + \text{Const}\sqrt{\alpha}\|\Delta\|_2\|E[\psi]1\|_2$$

$$\leqq \text{Const}\,\nu(\alpha)|I|.$$

2.4. In this paragraph, we estimate $A^{(1)}$. We have

$$A^{(1)} = \sum_{k=1}^{\infty}\int_U F\chi_k(x)\overline{E[\psi]1(x)}\,dx = \sum_{k=1}^{\infty}\int_{U-I_k} F\chi_k(x)\overline{E[\psi]1(x)}\,dx$$

$$+ \sum_{k=1}^{\infty}\int_{I_k} F\chi_k(x)\overline{E[\psi]\chi_{I-I_k}(x)}\,dx + \sum_{k=1}^{\infty}\int_{I_k} F\chi_k(x)\overline{E[\psi]\chi_k(x)}\,dx$$

$$(= A_1^{(1)} + A_2^{(1)} + A_3^{(1)}, \text{ say}).$$

Since $\psi(x)$ is a constant at each I_k, we have, with $H(x, y) = 1/(x-y)$ and a sequence $\{d_k\}_{k=1}^{\infty}$ of real numbers,

$$|A_3^{(1)}| = \left|\sum_{k=1}^{\infty} e^{-id_k}\int_{I_k} F\chi_k(x)H\chi_k(x)\,dx\right|$$

$$\leqq \sum_{k=1}^{\infty}\|F\chi_k\|_{2,I_k}\|H\chi_k\|_{2,I_k} \leqq \text{Const}\,\nu(\alpha)|I|.$$

We have, by (3),

$$|A_2^{(1)}| = \left|\sum_{k=1}^{\infty}\int_{I_k} F\chi_k(x)\{\overline{E[\psi]\chi_{I-I_k}(x)} - \overline{E[\psi]\chi_{I-I_k}(x_k)}\}\,dx\right|$$

$$\leqq \sum_{k=1}^{\infty}\int_{I_k}|F\chi_k(x)|\left[\left\{\int_{I_k^*-I_k} + \int_{I_k^{*c}}\right\}|E[\psi](x, y) - E[\psi](x_k, y)|dy\right]dx$$

$$\leqq \text{Const}\sum_{k=1}^{\infty}\int_{I_k}|F\chi_k(x)|\{\mu_2^{(k)}(x) + \sqrt{\alpha}\}dx \leqq \text{Const}\,\nu(\alpha)|I|.$$

We have

$$A_1^{(1)} = \sum_{k=1}^{\infty}\int_{I_k^*-I_k} F\chi_k(x)\overline{E[\psi]1(x)}\,dx + \sum_{k=1}^{\infty}\int_{I_k^{*c}\cap U} F\chi_k(x)\overline{E[\psi]1(x)}\,dx$$

$$(= A_{11}^{(1)} + A_{12}^{(1)}, \text{ say}).$$

We have, by (6) and Lemmas 2, 3,

$$|A_{12}^{(1)}| \leqq \sum_{k=1}^{\infty}\sum_{l\neq k}\int_{I_k^{*c}\cap I_l}|F\chi_k(x)E[\psi]1(x)|dx$$

$$\leqq \sum_{k=1}^{\infty}\sum_{l\neq k}\int_{I_k^{*c}\cap I_l}|F\chi_{I_k\cap I_l^{*c}}(x)E[\psi]1(x)|dx$$

$$+ \sum_{k=1}^{\infty}\sum_{l\neq k}\int_{I_k^{*c}\cap I_l}|F\chi_{I_k\cap I_l^*}(x)E[\psi]1(x)|dx$$

$$\leqq \text{Const}\sqrt{\alpha}\sum_{k=1}^{\infty}\sum_{l\neq k}\int_{I_k^{*c}\cap I_l}\left[\int_{I_k\cap I_l^{*c}}\{\sqrt{|I_l|} + \sqrt{|I_k|}\}/|x-y|^{3/2}dy\right]|E[\psi]1(x)|dx$$

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$$+\sum_{l=1}^{\infty}\int_{I_l}\{\sum_{k\neq l}|F\chi_{I_k\cap I_l^*}(x)|\}|E[\psi]1(x)|dx$$
$$\leqq \text{Const}\sqrt{\alpha}\sum_{l=1}^{\infty}\int_{I_l}\Big[\sum_{k\neq l}\int_{I_k}\{P_l(y)+P_k(x)\}dy\Big]|E[\psi]1(x)|dx$$
$$+2\sum_{l=1}^{\infty}\int_{I_l}\mu_2^{(l)}(x)|E[\psi]1(x)|dx$$
$$\leqq \text{Const}\sqrt{\alpha}\sum_{l=1}^{\infty}\int_{I_l}\{\|P_l\|_1+\Delta(x)\}|E[\psi]1(x)|dx$$
$$+2\sum_{l=1}^{\infty}\|\mu_2^{(l)}\|_{2,I_l}\|E[\psi]1\|_{2,I_l}$$
$$\leqq \text{Const}\sqrt{\alpha}\{\|E[\psi]1\|_1+\|\Delta\|_2\|E[\psi]1\|_2\}+\text{Const}\,\nu(\alpha)|I|$$
$$\leqq \text{Const}\,\nu(\alpha)|I|.$$

We have, by Lemmas 2 and 4,

$$|A_{11}^{(1)}|\leqq 2\sum_{k=1}^{\infty}\int_{I_k^*-I_k}\mu_1^{(k)}(x)|E[\psi]1(x)|dx$$
$$\leqq 2\sum_{k=1}^{\infty}\|\mu_1^{(k)}\|_{2,I_k^*-I_k}\|E[\psi]1\|_{2,I_k^*}\leqq \text{Const}\,\nu(\alpha)|I|.$$

Thus $|A_1^{(1)}|\leqq \text{Const}\,\nu(\alpha)|I|$. Consequently,

(9) $$|A|\leqq\{|A_1^{(1)}|+|A_2^{(1)}|+|A_3^{(1)}|\}+|A^{(2)}|+|A^{(3)}|\leqq \text{Const}\,\nu(\alpha)|I|.$$

2.5. In this paragraph, we estimate B. We have, by (4),

$$B=\int_U F\chi_U(x)\overline{F1(x)}\,dx+\int_V F\chi_U(x)\overline{F1(x)}\,dx+\int_U F\chi_V(x)\overline{F1(x)}\,dx$$
$$(=B^{(1)}+B^{(2)}+B^{(3)},\ \text{say}).$$

In the same manner as in $A^{(3)}$, $|B^{(3)}|\leqq \text{Const}\,\nu(\alpha)|I|$. In the same manner as in $A^{(2)}$, $|B^{(2)}|\leqq \text{Const}\,\nu(\alpha)|I|$. Now we estimate $B^{(1)}$. We have

$$B^{(1)}=\sum_{k=1}^{\infty}\int_{U-I_k}F\chi_k(x)\overline{F1(x)}\,dx+\sum_{k=1}^{\infty}\int_{I_k}F\chi_k(x)\overline{F\chi_{I-I_k}(x)}\,dx$$
$$+\sum_{k=1}^{\infty}\int_{I_k}|F\chi_k(x)|^2dx\quad(=B_1^{(1)}+B_2^{(1)}+B_3^{(1)},\ \text{say}).$$

We have

$$B_3^{(1)}=\sum_{k=1}^{\infty}\int_{I_k}|E[\phi]\chi_k(x)-e^{id_k}H\chi_k(x)|^2dx$$
$$=\sum_{k=1}^{\infty}\tilde{\sigma}(\phi,I_k)-2\,\text{Re}\sum_{k=1}^{\infty}e^{-id_k}\int_{I_k}E[\phi]\chi_k(x)H\chi_k(x)\,dx+\sum_{k=1}^{\infty}\int_{I_k}|H\chi_k(x)|^2dx.$$

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Thus

$$B_3^{(1)} \leqq \sum_{k=1}^{\infty} \tilde{\sigma}(\phi, I_k) + \text{Const}\, \nu(\alpha)|I|.$$

In the same manner as in $A_2^{(1)}$, $|B_2^{(1)}| \leqq \text{Const}\, \nu(\alpha)|I|$. In the same manner as in $A_1^{(1)}$, $|B_1^{(1)}| \leqq \text{Const}\, \nu(\alpha)|I|$. Consequently,

$$\begin{aligned} B &\leqq \{|B_1^{(1)}| + |B_2^{(1)}| + B_3^{(1)}\} + |B^{(2)}| + |B^{(3)}| \\ &\leqq \sum_{k=1}^{\infty} \tilde{\sigma}(\phi, I_k) + \text{Const}\, \nu(\alpha)|I|. \end{aligned} \tag{10}$$

Inequalties (8), (9) and (10) show (1). This completes the proof of our main lemma. As stated above, the required inequality is deduced from our main lemma. This completes the proof of our theorem.

Using $\sigma(\cdot, \cdot)$, Tchamitchian [5] showed that $\|C[\phi]\| \leqq \text{Const}\,(1 + \|\phi\|_{BMO})$. Replacing $\sigma(\cdot, \cdot)$ by $\tilde{\sigma}(\cdot, \cdot)$ in his proof, we obtain $\|C[\phi]\| \leqq \text{Const}\,(1+\sqrt{\|\phi\|_{BMO}})$.

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