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OPERATORS HAVING THE SYMMETRIZED BIDISC AS A SPECTRAL SET

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Abstract We characterize those commuting pairs of operators on Hilbert space that have the symmetrized bidisc as a spectral set in terms of the positivity of a hermitian operator pencil (without any assumption about the joint spectrum of the pair). Further equivalent conditions are that the pair has a normal dilation to the distinguished boundary of the symmetrized bidisc, and that the pair has the symmetrized bidisc as a *complete* spectral set. A consequence is that every contractive representation of the operator algebra $A(\Gamma)$ of continuous functions on the symmetrized bidisc analytic in the interior is completely contractive. The proofs depend on a polynomial identity that is derived with the aid of a realization formula for doubly symmetric hereditary polynomials, which are positive on commuting pairs of contractions.

Keywords: spectral set; commuting operators; hereditary functions interpolation; commutant lifting theorem

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1. Introduction

The *symmetrized bidisc* is the set

$$\Gamma = \{(\lambda_1 + \lambda_2, \lambda_1\lambda_2) : |\lambda_1| \leqslant 1,\ |\lambda_2| \leqslant 1\} \subset \mathbb{C}^2.$$

There are good reasons to study functions on Γ. From a pure mathematical viewpoint, functions on Γ are the same thing as symmetric functions on the bidisc; moreover, Γ has the virtue that it is possible to obtain explicit results despite the fact that Γ is neither convex nor smooth nor homogeneous, and hence lies outside the scope of many of the main positive results in the theory of several complex variables. There is also a more concrete incentive to study Γ, relating to possible applications in control engineering, and it is this that lies behind the operator-theoretic question we address in this paper. The problem of the robust stabilization of imperfectly known physical devices in H^∞ control theory [**7**, **9**] gives rise to a range of interpolation problems for analytic functions. One of these, the problem of μ-synthesis [**6**], is much studied but evidently intractable. A special case of this problem led us to questions about interpolation by analytic functions from the disc *into* Γ. A promising approach is to develop the operator theory associated with Γ; this idea led us to the discovery of a significant new type of necessary condition for spectral interpolation [**5**, Theorem 5.3]. This condition was a corollary of the main result of [**5**], restated as Theorem 1.1 below, which can be thought of as a commutant lifting theorem for Γ, or, alternatively, as a characterization of pairs of commuting operators on Hilbert space which have Γ as a complete spectral set. Just as results about functions on the disc can be obtained from an understanding of contractions, isometries and unitary operators (see, for example, [**8**]), so the elaboration of analogues of these families of operators associated with Γ provide us with an effective new technique. For example, they permit the explicit calculation of the Caratheodory distance for Γ, as we intend to show in a future paper. The Γ-analogue of a contraction is a commuting pair of operators having Γ as a spectral set. In [**5**] we began the development of the theory of such pairs; here we carry it further. A central question is as follows.

Which commuting pairs of operators have Γ as a spectral set?

Here we strengthen the main result of [**5**] in three ways. Firstly, we remove entirely an assumption about the joint spectrum of the pair of operators in question; since spectral assumptions are hard to check, this is a significant improvement. Secondly, we show that Γ is a spectral set for a particular commuting pair of operators if and only if it is a *complete* spectral set. This is equivalent to saying that every contractive representation of the analytic function algebra $A(\Gamma)$ is also completely contractive. The third addition to the theorem is an alternative characterization of pairs of commuting operators which have Γ as a complete spectral set, in terms of contractivity instead of positivity.

More specifically, let H be a complex Hilbert space and let $\mathcal{L}(H)$ denote the C^*-algebra of bounded linear transformations of H. For $S, P \in \mathcal{L}(H)$, define $\rho(S,P) \in \mathcal{L}(H)$ by the formula

$$\rho(S,P) = 2(1 - P^*P) - S + S^*P - S^* + P^*S. \tag{1.1}$$

For a compact set K in $\mathbb{C}^2$ and a commuting pair $T = (T_1, T_2)$ of operators in $\mathcal{L}(H)$, we say (following von Neumann) that K *is a spectral set for* T if $\sigma(T)$, the Taylor spectrum of T, is a subset of K and

$$\|f(T)\| \leqslant \max_{z\in K} |f(z)|, \tag{1.2}$$

for all functions f that are holomorphic on a neighbourhood of K. Following Arveson [**4**], we say that K is a *complete spectral set* for T if $\sigma(T) \subset K$ and

$$\|f(T)\| \leqslant \max_{z\in K} \|f(z)\|, \tag{1.3}$$

for all matrix-valued functions f that are holomorphic on a neighbourhood of K.

In the special case when K is polynomially convex, the Oka–Weil Theorem (see [**3**] for example) and the continuity of the Taylor functional calculus imply that K is a spectral set for T if and only if (1.2) holds for all polynomials f. Thus, in the case of polynomially convex K, the assumption in the definition of spectral set that $\sigma(T) \subset K$ is unnecessary. Furthermore, K is a complete spectral set if and only if (1.3) holds for all matrix polynomials. Thus, polynomially convex spectral sets and complete spectral sets can be understood without reference to either the Taylor spectrum or the Taylor functional calculus. It is easy to show that Γ is indeed polynomially convex [**5**, Lemma 2.1].

By the *distinguished boundary* of Γ we mean the set

$$b\Gamma \overset{\text{def}}{=} \{(\lambda_1 + \lambda_2, \lambda_1\lambda_2) : |\lambda_1| = |\lambda_2| = 1\}.$$

We now recall Theorem 1.2 from [**5**]. For S and P commuting operators, let $\sigma_{\text{pol}}(S,P)$ denote the spectrum of (S,P) in the norm closed algebra generated by the polynomials in S and P.

Theorem 1.1. *Let (S,P) be a pair of commuting operators on a Hilbert space H such that $\sigma_{\text{pol}}(S,P) \subset \Gamma$. The following statements are equivalent.*

(i) *$\rho(\alpha S, \alpha^2 P) \geqslant 0$ for all $\alpha \in \mathbb{D}$.*

(ii) *For every matrix polynomial f in two variables*

$$\|f(S,P)\| \leqslant \sup_{z\in\Gamma} \|f(z)\|.$$

(iii) *There exist Hilbert spaces H_-, H_+ and a commuting pair of normal operators $(\tilde{S},\tilde{P})$ on $K \overset{\text{def}}{=} H_- \oplus H \oplus H_+$, such that $\sigma(\tilde{S},\tilde{P})$ is contained in the distinguished boundary of Γ and $\tilde{S}$ and $\tilde{P}$ are expressible by operator matrices of the form*

$$\tilde{S} \sim \begin{bmatrix} * & * & * \\ 0 & S & * \\ 0 & 0 & * \end{bmatrix} \quad \text{and} \quad \tilde{P} \sim \begin{bmatrix} * & * & * \\ 0 & P & * \\ 0 & 0 & * \end{bmatrix}, \tag{1.4}$$

with respect to the orthogonal decomposition $K = H_- \oplus H \oplus H_+$.

We shall strengthen Theorem 1.1 in three ways. First we shall remove the hypothesis that $\sigma_{\text{pol}}(S,P) \subset \Gamma$. Specifically, we shall prove the following proposition.

Proposition 1.2. *Let (S,P) be a commuting pair of operators on a Hilbert space. If $\rho(\alpha S, \alpha^2 P) \geqslant 0$ for all $\alpha \in \mathbb{D}$, then $\sigma_{\text{pol}}(S,P) \subset \Gamma$.*

The second way in which we will strengthen Theorem 1.1 will be to show that statement (ii) (that Γ is a *complete* spectral set for (S,P)) is, in fact, equivalent to the *a priori* weaker statement that Γ is a spectral set for (S,P). This will be accomplished by proving the following proposition.

Proposition 1.3. *If Γ is a spectral set for a commuting pair of operators (S, P), then*

$$\rho(\alpha S, \alpha^2 P) \geqslant 0, \quad \textit{for all } \alpha \in \mathbb{D}.$$

A third enhancement of Theorem 1.1 is an alternative to (i) in which positivity of the hermitian operator pencil $\rho(\alpha S, \alpha^2 P)$ is replaced by the contractivity of a pencil of linear fractional transforms of the pair (S, P). For any $\alpha \in \mathbb{D}$, we introduce the linear fractional mapping ψ_α, given by

$$\psi_\alpha(z) = \frac{z_2 - \alpha z_1 + \alpha^2}{1 - \bar{\alpha} z_1 + \bar{\alpha}^2 z_2}. \tag{1.5}$$

The denominator does not vanish on Γ (if $z = (\lambda_1 + \lambda_2, \lambda_1\lambda_2)$, then the denominator is $(1 - \bar{\alpha}\lambda_1)(1 - \bar{\alpha}\lambda_2)$, and, since $\alpha \in \mathbb{D}$, this is non-zero for λ in the closed bidisc). Thus, ψ_α maps a neighbourhood of Γ analytically into $\mathbb{C}$; moreover, it has modulus one on the distinguished boundary of Γ, that is, ψ_α is an inner function on Γ. The connection between the ψ_α and ρ is given by the following proposition.

Proposition 1.4. *Let (S, P) be a commuting pair of operators on a Hilbert space and suppose that for every $\alpha \in \mathbb{D}$, $1 - \bar{\alpha} S + \bar{\alpha}^2 P$ is invertible and $\psi_\alpha(S, P)$ is a contraction. Then*

$$\rho(\alpha S, \alpha^2 P) \geqslant 0, \quad \textit{for all } \alpha \in \mathbb{D}.$$

Putting together Theorem 1.1 and Propositions 1.2, 1.3 and 1.4, we obtain the following result.

Theorem 1.5. *Let (S, P) be a pair of commuting operators on a Hilbert space H. The following conditions are equivalent.*

(i) *Γ is a spectral set for (S, P).*

(ii) *For every $\alpha \in \mathbb{D}$, $1 - \bar{\alpha} S + \bar{\alpha}^2 P$ is invertible and $\psi_\alpha(S, P)$ is a contraction.*

(iii) *$\rho(\alpha S, \alpha^2 P) \geqslant 0$ for all $\alpha \in \mathbb{D}$.*

(iv) *(S, P) has a dilation to the distinguished boundary of Γ (i.e. statement (iii) of Theorem 1.1 holds).*

(v) *Γ is a complete spectral set for (S, P).*

The proof of Theorem 1.5 consists of the following observations. (i) $\Rightarrow$ (iii) is Proposition 1.3. (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) follow from Theorem 1.1 and Proposition 1.2. (v) $\Rightarrow$ (i) is tautological, and so (i), (iii), (iv) and (v) are all equivalent. (ii) $\Rightarrow$ (iii) is Proposition 1.4. It remains to prove (iii) $\Rightarrow$ (ii). Suppose (iii). We shall need Lemma 3.4, proved below, which tells us that for every $\alpha \in \mathbb{D}$, $1 - \bar{\alpha} S + \bar{\alpha}^2 P$ is invertible and so $\psi_\alpha(S, P)$ is defined. We already know that (iii) $\Rightarrow$ (i), and so Γ is a spectral set for (S, P). Since, moreover, $|\psi_\alpha| \leqslant 1$ on Γ, we have $\|\psi_\alpha(S, P)\| \leqslant 1$ for all $\alpha \in \mathbb{D}$. Hence (iii) $\Rightarrow$ (ii).

The nub of this result is the equivalence (iii) $\Leftrightarrow$ (v). At the end of this section we shall briefly outline the proof (given in [**5**]) that the positivity condition (iii) implies (v), that Γ is a complete spectral set for (S, P).

One part of Theorem 1.5 is worth singling out. If $\mathcal{A}$ is an operator algebra, then a basic question that one can ask is the following (see, for example, [**11**] or [**10**, ch. 4]).

Is every contractive representation of $\mathcal{A}$ completely contractive?

A class of examples is furnished by the algebras $\mathcal{A} = A(K)$ for compact sets $K \subset \mathbb{C}^n$. Here, $A(K)$ denotes, as usual, the uniformly closed subalgebra of $C(K)$ comprising the functions that are holomorphic on the interior of K. It is known, for example, that the answer to the above question is yes when $\mathcal{A} = A(K)$ and K is the disc, the bidisc, or an annulus, but no when K is a tridisc. Theorem 1.5 gives a qualitatively new example to add to the pot.

Theorem 1.6. *Every contractive representation of $A(\Gamma)$ is completely contractive.*

Here we are regarding $A(\Gamma)$ as an algebra of operators on the Hardy space of the bidisc $H^2(\mathbb{D}^2)$ in a natural way. We define $\pi : \mathbb{D}^2 \to \Gamma$ by

$$\pi(\lambda_1, \lambda_2) = (\lambda_1 + \lambda_2, \lambda_1\lambda_2),$$

so that $h \mapsto h \circ \pi$ is a bijection between analytic functions h on Γ and symmetric analytic functions on the closed bidisc. Then $h \in A(\Gamma)$ is identified with the operator $M_{h\circ\pi}$ on $H^2(\mathbb{D}^2)$ of multiplication by $h \circ \pi$. In this way, $A(\Gamma)$ becomes a commuting algebra of operators on the Hilbert space $H^2(\mathbb{D}^2)$, and so it makes sense to speak of completely contractive maps on $A(\Gamma)$.

Proof. Let τ be a contractive representation of $A(\Gamma)$ on a Hilbert space H. Let s, p be the restrictions to Γ of the coordinate functions on $\mathbb{C}^2$, and let $\tau(s) = S$, $\tau(p) = P \in \mathcal{L}(H)$. Then $SP = PS$, and, for any polynomial function $h \in A(\Gamma)$, we have $\tau(h) = h(S, P)$. By the contractivity of τ, Γ is a spectral set for (S, P). Hence, by Theorem 1.5, Γ is a complete spectral set for (S, P). We claim that τ is completely contractive. Consider any matricial polynomial $h = [h_{ij}] \in M_n(\mathbb{C}) \otimes A(\Gamma)$. Regarded as an operator on $H^2(\mathbb{D}^2)^n$, h is identified with the multiplication operator $[M_{h_{ij}\circ\pi}] = M_{h\circ\pi}$, and so

$$\|h\| = \|M_{h\circ\pi}\| = \sup_{z\in\mathbb{D}^2} \|h \circ \pi(z)\|.$$

On the other hand, the representation τ_n of $M_n(\mathbb{C}) \otimes A(\Gamma)$ on H^n induced by τ satisfies

$$\tau_n(h) = [\tau(h_{ij})] = [h_{ij}(S, P)] = h(S, P),$$

and hence, since Γ is a complete spectral set for (S, P),

$$\|\tau_n(h)\| = \|h(S, P)\| \leqslant \sup_{z\in\Gamma} \|h(z)\| = \sup_{z\in\mathbb{D}^2} \|h \circ \pi(z)\| = \|h\|.$$

Since the polynomial functions are dense in $A(\Gamma)$, this inequality remains valid for all $h \in M_n(\mathbb{C}) \otimes A(\Gamma)$. Since it is true for all positive integers n, we conclude that τ is completely contractive. □

A natural question is: does every contractive representation of $A(\Gamma)$ extend to a contractive representation of the bidisc algebra? Equivalently, if (S, P) is a commuting pair of operators on H for which Γ is a spectral set, does it follow that there are commuting contractions A, B on H such that $S = A + B$, $P = AB$? The answer is no: there is an obstruction involving the existence of square roots. If such an A, B exist, then $S^2 - 4P = (A - B)^2$, so that $S^2 - 4P$ has a square root. Consider the case $S = 0$: it follows from Theorem 1.1 that Γ is a spectral set for $(0, P)$ if and only if P is a contraction. Hence, if we take P to be a contraction having no square root, then $(0, P)$ is a counter-example.

Propositions 1.2–1.4 will be proved in § 3 of the paper. It is interesting that the proofs of all three depend on the same algebraic identity.

Proposition 1.7. *If $\alpha \in \mathbb{D}$, then the following polynomial identity obtains:*

$$
\begin{aligned}
&(1 - \alpha y_1 + \alpha^2 y_2)(1 - \bar{\alpha} x_1 + \bar{\alpha}^2 x_2) - (y_2 - \bar{\alpha} y_1 + \bar{\alpha}^2)(x_2 - \alpha x_1 + \alpha^2) \\
&\quad = \tfrac{1}{4}(1 - |\alpha|^2)\Bigg[|1 - \alpha|^2\left(2(1 - y_2 x_2) - \frac{1 - \alpha}{1 - \bar{\alpha}}(y_2 x_1 - y_1) - \frac{1 - \bar{\alpha}}{1 - \alpha}(y_1 x_2 - x_1)\right) \\
&\qquad\qquad + |1 + \alpha|^2\left(2(1 - y_2 x_2) + \frac{1 + \alpha}{1 + \bar{\alpha}}(y_2 x_1 - y_1) + \frac{1 + \bar{\alpha}}{1 + \alpha}(y_1 x_2 - x_1)\right)\Bigg].
\end{aligned}
$$

The connection between this identity and (ii) ⇔ (iii) in Theorem 1.5 will become apparent later: in Lemma 3.1 below, we substitute S, P and their adjoints into it to obtain an operator identity that relates $\psi_\alpha(S, P)$ and $\rho(\omega S, \omega^2 P)$ for a suitable ω of unit modulus.

One proof of Proposition 1.7 is easily executed: expand both sides into sums of monomials, collect terms, equate coefficients. A second approach would be to notice that if the identity is multiplied by $|1 - \alpha|^2 |1 + \alpha|^2$, it becomes a polynomial identity in the six variables x_1, x_2, y_1, y_2, α and $\bar{\alpha}$, an identity that can be verified either by hand or on a computer.

Here we shall adopt a third approach. As with most identities that have interesting applications, the hard thing is to find them, not to prove them. In § 2 of the paper we shall derive the identity by explicitly computing the symmetric realization (see [**5**, Theorem 3.5]) of the 'simplest' symmetric inner functions on the bidisc. For this purpose we shall require the elements of the hereditary functional calculus for operators. All of the paper except the derivation of the above identity can be understood without this theory, but some of the point of the paper would be lost. A brief introduction to hereditary functions can be found in § 3 of [**5**], and a fuller one in [**2**]. In brief, a hereditary polynomial in a commuting pair of operators $T = (T_1, T_2)$ is a finite sum

$$h(T) = h(T_1, T_2) = \sum_{r,s,t,u} c_{rstu} T_1^{*^r} T_2^{*^s} T_1^t T_2^u.$$

We think of this as the result of applying the scalar polynomial in four variables

$$h(x_1, x_2, y_1, y_2) = \sum_{r,s,t,u} c_{rstu} y_1^r y_2^s x_1^t x_2^u$$

to T. We also consider matricial polynomials h.

A central idea in the hereditary polynomial approach is that a family of commuting pairs of operators can be understood in terms of the cone of hereditary polynomials, which are positive on all members of the family. If the family is such that this 'dual cone' admits a simple description, for example, in terms of a small generating set, one typically obtains powerful representation results. We illustrate with a broad brush sketch of the proof that (iii) $\Rightarrow$ (v) in Theorem 1.5 above. Let $\bar{\mathbb{D}}$ denote the closed unit disc. Suppose the commuting pair of operators (S, P) is such that $\rho(\alpha S, \alpha^2 P) \geqslant 0$ for all $\alpha \in \mathbb{D}$. Consider any matricial polynomial that is bounded by 1 on Γ; we must show that $g(S, P)$ is a contraction. Since $g \circ \pi$ is bounded by 1 on $\bar{\mathbb{D}}^2$, it follows from Ando's theorem [**12**] that the hereditary polynomial

$$h : (x, y) \mapsto 1 - g(\bar{y}_1 + \bar{y}_2, \bar{y}_1\bar{y}_2)^* g(x_1 + x_2, x_1 x_2)$$

is positive on pairs of commuting contractions. This hereditary polynomial is also *doubly symmetric*, that is, invariant under the transposition of coordinates in both x and y independently. By Theorem 2.2 below and the spectral theorem for unitary operators, h can be approximated uniformly on compact subsets of $\mathbb{D}^2$ by finite sums of the form

$$G \circ \pi(\bar{y})^* \nu_\omega(\pi(x), \pi(\bar{y})) G \circ \pi(x),$$

for some analytic operator-valued function G on Γ and certain hereditary polynomials ν_ω indexed by $\omega \in \mathbb{T}$; the ν_ω constitute the desired small set of generators for the appropriate dual cone in this case. Put

$$\pi(x) = (s, p), \qquad \pi(\bar{y}) = (s_*, p_*),$$

to get

$$1 - g(s_*, p_*)^* g(s, p) \approx \sum_j G_j(s_*, p_*)^* \nu_{\omega_j}(s, p, s_*, p_*) G_j(s, p).$$

The hereditary polynomial ν_ω has the property that

$$\nu_\omega(S, P, S^*, P^*) = \rho(\omega S, \omega^2 P),$$

and hence,

$$1 - g(S, P)^* g(S, P) \approx \sum_j G_j(S, P)^* \rho(\omega_j S, \omega_j^2 P) G_j(S, P).$$

Since each $\rho(\omega_j S, \omega_j^2 P) \geqslant 0$, it is clear that

$$1 - g(S,P)^* g(S,P) \geqslant 0,$$

that is,

$$\|g(S,P)\| \leqslant 1.$$

Thus Γ is a complete spectral set for (S,P).

2. An identity

In this section we shall show in detail how the doubly symmetric hereditary realization formula from [**5**] can be used to derive the identity in Proposition 1.7. Recall from [**5**, § 2] that a hereditary function h on $\mathbb{D}^2$ is said to be *positive on contractions* if $h(T) \geqslant 0$ whenever T is a pair of commuting contractions. The following result is proved in [**2**].

Theorem 2.1. *Let h be an $\mathcal{L}(\mathcal{C})$-valued hereditary function on $\mathbb{D}^2$. Then h is positive on contractions if and only if there exist Hilbert spaces $\mathcal{C}_1, \mathcal{C}_2$ and analytic functions*

$$f_j : \mathbb{D}^2 \to \mathcal{L}(\mathcal{C}, \mathcal{C}_j), \quad j = 1, 2,$$

such that for all $\lambda, \mu \in \mathbb{D}^2$,

$$h(\lambda, \bar{\mu}) = (1 - \bar{\mu}_1 \lambda_1) f_1(\mu)^* f_1(\lambda) + (1 - \bar{\mu}_2 \lambda_2) f_2(\mu)^* f_2(\lambda).$$

For $\lambda = (\lambda_1, \lambda_2) \in \mathbb{C}^2$, let $\lambda^\sigma = (\lambda_2, \lambda_1)$. For h a hereditary function on $\mathbb{D}^2$, say h is *doubly symmetric* if $h(\lambda^\sigma, \bar{\mu}) = h(\lambda, \bar{\mu}) = h(\lambda, \bar{\mu}^\sigma)$ for all $\lambda, \mu \in \mathbb{C}^2$. Evidently, hereditary functions on Γ induce doubly symmetric hereditary functions on $\mathbb{D}^2$. It turns out that the hereditary functions on Γ that are positive on pairs (S,P) satisfying the equivalent conditions of Theorem 1.1 are exactly those that induce doubly symmetric hereditary functions on $\mathbb{D}^2$ that are positive on contractions. Furthermore, this fact is equivalent to Theorem 1.1 modulo various abstract model theory manoeuvres and technical details. For this reason one wants to obtain a realization formula for doubly symmetric hereditary functions that are positive on contractions.

The strategy taken in [**5**] was to state the doubly symmetric realization formula [**5**, Theorem 2.5] and then to prove it. Here we shall derive the formula and, in the process, not only prove it but explain how it was discovered. We shall then specialize the derivation to a particular h and in the process discover the identity from § 1.

Accordingly, let h be an $\mathcal{L}(\mathcal{C})$-valued hereditary function on $\mathbb{D}^2$ such that h is doubly symmetric and positive on contractions. By Theorem 2.1, there exist a pair of Hilbert spaces, $\mathcal{C}_1$ and $\mathcal{C}_2$, and a pair of analytic functions, f_1 and f_2, such that $f_i : \mathbb{D}^2 \to \mathcal{L}(\mathcal{C}, \mathcal{C}_i)$ and

$$h(\lambda, \bar{\mu}) = (1 - \bar{\mu}_1 \lambda_1) f_1(\mu)^* f_1(\lambda) + (1 - \bar{\mu}_2 \lambda_2) f_2(\mu)^* f_2(\lambda), \tag{2.1}$$

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for all $\lambda \in \mathbb{D}^2$. It follows (as in the proof of Lemma 3.3 of [**5**]) that if a Hilbert space $\mathcal{E}$ is defined by

$$\mathcal{E} = \mathcal{C}_1 \oplus \mathcal{C}_2, \tag{2.2}$$

and an analytic $\mathcal{L}(\mathcal{C}, \mathcal{E})$-valued function on $\mathbb{D}^2$ is defined by

$$F(\lambda) = (1/\sqrt{2}) f_1(\lambda) \oplus f_2(\lambda^\sigma), \tag{2.3}$$

then

$$h(\lambda, \bar{\mu}) = (1 - \bar{\mu}_1 \lambda_1) F(\mu)^* F(\lambda) + (1 - \bar{\mu}_2 \lambda_2) F(\mu^\sigma)^* F(\lambda^\sigma), \tag{2.4}$$

for all $\lambda, \mu \in \mathbb{D}^2$. Also, as in the proof of Lemma 2.4 of [**5**], there exists a unitary operator L on $\mathcal{E}$ such that

$$L(F(\lambda) - F(\lambda^\sigma)) = \lambda_1 F(\lambda) - \lambda_2 F(\lambda^\sigma), \tag{2.5}$$

for all $\lambda \in \mathbb{D}^2$.

Now define $G : \mathbb{D}^2 \to \mathcal{L}(\mathcal{E})$ by setting

$$G(\lambda) = (\lambda_2 - L)^{-1} F(\lambda). \tag{2.6}$$

With this definition, (2.5) implies that G is symmetric and (2.4) becomes

$$h(\lambda, \bar{\mu}) = G(\mu)^* [(1 - \bar{\mu}_1 \lambda_1)(\mu_2 - L)^* (\lambda_1 - L) + (1 - \bar{\mu}_2 \lambda_2)(\mu_1 - L)^* (\lambda_2 - L)] G(\lambda). \tag{2.7}$$

Note that (2.7) holds for any choice of unitary L satisfying (2.5) provided G is defined by (2.6).

Next we see that if the bracketed expression in (2.7) is expanded and the terms are grouped appropriately, then

$$h(\lambda, \bar{\mu}) = G(\mu)^* [2(1 - \overline{\mu_1 \mu_2} \lambda_1 \lambda_2) + (\overline{\mu_1 \mu_2}(\lambda_1 + \lambda_2) - (\overline{\mu_1 + \mu_2}))L \\ + ((\overline{\mu_1 + \mu_2}) \lambda_1 \lambda_2 - (\lambda_1 + \lambda_2)) L^*] G(\lambda). \tag{2.8}$$

To simplify the appearance of (2.8) somewhat we introduce the polynomial ν, defined by

$$\nu(x, y) = 2(1 - y_2 x_2) + y_2 x_1 - y_1 + y_1 x_2 - x_1. \tag{2.9}$$

Recall that π is defined by

$$\pi(\lambda) = (\lambda_1 + \lambda_2, \lambda_1 \lambda_2),$$

so that the bracketed expression in (2.8) simplifies to

$$\nu(\pi(\lambda L^*), \pi(\bar{\mu} L)).$$

Here we are using the notation

$$\lambda L^* = (\lambda_1 L^*, \lambda_2 L^*),$$

so that

$$\pi(\lambda L^*) = ((\lambda_1 + \lambda_2)L^*, (\lambda_1\lambda_2)L^{*2}).$$

We summarize the previous calculation in the following theorem (cf. [**5**, Theorem 3.5])

Theorem 2.2. *Let $h : \mathbb{D}^2 \times \mathbb{D}^2 \to \mathcal{L}(\mathcal{C})$ be a doubly symmetric hereditary function that is positive on contractions. There exist a Hilbert space $\mathcal{E}$, a unitary operator L on $\mathcal{E}$, and a holomorphic symmetric $\mathcal{L}(\mathcal{C}, \mathcal{E})$-valued function G on $\mathbb{D}^2$ such that*

$$h(\lambda, \bar{\mu}) = G(\mu)^* \nu(\pi(\lambda L^*), \pi(\bar{\mu} L))G(\lambda). \tag{2.10}$$

Note that the converse of Theorem 2.2 follows immediately from (2.7), Theorem 2.1 and basic algebraic properties of the hereditary functional calculus.

We shall now repeat the derivation of (2.10) for a particular function h. Our goal will be to calculate F, G and L explicitly. For fixed $\alpha \in \mathbb{D}$, define functions s_α and B_α on $\mathbb{D}$, and a function φ_α on $\mathbb{D}^2$, by the formulae

$$s_\alpha(z) = (1 - |\alpha|^2)^{1/2} \frac{1}{1 - \bar{\alpha} z},$$
$$B_\alpha(z) = \frac{z - \alpha}{1 - \bar{\alpha} z}$$

and

$$\varphi_\alpha(\lambda) = B_\alpha(\lambda_1) B_\alpha(\lambda_2).$$

We remark that s_α is the normalized Szegő kernel function for the point α, and that B_α is the Blashke factor with zero at α. Since φ_α is an inner function on $\mathbb{D}^2$, Ando's Theorem [**12**] guarantees that φ_α maps pairs of commuting contractions to contractions. Thus, since φ_α is also symmetric, there exist G and L such that (2.8) holds with

$$h(\lambda, \bar{\mu}) = 1 - \overline{\varphi_\alpha(\mu)} \varphi_\alpha(\lambda).$$

To compute G and L for this choice of h, we start with the computation of (2.1). A well-known identity on the disc is

$$1 - \overline{B_\alpha(\mu)} B_\alpha(\lambda) = (1 - \bar{\mu}\lambda) \overline{s_\alpha(\mu)} s_\alpha(\lambda).$$

Hence,

$$\begin{aligned} 1 - \overline{\varphi_\alpha(\mu)}\varphi_\alpha(\lambda) &= 1 - \overline{B_\alpha(\mu_1) B_\alpha(\mu_2)} B_\alpha(\lambda_1) B_\alpha(\lambda_2) \\ &= 1 - \overline{B_\alpha(\mu_1)} B_\alpha(\lambda_1) + \overline{B_\alpha(\mu_1)}(1 - \overline{B_\alpha(\mu_2)} B_\alpha(\lambda_2)) B_\alpha(\lambda_1) \\ &= (1 - \bar{\mu}_1\lambda_1)\overline{s_\alpha(\mu_1)} s_\alpha(\lambda_1) + (1 - \bar{\mu}_2\lambda_2)\overline{B_\alpha(\mu_1) s_\alpha(\mu_2)} B_\alpha(\lambda_1) s_\alpha(\lambda_2), \end{aligned}$$

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and we see that (2.1) holds with

$$\mathcal{C} = \mathcal{C}_1 = \mathcal{C}_2 = \mathbb{C},$$
$$f_1(\lambda) = s_\alpha(\lambda_1)$$

and

$$f_2(\lambda) = B_\alpha(\lambda_1)s_\alpha(\lambda_2).$$

From (2.2) and (2.3), we deduce that (2.4) holds with the choices

$$\mathcal{E} = \mathbb{C}^2 \quad \text{and} \quad F(\lambda) = \frac{1}{\sqrt{2}}\begin{bmatrix} s_\alpha(\lambda_1) \\ B_\alpha(\lambda_2)s_\alpha(\lambda_1)\end{bmatrix}. \tag{2.11}$$

We now need to compute a unitary L acting on $\mathbb{C}^2$ such that (2.5) holds. Equation (2.11) and a little linear algebra gives the following description of an L:

$$L = \omega_1 u_1 \otimes u_1 + \omega_2 u_2 \otimes u_2, \quad \text{where}$$
$$\omega_1 = -\frac{1-\alpha}{1-\bar{\alpha}}, \qquad \omega_2 = \frac{1+\alpha}{1+\bar{\alpha}}, \qquad u_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}, \qquad u_2 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}. \tag{2.12}$$

Next, we see from (2.11) that (2.10) holds if L is as in (2.12) and G is defined by

$$G(\lambda) = \frac{1}{\sqrt{2}}s_\alpha(\lambda_1)(\lambda_2 - L)^{-1}\begin{bmatrix}1\\B_\alpha(\lambda_2)\end{bmatrix}. \tag{2.13}$$

Thus, viewing G as vector-valued (rather than $\mathcal{L}(\mathbb{C},\mathbb{C}^2)$-valued) we deduce that

$$1 - \overline{\varphi_\alpha(\mu)}\varphi_\alpha(\lambda) = \langle \nu(\pi(\lambda L^*), \pi(\bar{\mu}L))G(\lambda), G(\mu)\rangle. \tag{2.14}$$

The remainder of the computation will be achieved by expressing G in terms of u_1 and u_2, the eigenvectors of L. Note that (2.12) implies that

$$\begin{aligned}\begin{bmatrix}1\\B_\alpha(\lambda_2)\end{bmatrix} &= \left\langle \begin{bmatrix}1\\B_\alpha(\lambda_2)\end{bmatrix}, u_1\right\rangle u_1 + \left\langle \begin{bmatrix}1\\B_\alpha(\lambda_2)\end{bmatrix}, u_2\right\rangle u_2 \\ &= \frac{1}{\sqrt{2}}((1+B_\alpha(\lambda_2))u_1 + (1-B_\alpha(\lambda_2))u_2), \\ 1 + B_\alpha(\lambda_2) &= \frac{1}{(1-|\alpha|^2)^{1/2}}s_\alpha(\lambda_2)(1-\bar{\alpha})(\lambda_2-\omega_1),\end{aligned}$$

and

$$1 - B_\alpha(\lambda_2) = -\frac{1}{(1-|\alpha|^2)^{1/2}}s_\alpha(\lambda_2)(1+\bar{\alpha})(\lambda_2-\omega_2).$$

Hence, from (2.12) and (2.13), we see that

$$\begin{aligned}
G(\lambda) &= \frac{1}{\sqrt{2}} s_\alpha(\lambda_1)(\lambda_2 - L)^{-1} \begin{bmatrix} 1 \\ B_\alpha(\lambda_2) \end{bmatrix} \\
&= \tfrac{1}{2} s_\alpha(\lambda_1)((1 + B_\alpha(\lambda_2))(\lambda_2 - L)^{-1} u_1 + (1 - B_\alpha(\lambda_2))(\lambda_2 - L)^{-1} u_2) \\
&= \tfrac{1}{2} s_\alpha(\lambda_1)((1 + B_\alpha(\lambda_2))(\lambda_2 - \omega_1)^{-1} u_1 + (1 - B_\alpha(\lambda_2))(\lambda_2 - \omega_2)^{-1} u_2) \\
&= \frac{1}{2} \frac{1}{(1 - |\alpha|^2)^{1/2}} s_\alpha(\lambda_1) s_\alpha(\lambda_2)((1 - \bar{\alpha}) u_1 - (1 + \bar{\alpha}) u_2).
\end{aligned}$$

Since

$$\nu(\pi(\lambda L^*), \pi(\bar{\mu} L)) u_i = \nu(\pi(\bar{\omega}_i \lambda), \pi(\omega_i \bar{\mu})) u_i,$$

we obtain from (2.14) the equation

$$\begin{aligned}
1 - \overline{\varphi_\alpha(\mu)} \varphi_\alpha(\lambda) = {} & \frac{1}{4} \frac{1}{1 - |\alpha|^2} \overline{s_\alpha(\mu_1) s_\alpha(\mu_2)} s_\alpha(\lambda_1) s_\alpha(\lambda_2) \\
& \times [|1 - \alpha|^2 \nu(\pi(\bar{\omega}_1 \lambda), \pi(\omega_1 \bar{\mu})) + |1 + \alpha|^2 \nu(\pi(\bar{\omega}_2 \lambda), \pi(\omega_2 \bar{\mu}))]. \quad (2.15)
\end{aligned}$$

Finally, notice that if we multiply this last formula by $\overline{(1 - \bar{\alpha}\mu_1)(1 - \bar{\alpha}\mu_2)}(1 - \bar{\alpha}\lambda_1)(1 - \bar{\alpha}\lambda_2)$ and use (2.9), we obtain Proposition 1.7 after the substitutions

$$\begin{aligned}
\lambda_1 + \lambda_2 &= x_1, \\
\lambda_1 \lambda_2 &= x_2, \\
\overline{\mu_1 + \mu_2} &= y_1
\end{aligned}$$

and

$$\overline{\mu_1 \mu_2} = y_2.$$

3. The proofs

In this section we shall prove Propositions 1.2, 1.3 and 1.4 from § 1. For $\alpha \in \mathbb{C}$, we define polynomials g_α and h_α in two variables by the formulae

$$\left.\begin{aligned}
g_\alpha(x) &= 1 - \bar{\alpha} x_1 + \bar{\alpha}^2 x_2, \\
h_\alpha(x) &= x_2 - \alpha x_1 + \alpha^2.
\end{aligned}\right\} \quad (3.1)$$

Note that g_α and h_α are such that

$$\varphi_\alpha(\lambda) = \frac{h_\alpha(\pi(\lambda))}{g_\alpha(\pi(\lambda))}, \quad \lambda \in \mathbb{D}^2, \quad (3.2)$$

and ψ_α (see equation (1.5)) satisfies

$$\psi_\alpha(z) = \frac{h_\alpha(z)}{g_\alpha(z)} = \frac{z_2 - \alpha z_1 + \alpha^2}{1 - \bar{\alpha} z_1 + \bar{\alpha}^2 z_2}. \quad (3.3)$$

Note that ψ_α is an inner function on Γ which is analytic on a neighbourhood of Γ whenever $\alpha \in \mathbb{D}$. For $\omega \in \mathbb{C}$ we define a hereditary polynomial ν_ω by the formula

$$\nu_\omega(x,y) = 2(1 - y_2 x_2) + \omega(y_2 x_1 - y_1) + \bar{\omega}(y_1 x_2 - x_1).$$

The polynomials ν_ω were heavily used in § 2 of [**5**] and we note here the facts that if ν is the polynomial defined in (2.9), then

$$\nu_\omega(\pi(\lambda), \overline{\pi(\mu)}) = \nu(\pi(\bar{\omega}\lambda), \pi(\omega\bar{\mu})),$$

for all $\lambda, \mu \in \mathbb{C}^2$ and that if (S,P) is a commuting pair of operators and ρ is defined as in (1.1), then the hereditary functional calculus yields

$$\nu_\omega(S,P) = \rho(\bar{\omega}S, \bar{\omega}^2 P),$$

for all $\omega \in \mathbb{C}$ with $|\omega| = 1$ (this may be taken to be the *definition* of $\nu_\omega(S,P)$).

For any polynomial function g we define the conjugate polynomial $g^\vee$ by

$$g^\vee(x) = \overline{g(\bar{x})},$$

for all $x \in \mathbb{C}^n$. The identity in Proposition 1.7 from § 1 can be written in terms of our new notations in the following way:

$$g_\alpha^\vee(y) g_\alpha(x) - h_\alpha^\vee(y) h_\alpha(x) = \tfrac{1}{4}(1 - |\alpha|^2)[|1-\alpha|^2 \nu_{\omega_1}(x,y) + |1+\alpha|^2 \nu_{\omega_2}(x,y)],$$

where ω_1, ω_2 are as in equation (2.12). Hence, by the hereditary functional calculus, we have the following lemma.

Lemma 3.1. *If (S,P) is a commuting pair of operators on a Hilbert space and $\alpha \in \mathbb{D}$, then*

$$\begin{aligned} &g_\alpha(S,P)^* g_\alpha(S,P) - h_\alpha(S,P)^* h_\alpha(S,P) \\ &= \tfrac{1}{4}(1-|\alpha|^2)\left[|1-\alpha|^2 \rho\left(-\frac{1-\bar{\alpha}}{1-\alpha}S, \left(\frac{1-\bar{\alpha}}{1-\alpha}\right)^2 P\right) + |1+\alpha|^2 \rho\left(\frac{1+\bar{\alpha}}{1+\alpha}S, \left(\frac{1+\bar{\alpha}}{1+\alpha}\right)^2 P\right)\right]. \end{aligned} \tag{3.4}$$

To prove the propositions we need the following simple lemmas.

Lemma 3.2. *Let $z \in \mathbb{C}^2$. Then $z \in \Gamma$ if and only if $g_\alpha(z) \neq 0$ for all $\alpha \in \mathbb{D}$.*

Proof. If $z = \pi(\lambda)$, then $z \in \Gamma$ if and only if $\lambda \in \operatorname{clos}\mathbb{D}^2$, while $g_\alpha(z) \neq 0$ for all $\alpha \in \mathbb{D}$ if and only if $(1 - \bar{\alpha}\lambda_1)(1 - \bar{\alpha}\lambda_2) \neq 0$ for all $\alpha \in \mathbb{D}$, that is, if and only if $\lambda_i \neq (1/\bar{\alpha})$ for all $\alpha \in \mathbb{D}$ and $i = 1, 2$, hence, if and only if $\lambda_1, \lambda_2 \in \operatorname{clos}\mathbb{D}$. □

Lemma 3.3. *For any commuting pair (S,P) of operators, $\sigma(S,P) \subset \Gamma$ if and only if $g_\alpha(S,P)$ is invertible for all $\alpha \in \mathbb{D}$.*

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Proof. By Lemma 3.2, $\sigma(S,P) \subset \Gamma$ if and only if g_α does not vanish anywhere on $\sigma(S,P)$, that is, $0 \notin g_\alpha(\sigma(S,P))$. By the spectral mapping theorem, this is equivalent to $0 \notin \sigma(g_\alpha(S,P))$, that is, $g_\alpha(S,P)$ is invertible for all $\alpha \in \mathbb{D}$. □

The final fact we need is the following perturbation result.

Lemma 3.4. *Let (S,P) be a commuting pair of operators on a Hilbert space H. If*

$$\rho(\alpha S, \alpha^2 P) \geqslant 0, \quad \textit{for all } \alpha \in \mathbb{D}, \tag{3.5}$$

then

$$g_\alpha(S,P) \textit{ is invertible for all } \alpha \in \mathbb{D}. \tag{3.6}$$

Proof. First assume that (3.5) holds and that we are in the special case where $\|P\| < 1$. Let

$$\mathcal{O} = \{\alpha \in \mathbb{D} : g_\alpha(S,P) \text{ is invertible}\},$$

and note that $\mathcal{O}$ is an open non-empty subset of $\mathbb{D}$, since $0 \in \mathcal{O}$. Hence, (3.6) will follow if we show that

$$\partial\mathcal{O} \subset \partial\mathbb{D}. \tag{3.7}$$

To prove (3.7) we argue by contradiction. Accordingly, assume that $\beta \in \mathbb{D}\backslash\mathcal{O}$ and that there exists a sequence $\{\alpha_n\}$ in $\mathcal{O}$ such that $\alpha_n \to \beta$. Thus, $g_{\alpha_n}(S,P)$ is invertible for each n, yet

$$g_\beta(S,P) = \lim_{n\to\infty} g_{\alpha_n}(S,P)$$

is not invertible, i.e. $g_\beta(S,B)$ lies in the boundary of the set of invertible elements of $\mathcal{L}(H)$. Consequently, there exists a sequence $\{u_n\}$ in H with $\|u_n\| = 1$ and

$$\|g_\beta(S,P)u_n\| \to 0. \tag{3.8}$$

Now notice from (3.4) and (3.5) that

$$h_\beta(S,P)^* h_\beta(S,P) \leqslant g_\beta(S,P)^* g_\beta(S,P).$$

Hence, (3.8) implies that

$$\|h_\beta(S,P)u_n\| \to 0. \tag{3.9}$$

Observe from (3.1) that

$$\alpha g_\alpha(x) - \bar{\alpha} h_\alpha(x) = \bar{\alpha}(|\alpha|^2 - 1)(x_2 - (\alpha/\bar{\alpha})).$$

Hence (3.8) and (3.9) imply that

$$\bar{\beta}(|\beta|^2 - 1)(P - (\beta/\bar{\beta}))u_n \to 0.$$

But, since $\beta \in \mathbb{D}$, $|\beta|^2 - 1 \neq 0$, and since $\beta \notin \mathcal{O}$, $\beta \neq 0$. Thus,

$$(P - (\beta/\bar{\beta}))u_n \to 0$$

and we see that $\beta\bar{\beta}^{-1} \in \sigma(P)$, contradicting the fact that $\|P\| < 1$.

We have shown that Lemma 3.4 holds when $\|P\| < 1$. To prove Lemma 3.4 in general now assume that (S, P) is a pair such that (3.5) holds. Fix $\alpha \in \mathbb{D}$. Choose $r < 1$ such that $(\alpha/r) \in \mathbb{D}$. Since the pair (rS, r^2P) satisfies the hypothesis of Lemma 3.4 and $\|r^2P\| < 1$, by the special case of the lemma just proved,

$$g_\beta(rS, r^2P) \text{ is invertible for all} \beta \in \mathbb{D}.$$

In particular, if $\beta = (\alpha/r)$, we have that $g_\beta(rS, r^2P) = g_\alpha(S, P)$ is invertible, and the proof of Lemma 3.4 is complete. □

We are ready to complete the proof of Theorem 1.5. We must prove Propositions 1.2, 1.3 and 1.4.

To prove Proposition 1.2, fix a commuting pair (S, P) such that

$$\rho(\alpha S, \alpha^2 P) \geqslant 0,$$

for all $\alpha \in \mathbb{D}$. By Lemma 3.4, it follows that $g_\alpha(S, P))$ is invertible for all $\alpha \in \mathbb{D}$. By Lemma 3.3, $\sigma(S, P) \subset \Gamma$. Since Γ is polynomially convex [**5**, Lemma 2.1], it follows from the Oka–Weil Theorem (see, for example, [**3**]) that $\sigma_{\mathrm{pol}}(S, P) \subset \Gamma$. This proves Proposition 1.2.

We prove Proposition 1.4. Let (S, P) be a commuting pair such that $\psi_\alpha(S, P)$ is a contraction for all $\alpha \in \mathbb{D}$. For any $\beta \in \mathbb{T}$ we have

$$\psi_\alpha(\beta S, \beta^2 P) = \beta^2 \psi_{\alpha\bar{\beta}}(S, P),$$

and hence,

$$\|\psi_\alpha(\beta S, \beta^2 P)\| \leqslant 1, \tag{3.10}$$

for all $\alpha \in \mathbb{D}$ and $\beta \in \mathbb{T}$. Since $\beta \mapsto \psi_\alpha(\beta S, \beta^2 P)$ is analytic on a neighbourhood of the closed disc, the maximum principle implies that the inequality (3.10) holds for all $\alpha, \beta \in \mathbb{D}$, and so

$$1 - \psi_\alpha(\beta S, \beta^2 P)^* \psi_\alpha(\beta S, \beta^2 P) \geqslant 0,$$

for all $\alpha, \beta \in \mathbb{D}$. From (3.3), we deduce that

$$g_\alpha(\beta S, \beta^2 P)^* g_\alpha(\beta S, \beta^2 P) - h_\alpha(\beta S, \beta^2 P)^* h_\alpha(\beta S, \beta^2 P) \geqslant 0,$$

for all $\alpha, \beta \in \mathbb{D}$. Hence, by Lemma 3.1,

$$|1-\alpha|^2 \rho\left(-\frac{1-\bar{\alpha}}{1-\alpha}\beta S, \left(\frac{1-\bar{\alpha}}{1-\alpha}\right)^2 \beta^2 P\right) + |1+\alpha|^2 \rho\left(\frac{1+\bar{\alpha}}{1+\alpha}\beta S, \left(\frac{1+\bar{\alpha}}{1+\alpha}\right)^2 \beta^2 P\right) \geqslant 0,$$

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for all $\alpha, \beta \in \mathbb{D}$. In this last inequality let α be real and let $\alpha \to 1$, to obtain

$$\rho(\beta S, \beta^2 P) \geqslant 0,$$

for all $\beta \in \mathbb{D}$. This proves Proposition 1.4.

Proposition 1.3 follows immediately. Suppose Γ is a spectral set for the commuting pair (S, P). Since ψ_α has supremum norm on Γ equal to 1, we have

$$\|\psi_\alpha(S, P)\| \leqslant 1,$$

and so, by Proposition 1.4, $\rho(\alpha S, \alpha^2 P) \geqslant 0$ for all $\alpha \in \mathbb{D}$.

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