AN ORDER-PRESERVING REPRESENTATION THEOREM FOR COMPLEX BANACH ALGEBRAS AND SOME EXAMPLES

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Let A be a complex Banach algebra with unit e of norm one. We show that A can be represented on a compact Hausdorff space Ω which arises entirely out of the algebraic and norm structures of A. This space induces an order structure on A that is preserved by the representation. In the commutative case, Ω is the spectrum of A, and we have a generalization of Gelfand's representation theorem for commutative complex Banach algebras with unit. Various aspects of this representation are illustrated by considering algebras of $n \times n$ complex matrices.

This paper is a revised and expanded version of the unpublished work [8] by the first author.

The following notations are used throughout the paper: cl denotes closure, co denotes convex span of, A' is the dual space of A, w^*-topology $= \sigma(A', A)$-topology, and maximal left ideal is abbreviated to m.l.i.

1. A wedge constructed from the algebraic and norm structures of A. Let J be a m.l.i. of A. Since J is closed and e is at unit distance from J, there exists (by the Hahn–Banach theorem) an element f in A' such that $f(J) = (0)$ and $f(e) = 1 = \|f\|$. Hence the following sets are nonempty:

$$B = \{f \in A' : f(e) = \|f\| = 1\},$$

$$M = \{f \in B : f(J) = (0) \quad \text{for some m.l.i. } J \text{ of } A\},$$

$$\Omega = w^*\text{-cl}\, M,$$

$$S = w^*\text{-cl co}\, M,$$

$$C' = \bigcup \{\lambda S : \lambda \geqq 0\},$$

$$K' = C' - iC',$$

$$K = \{x \in A : \operatorname{Re} f(x) \geqq 0 \quad \text{for all} \quad f \in K'\}.$$

The sets B, S, and Ω are compact and Hausdorff in the w^*-topology. We have $\Omega \subseteq S \subseteq B$ and B and S are convex. Our aim is to represent A as a subspace of $C(\Omega)$, the space of all complex-valued continuous functions on Ω with sup norm, and to consider various aspects of this representation (Examples: In general, M is not w^*-closed and not in one-to-one correspondence with the maximal left ideals).

We observe here that, if we let $H(A) = \{x \in A : f(x) \text{ is real for all } f \text{ in } B\}$ and $K(A) = \{x \in A : f(x) \geqq 0 \text{ for all } f \text{ in } B\}$, then $K(A) + iK(A) \subseteq K$. In fact, $K(A) + iK(A) \subseteq K_B \subseteq K$,

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where $K_B = \{x \in A : \operatorname{Re} f(x) \geqq 0,\ \operatorname{Im} f(x) \geqq 0$, for all f in $B\}$. We show below (§ 2) that $A = K - K$ so that, if $K(A) + iK(A) = K$, then $A = H(A) + iH(A)$, and hence [6] A is a C^*-algebra. Conversely, if A is a C^*-algebra, then $A = H(A) + iH(A)$ and also (see, for example, [3, Proposition 2.9.1, Theorem 2.9.5]) $M = \operatorname{ext} B$, so that $K_B = K$ and, if $x \in K$, then $x = y + iz$ ($y, z \in H(A)$), which gives, for all f in B, $f(y) = \operatorname{Re} f(x) \geqq 0$, $f(z) = \operatorname{Im} f(x) \geqq 0$, and so $K \subseteq K(A) + iK(A)$. Thus A is a C^*-algebra if and only if $K(A) + iK(A) = K$.

The set M also arises in a somewhat different context. In connection with the notion of numerical ranges, F. F. Bonsall [2] considers the subset of B given by

$$D = \{f \in B : L_f \text{ is an m.l.i. of } A\},$$

where

$$L_f = \{x \in A : f(J_x) = (0)\},$$

J_x being the principal left ideal generated by x. It is easy to show that $M = D$. We remark that the definition of M is more direct than that of D.

Since $0 \notin S$, C' and K' are cones; C' is contained in the set of all those functionals f which support the unit ball at e; the intersection of the associated hyperplanes $(\{x : f(x) = \|f\| = f(e)\})$ is $\{e\}$ (Bohnenblust and Karlin [1]), that is, e is a vertex of the unit ball.

The set $S' = \operatorname{co}\{S \cup (-iS)\}$ is a w^*-compact base for K'. Since w^*-convergent nets are norm bounded and S, S' are w^*-closed, it follows that such nets in C' or K' are contained in suitable nonnegative multiples of S or S' which are evidently w^*-closed and therefore contain the limits of these nets. Thus C' and K' are w^*-closed. Excepting M, the sets discussed so far are also norm-closed. Using the facts that e and ie are in K (§ 2), it can be shown that C' and K' are also normal. It is clear that $K + K \subseteq K$ and $\lambda K \subseteq K$ for all $\lambda \geqq 0$, that is, K is a wedge. Here, in letting K' induce K, we are following the definitions of H. H. Schaefer [7, p. 214].

2. Properties of the wedge K. Let $x \in A$. Then

(1) $x \in K$ if and only if $\operatorname{Re} f(x) \geqq 0$ and $\operatorname{Im} f(x) \geqq 0$ for all f in C' (or, equivalently, S or Ω or M);

(2) $x \in (-K) \cap K$ if and only if $f(x) = 0$ for all f in K' (or, equivalently, C' or S or Ω or M);

(3) since $f(e) = \|f\|$ for all f in C', (1) shows that e, ie and (hence) $u = e + ie$ are in K;

(4) $\operatorname{Re} f(u) = \operatorname{Im} f(u) = 1$ for all f in M;

(5) K is weakly closed and (hence) norm-closed.

Let $f = f_1 - if_2 \in K'$, so that f_1 and f_2 are in C'. From (4) we have, for all x in A,

$$\begin{aligned}\operatorname{Re} f(u+x) &= f_1(e) + f_2(e) + \operatorname{Re} f_1(x) + \operatorname{Im} f_2(x)\\ &\geqq (\|f_1\| + \|f_2\|)(1 - \|x\|).\end{aligned}$$

This shows that K contains the u-translate of the unit ball. Therefore u is in $\operatorname{int} K$ (interior of K); hence it is an order unit, and K generates A.

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It is easy to see that, if $\alpha u - x \in K$ for all $\alpha > 0$, then $-x \in K$, so that K induces an *Archimedean ordering* denoted by $\leqq$.

If $x \in \operatorname{int} K$, then $x - \lambda u \in K$ for some $\lambda > 0$; hence, from (1) and (4), we have, for all f in M, $f(x) \neq 0$ (indeed, $0 < \lambda \leqq \min\{\operatorname{Re} f(x), \operatorname{Im} f(x)\}$). Since, for each m.l.i. J of A, $f(J) = (0)$ for some f in M, one has that x is not in any m.l.i. Therefore x is left regular. Thus $\operatorname{int} K$ *is contained in the principal component of the set of all left regular elements of* A.

For each $x \in A$, the set $\hat{x}(M) = \{f(x) : f \in M\}$ contains the set $\sigma_l(x) = \{\lambda \in \mathbb{C} : x - \lambda e$ is not left regular$\}$, which may be called the *left spectrum* of x, and which, in the commutative case, is the spectrum of x.

Since $ie \in K$ and $-e = (ie)^2 \notin K$, K is not closed under multiplication; in contrast, Kung-Fu Ng [5] obtains representation theorems for Banach algebras with given cones closed under multiplication.

If x is in every m.l.i., then $f(x) = 0$ for all f in M, so that, by (2) above, $x \in (-K) \cap K$. This shows that, if K is a cone, that is, if $(0) = (-K) \cap K$, then A is semisimple.

3. Two seminorms on A and their properties. Since u is an order unit for A and (therefore) for A_0, the underlying real space of A, the following is an *order unit seminorm* on A_0:

$$\|x\|_u = \inf\{\alpha > 0 : -\alpha u \leqq x \leqq \alpha u\}.$$

Since the ordering is Archimedean, this seminorm is a norm on A_0 if and only if K is a cone. For the same reason, the $\|\ \|_u$-unit ball is the order-interval $[-u, u]$. Since K contains the u-translate of the unit ball, $\|x\|_u \leqq \|x\|$, so that $[-u, u]$ contains the given unit ball.

Since Ω is w^*-compact, the usual embedding of A into $C(\Omega)$ and the restriction of the sup norm to A, yield the following *function seminorm* on A:

$$\|x\|_\Omega = \sup\{|f(x)| : f \in \Omega\}.$$

Here Ω may be replaced by M or S. This is a norm if and only if $K' - K'$ is w^*-dense in A', which is so (by property (2) of the wedge K) if and only if K is a cone. We also have $\|x\|_\Omega \leqq \|x\|$, for all x in A. However, unlike the order unit seminorm, the function seminorm is absolutely homogeneous for complex scalars.

PROPOSITION 1. *On A_0, $\|\cdot\|_u$ and $\|\cdot\|_\Omega$ are equivalent seminorms.*

Proof. For all f in Ω, $\operatorname{Re} f(u) = f(e) = 1$; hence, if $-\alpha u \leqq x \leqq \alpha u$ for some $\alpha > 0$, then $\alpha u \pm x \in K$, or, equivalently, $|\operatorname{Re} f(x)| \leqq \alpha$ and $|\operatorname{Im} f(x)| \leqq \alpha$. Thus $|f(x)| \leqq \sqrt{2}\alpha$ and

$$\|x\|_\Omega \leqq \sqrt{2}\|x\|_u.$$

On the other hand, for all f in Ω, $|\operatorname{Re} f(x)| \leqq |f(x)| \leqq \|x\|_\Omega$; equivalently, $\operatorname{Re} f(-\|x\|_\Omega u) \leqq \operatorname{Re} f(x) \leqq \operatorname{Re} f(\|x\|_\Omega u)$. These inequalities hold with Im in place of Re; hence $-\|x\|_\Omega u \leqq x \leqq \|x\|_\Omega u$, that is,

$$\|x\|_u \leqq \|x\|_\Omega.$$

The assertion follows.

The next proposition is, essentially, well known. We are indebted to the referee for the proof given here.

PROPOSITION 2. *Let the wedge K be a cone. Then the following statements are equivalent*:

(a) *K is normal with respect to the given norm.*

(b) $\|\cdot\|$ *and* $\|\cdot\|_u$ *are equivalent norms on* A_0.

(c) *The* $\|\cdot\|$ *and* $\|\cdot\|_u$ *duals of* A_0 *are identical.*

Proof. Since K is a cone and the ordering is Archimedean, $\|\cdot\|_u$ is a norm on A_0. Hence, if K is normal, $[-u, u]$ is norm bounded, i.e., $\|x\| \leqq m\|x\|_u$ for all x in A_0 and for some constant $m > 0$. Since we already have $\|x\|_u \leqq \|x\|$, (a)$\Rightarrow$(b). Conversely, since $\|\cdot\|_u$ is monotonic on K, if $\|\cdot\|$ is equivalent to $\|\cdot\|_u$, then K is normal.

That (b)$\Rightarrow$(c) is evident. Conversely, if the dual spaces are identical, then the dual norms are equivalent (by the closed graph theorem and the fact that there is an inequality one way) and hence, by the Hahn–Banach theorem, the norms on A_0 are equivalent.

4. The Representation Theorem.

THEOREM. *Let A be a complex Banach algebra with unit e of norm one and let the sets defined in § 1 be constructed. Then there is a mapping ϕ from A into $C(\Omega)$ such that*

(a) *ϕ is a continuous linear homomorphism from A to a subspace of $C(\Omega)$*;

(b) *with respect to the cone P of functions in $C(\Omega)$ with nonnegative real and imaginary parts, the homomorphism ϕ is also an order homomorphism*;

(c) *the homomorphism ϕ is an isomorphism if and only if K is a cone*;

(d) *if K is a cone, then ϕ is a homeomorphism if and only if K is normal.*

Proof. Let ϕ be the evaluation mapping of A into $C(\Omega)$; i.e., let $\phi(x) = \hat{x}$, where $\hat{x}(f) = f(x)$. Then use the fact that $\|\cdot\|_\Omega \leqq \|\cdot\|$ to establish that ϕ is continuous and hence that (a) holds.

(b) follows by observing that $x \in K$ if and only if $0 \leqq \operatorname{Re} \hat{x}(f)$ and $0 \leqq \operatorname{Im} \hat{x}(f)$ for all f in Ω, which is so if and only if $\hat{x} \in P$ (as an element of $C(\Omega)$).

In § 2, property (2) of K shows that K is a cone if and only if $x = 0$ whenever $\hat{x}(f) = 0$ for all $f \in \Omega$, which is so if and only if $x = 0$ whenever $\hat{x}|_\Omega = 0$; (c) follows.

Let K be a cone. From (c), the evaluation mapping is a continuous isomorphism. It is a homeomorphism if and only if the norms $\|\cdot\|$ and $\|\cdot\|_\Omega$ are equivalent, which is so if and only if $\|\cdot\|$ and $\|\cdot\|_u$ are equivalent norms on A_0 (since $\|\cdot\|_u$ and $\|\cdot\|_\Omega$ are equivalent on A_0, by Proposition 1). By Proposition 2, this is so if and only if K is normal, and (d) follows.

Remark. If A is commutative, then M is precisely the set of nonzero multiplicative linear functionals, which is w^*-closed, so that $M = \Omega$. The evaluation mapping reduces to the Gelfand transform and we thus have a generalization of Gelfand's representation theorem.

5. Examples. These are drawn from the algebra $E = M_n(\mathbb{C})$ of all $n \times n$ ($n \geqq 2$) complex matrices. It can be considered as the space of linear operators on $\mathbb{C}^n$, the space of n-tuples of complex numbers.

In what follows, an element of E will be denoted either by its entries or by its column vectors enclosed within square brackets. Thus $x = (a_{ij}) = [b_1, b_2, \ldots, b_n]$ denotes an arbitrary element x of E with column vectors $b_1, b_2, \ldots, b_n$ and $f = (p_{ij})$ denotes an arbitrary element of E', the algebraic dual of E; unless the contrary is stated, the indices range from 1 to n.

One can verify that the operator norm on E induced by the maximum modulus norm on $\mathbb{C}^n$, its dual norm, and $f(x)$ are given by

$$\|x\| = \max_i \{\sum_j |a_{ij}|\},$$

$$\|f\| = \sum_i \max_j \{|p_{ij}|\},$$

$$f(x) = \sum_i \sum_j a_{ij} p_{ij}.$$

With respect to this norm, E is a noncommutative Banach algebra with unit I (the identity matrix) of norm one.

The maximal left ideals (m.l.i.) of E are its principal left ideals generated by elements of rank $(n-1)$ (see, for example, [4], pp. 230–231). These generating elements can be chosen in " row-reduced echelon form " as follows:

$$\begin{aligned} x_0 &= [0, e_1, \ldots, e_{n-1}], \\ x_s &= [e_1, \ldots, e_s, \sum_t \alpha_t e_t, e_{s+1}, \ldots, e_{n-1}], \end{aligned} \tag{1}$$

where the unit basis vectors e_i are regarded as column vectors, $1 \leqq t \leqq s \leqq n-1$, and $\alpha_t \in \mathbb{C}$.

In Propositions 3, 4, and 5 below, we show that certain properties of the set M which hold in commutative complex algebras are not true in general.

PROPOSITION 3. *In general, the set M and the m.l.i.'s are not in one-to-one correspondence.*

Proof. The set B (see § 1) for the Banach algebra E is given by

$$B = \{f = (p_{ij}) \in E' : |p_{ij}| \leqq p_{ii} \leqq 1 = \sum_r p_{rr}\}.$$

The extreme points of B are given by

$$g_i = [\alpha_1 e_i, \alpha_2 e_i, \ldots, e_i, \ldots, \alpha_n e_i],$$

where $\alpha_i = 1 = |\alpha_t|$ $(t \neq i)$.

The assertion follows by proving that each g_i determines and belongs properly to a convex subset of B, each element of which annihilates the same m.l.i.

It is enough to consider g_1, for which $\alpha_1 = 1$ and $|\alpha_t| = 1$ $(2 \leqq t \leqq n)$. Define λ_i by $\lambda_1 = (1/\alpha_n)$, $\lambda_i = \lambda_1 \alpha_1$ $(2 \leqq i \leqq n-1)$, and $\lambda_n = 1$. Thus $|\lambda_i| = 1$ for all i. The following are n elements of ext B (the set of extreme points of B):

$$f_r = \lambda_r^{-1}[\lambda_1 e_r, \lambda_2 e_r, \ldots, \lambda_n e_r] \qquad (r = 1, 2, \ldots, n). \tag{2}$$

The element

$$x = [e_1, e_2, \ldots, e_{n-1}, -\sum_{j=1}^{n-1} \lambda_j e_j] \tag{3}$$

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has rank $(n-1)$, so that the principal left ideal J_x generated by x is maximal. An element c of J_x is given by

$$c = [c_1, c_2, \ldots, c_{n-1}, -\sum_{j=1}^{n-1} \lambda_j c_j],$$

where c_j are arbitrary vectors of $\mathbb{C}^n$. Regarding e_r in (2) as linear functionals on $\mathbb{C}^n$, one has

$$f_r(c) = \lambda_r^{-1}\{\sum_{j=1}^{n-1} \lambda_j(e_r, c_j) - (e_r, \sum_{j=1}^{n-1} \lambda_j c_j)\} = 0.$$

Therefore each f_r and all linear combinations of $f_1, \ldots, f_n$ annihilate J_x. Conversely, each m.l.i. generated by an element such as x in (3) above is annihilated by elements such as f_r in (2) above, and their linear combinations.

The assertion follows.

Remarks. (a) In the above example, some m.l.i.'s have unique annihilators in B. The annihilator of J_{x_0} (see (1) above) is $f = (p_{ij})$, whose only nonzero entry is $p_{11} = 1$; that of J_{x_s}, for $|\alpha_t| < 1$ for all t, is $f = (p_{ij})$ such that $p_{(s+1)t} = \alpha_t$ $(1 \leqq t \leqq s)$, $p_{(s+1)(s+1)} = 1$, and $p_{ij} = 0$ otherwise.

(b) Since $\operatorname{ext} B \subseteq M$, we have $S = B$, C' = the cone generated by B, and $K' = C' - iC'$. The wedge K is described by

$$K = \{x = (a_{ij}) \in E : \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| \leqq \min\{\operatorname{Re} a_{ii}, \operatorname{Im} a_{ii}\}\}.$$

It is clear that K is a cone. Hence $\|\cdot\|_u$ and $\|\cdot\|_\Omega$ are norms. Indeed, the latter is the given norm and

$$\|x\|_u = \max\{\sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| + \max\{|\operatorname{Re} a_{ii}|, |\operatorname{Im} a_{ii}|\} : i = 1, 2, \ldots, n\}.$$

Hence the representation of E on Ω is an isometric order isomorphism of E into $C(\Omega)$ and K is a normal, closed, generating cone.

PROPOSITION 4. *In general, the set M depends on the norm.*

Proof. For $M_2(\mathbb{C})$ with the above norm, it is readily verified that $f_0 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ annihilates the principal left ideal generated by $\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ and is in M. If, now, we consider $M_2(\mathbb{C})$ with the l_1-operator norm, f_0 is no longer of norm 1 in the dual space and so is not in the new set M. Conversely, $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ is in the new set but not in the former one.

Remark. This example corrects the oversight in [6] by which it was assumed that M is independent of all equivalent Banach algebra norms p such that $p(e) = 1$. This oversight made it appear that K is a cone if and only if A has no nonzero topologically nilpotent elements. Clearly, $E = M_n(\mathbb{C})$ has such elements and we have seen that the wedge K (above) is a cone.

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PROPOSITION 5. *In general, the set M is not w^*-closed.*

Proof. We show that M can even be dense in B. Let E be the noncommutative algebra of all 2×2 upper triangular complex matrices. This is a Banach algebra for the l_∞-operator norm. The m.l.i.'s are of the following form:

J_1: with elements of the form $\begin{bmatrix} 0 & b \\ 0 & c \end{bmatrix}$,

J_{2k}: with elements of the form $\begin{bmatrix} a & -ka \\ 0 & 0 \end{bmatrix}$ $(k\in\mathbb{C})$.

The annihilators of these in B are respectively given by

$$f_0=\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\qquad g_0=\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\quad (|k|<1)$$

and

$$\operatorname{co}\left\{\begin{bmatrix} k & 1/k \\ 0 & 0 \end{bmatrix}, g_0\right\}\quad (|k|\geqq 1).$$

Therefore no point of int co $\{f_0, g_0\}$ annihilates either J_1 or J_{2k}. The assertion follows, as M is the complement of this interior in B.

We observe that throughout the preceding discussion maximal left ideals can be replaced by maximal right ideals (m.r.i.). Corresponding to these two cases, we now let the sets M be denoted by M_l and M_r, respectively.

PROPOSITION 6. (a) *In general, $M_l\neq M_r$;* (b) *however, there exist unital noncommutative Banach algebras for which $M_l=M_r=M$.*

Remark. If $M_l=M_r$, then, whether or not the algebra is commutative, the spectrum of each element x of the algebra is contained in the set $\{f(x): f\in M\}$ and int K is contained in the principal component of the set of all regular elements.

Proof. (a) This follows from the previous example, whose m.r.i.'s are I_1 and I_{2k} with elements respectively of the forms $\begin{bmatrix} 0 & 0 \\ 0 & a \end{bmatrix}$ and $\begin{bmatrix} 0 & a \\ 0 & -ka \end{bmatrix}$ $(a, k\in\mathbb{C})$.

Let $f=\begin{bmatrix} \alpha & \beta \\ 0 & \gamma \end{bmatrix}\in B$. If $\gamma=0$, then f annihilates I_1; if $\gamma\neq 0$, then it annihilates I_{2k} with $k=\beta\gamma^{-1}$. Hence $M_r=B$, which properly contains M_l.

(b) This is true for $M_2(\mathbb{C})$ with the l_∞-operator norm. For its m.r.i.'s are I_1 and I_{2k} with elements respectively of the form

$$\begin{bmatrix} 0 & 0 \\ 0 & a \end{bmatrix}\quad\text{and}\quad\begin{bmatrix} a & b \\ -ka & -kb \end{bmatrix}\quad (a, b, k\in\mathbb{C}).$$

The annihilators in B of these ideals are respectively of the forms

$$(I_1): \begin{bmatrix} 1 & q \\ 0 & 0 \end{bmatrix} \quad (|q| \leqq 1);$$

$$(I_{20}): \begin{bmatrix} 0 & 0 \\ r & 1 \end{bmatrix} \quad (|r| \leqq 1);$$

$$\underset{(k \neq 0)}{(I_{2k})}: \begin{bmatrix} p & q \\ r & s \end{bmatrix}$$

where $$p = |k|/k',\ r = |k|/kk',\ q = k/k',\ s = 1/k',\ k' = 1+|k|.$$

I_{2k} $(k \neq 0)$ have, unlike I_1 and I_{20}, unique annihilators.
Therefore

$$M_r = \bigcup M_i \,(1 \leqq i \leqq 3),$$

where

$$M_1 = \mathrm{co}\left\{\begin{bmatrix} 1 & q \\ 0 & 0 \end{bmatrix} : |q| = 1\right\},$$

$$M_2 = \mathrm{co}\left\{\begin{bmatrix} 0 & 0 \\ r & 1 \end{bmatrix} : |r| = 1\right\},$$

$$M_3 = \bigcup_{|k|=1} M_{3k},$$

with

$$M_{3k} = \mathrm{int\,co}\left\{\begin{bmatrix} 1 & k \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ \bar{k} & 1 \end{bmatrix}\right\} \quad (|k| = 1)$$

($\bar{k}$ is the conjugate of k).

M_l can be computed from the general case discussed above and shown to be equal to M_r.

REFERENCES

1. H. F. Bohnenblust and S. Karlin, Geometrical properties of the unit sphere of Banach algebras, *Ann. of Math.* **62** (1955), 217–229.

2. F. F. Bonsall, Some axioms for numerical range, Conference on numerical ranges, Aberdeen, 5th–10th July, 1971.

3. J. Dixmier, *Les C*-algebras et leurs representations* (Paris, 1964).

4. N. Jacobson, *Lectures in abstract algebra*, Vol. II (Princeton, 1953).

5. Kung-Fu Ng, A representation theorem for partially ordered Banach algebras, *Proc. Cambridge Philos. Soc.* **64** (1968), 53–59.

6. T. W. Palmer, Characterizations of C^*-algebras, *Bull. Amer. Math. Soc.* **74** (1968), 538–540.

7. H. H. Schaefer, *Topological Vector Spaces* (New York, 1966).

8. A. C. Thompson, On the geometry of Banach algebras, *Notices Amer. Math. Soc.* **16** (1969), 274.

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