SOME GENERAL INSTABILITY CRITERIA FOR ORDINARY DIFFERENTIAL EQUATIONS

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1. Introduction. We consider a system of differential equations

$$\dot{x} = P(x, y)$$
(1)
$$\dot{y} = Q(x, y)$$

where $0 = (o, o)$ is an isolated singular point. Thus, there exists $B > o$ such that $S(0, B)$ contains only one singular point. Here, $S(0, B)$ denotes a sphere centered at 0 with radius B. We shall denote the boundary of $S(0, B)$ by $\partial S(0, B)$. Let us assume that P and Q satisfy a Lipschitz condition with respect to x and y in $S(0, B)$.

We establish criteria for determining instability of the equilibrium solution to (1) simply by examining the set of points on which P or Q vanish. This theorem is generalized and an example is given. Finally, we extend the criteria to nth order systems. Throughout this paper stability will be defined as follows for any system of order n.

Definition. Let $0 = (o, \ldots, o)$ and $X = (x_1, \ldots, x_n)$. Then the solution $X = 0$ is Liapunov stable if for every $e > o$ there exists $d > o$ such that $|X_o| < d$ implies that the solu-

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tion $f(X_o, t)$ through X_o remains in $S(0,e)$ for all $t \geq o$.

2. Instability of second order systems. In the following definition we assume that $A < B$.

Definition. A curve lying in $\overline{S(0,A)}$ which intersects 0 and $\partial S(0,A)$ for some $A > o$ is x-functional (y-functional) if it can be represented as the locus of $y = f(x)$ $(x = f(y))$ where f is some continuous function.

THEOREM 1. If $Q = o$ on the y-functional curves c_1 and c_2 in quadrants I and IV respectively, while $P \geq o$ in the region R bounded by c_1, c_2 and $\partial S(0,A)$, then 0 is unstable.

Proof. Note that the facts $Q = o$ on c_1 and c_2, $P \geq o$, and c_1 and c_2 are graphs of well defined functions imply that any trajectory in the interior of R cannot leave R through c_1 and c_2. By definition of R, if (x_o, y_o) is in the interior of R, then $x_o > o$. Let the solution through (x_o, y_o) be $(x(t), y(t))$. Since $\dot{x} = P \geq o$ in R then $x(t) \not\to o$ in R. That is, if $(x(t), y(t)) \to (o, o)$ then it must first leave R through $\partial S(0,A)$. By the covering assumption for the whole paper there exist no singular points in the interior of R, so there exist no closed orbits there. By the Poincaré-Bendixson theorem this solution must leave $S(0,A)$, and thus the null solution is unstable.

This theorem will be generalized, but in order to see the possibilities of extension we shall state a few corollaries. The close dependence on a coordinate system is retained for ease in application, although this dependence will be dropped in Theorem 2.

COROLLARY 1. If $P = o$ on the x-functional curve c_1 in quadrant I, $Q = o$ on the y-functional curve c_2 in quadrant IV, while $P \geq o$ and $Q \leq o$ in the region R bounded by c_1,

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c_2 and $\partial S(0, A)$, then 0 is unstable.

COROLLARY 2. If $Q = o$ on the y-functional curve c_1, $P = o$ on the x-functional curve c_2, with c_1 and c_2 in quadrant I and c_1 above c_2, while P and Q are nonnegative in the region R bounded by c_1, c_2 and $\partial S(0, A)$, then 0 is unstable.

If we require that a curve be both x- and y-functional we may relax the condition that P or Q vanishes on the curve as follows.

COROLLARY 3. If $P = o$ on the x-functional curve c_1 in quadrant I, and if there exists a y-functional and x-functional curve c_2 in quadrant II on which $P \geq o$, while $Q > o$ in the region R bounded by c_1, c_2 and $\partial S(0, A)$, then 0 is unstable.

COROLLARY 4. If $Q = o$ on the y-functional curve c_1 in quadrant I, $P \geq o$ on some y-functional and x-functional curve c_2 in quadrant IV, while both P and Q are nonnegative in the region R bounded by c_1, c_2 and $\partial S(0, A)$, then 0 is unstable.

More corollaries could be given, but the idea behind them should now be clear. By rotation of coordinates one may obtain similar theorems without "functionality" assumptions. The virtue of theorems such as these is that the identification of the loci of $P = o$ and $Q = o$ is usually relatively easy.

As an example consider the system

$$\dot{x} = x^2 - y^3,$$

$$\dot{y} = x^2 + y^2 - y^4.$$

The curves c_1 and c_2 of Corollary 3 are given by $x^2 = y^3$.

We may take $A = 1/2$. Then (o, o) is unstable. Note that instability could not have been shown by a linearization argument in this case.

3. Instability of higher order systems. In generalizing Theorem 1 and its corollaries to higher order systems there are two main difficulties. The geometrical picture is no longer so clear, but what is more important is that the Poincaré-Bendixson theorem no longer can be used.

Consider the system

$$(2) \qquad \dot{X} = F(X)$$

where $X = (x_1, \ldots, x_n)$ as before and $F = (f_1, \ldots, f_n)$. Assume that 0 is the only singular point in $S(0, B)$ for some $B > o$. Let F satisfy a Lipschitz condition with respect to X in $S(0, B)$.

We shall say that a closed set K, with 0 in the closure of the interior of K, is positively invariant near 0 if there exists $A > o$ and $A < B$ such that if X_o belongs to the interior of $R_A = K \cap \overline{S(0,A)}$ then $f(X_o, t)$, the solution through X_o, can leave R_A only through $\partial S(0, A)$, (e.g., this happens for any $A > o$ if K is closed and positively invariant).

THEOREM 2. If K is positively invariant near 0 and h is a vector such that the inner product $(h, F) > o$ at all points of R_A (with A chosen as above) except 0, then 0 is unstable.

Proof. Let X_o be in the interior of R_A and suppose that $(h, X_o) \neq o$ but that X_o is as close as one wishes to 0. $f(X_o, t)$ does not leave R_A through the boundary of K by definition of K. Then $R_A \cap H$ is compact, where H is that closed half space determined by $(h, X - X_o) = o$ such that 0 is not in H. There then exists $d > o$ such that $|F| \geq d$ on $R_A \cap H$. Now unless $f(X_o, t)$ leaves $R_A \cap H$ through $\partial S(0, A)$

we have $f(X_o, t)$ in $R_A \cap H$ for all positive t. Hence $|f(X_o, t) - X_o| \geq (t - t_o)d$ and so $f(X_o, t)$ leaves the compact set R_A.

As an example consider the system

$$\dot{x} = x^2 + y^2 - z ,$$

$$\dot{y} = -y(x^2 + 1) ,$$

and

$$\dot{z} = x^4(z^2 + 1) + z .$$

Let K be generated by the paraboloid $z = x^2 + y^2$. The inward normal of $z = x^2 + y^2$ is $(-2x, -2y, 1)$. The inner product of this with $(\dot{x}, \dot{y}, \dot{z})$ is positive on $z = x^2 + y^2$ except at 0. Thus K is positively invariant. $\dot{z} > o$ if $z > o$, so we may let $h = (o, o, 1)$. Then 0 is unstable by Theorem 2.

REFERENCE

1. T. Burton, A generalization of Liapunov's Direct Method (unpublished Ph. D. thesis), Washington State University, Pullman, Washington, 1964.

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