The "Fourier" Theory of the Cardinal Function.

By J. M. Whittaker.

(*Read* 2*nd December* 1927. *Received* 30*th March* 1928.)

§ 1. *Introduction.*

The generalised Riesz-Fischer theorem[1] states that if

$$\sum_{n=1}^{\infty} (|a_n|^p + |b_n|^p) \qquad \ldots\ldots\ldots\ldots\ldots(1)$$

is convergent, with $1 < p \leqslant 2$, then

$$\tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \ldots\ldots\ldots\ldots\ldots(2)$$

is the Fourier series of a function of class $L^{\frac{p}{p-1}}$. When $p > 2$ the series (2) is not necessarily a Fourier series; neither is it necessarily a Fourier D-series.[2] It will be shown below that it must however be what may be called a "Fourier Stieltjes" series. That is to say, the condition (1) with $(p > 1)$ implies that there is a continuous function $F(x)$ such that

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} dF(x),\ a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos nx\, dF(x),\ b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin nx\, dF(x)\ \ldots(3)$$

The necessary and sufficient condition that (2) should be a "Fourier-Stieltjes" series is given in Theorem 1 below. In itself the result is almost trivial but it has an interesting application to the theory of the cardinal interpolation function,[3] in particular to Ferrar's "consistency" property.

---

[1] Hobson. *Functions of a Real Variable* (1926), vol. II., p. 599.

[2] *ibid.*, p. 488.

[3] *cf.* E. T. Whittaker, *Proc. Roy. Soc. Edin.*, **35** (1915), 181. W. L. Ferrar, *ibid.*, **45** (1925), 269; **46** (1926), 323, and **47** (1927), 230. J. M. Whittaker, *Proc. Edin. Math. Soc.* (2), **1** (1927), 41. E. T. Copson, *ibid.*, p. 129.

In §§ 5, 6 the connection between the cardinal function and Pollard's "Fourier integrals of finite type" and de la Vallée Poussin's "formule fondamentale d'interpolation" is discussed.

Throughout the paper it is assumed that the numbers concerned are all real.

§ 2. *A theorem on "Fourier-Stieltjes" series.*

THEOREM. 1. *The necessary and sufficient condition that* (2) *should be the "Fourier-Stieltjes" series of a continuous function* $F(x)$ *is that*

$$\sum_{n=1}^{\infty} \frac{1}{n} (a_n \sin nx - b_n \cos nx) \quad \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

should be the Fourier series of a continuous function $G(x)$, *the two functions being connected by the equation*

$$F(x) = G(x) + \tfrac{1}{2} a_0 x \quad \ldots\ldots\ldots\ldots\ldots\ldots (5)$$

To define the integrals (3) we adopt the original definition of Stieltjes.[1] To prove the necessity of the condition, integrate by parts in (3)

$$a_n = \left[\frac{\cos nx\, F(x)}{\pi}\right]_{-\pi}^{\pi} + \frac{n}{\pi}\int_{-\pi}^{\pi} F(x) \sin nx\, dx$$

$$= (-)^n a_0 + \frac{n}{\pi}\int_{-\pi}^{\pi} F(x) \sin nx\, dx = \frac{n}{\pi}\int_{-\pi}^{\pi} G(x) \sin nx\, dx \quad \ldots\ldots (6)$$

since

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} dF(x) = \frac{F(\pi) - F(-\pi)}{\pi}.$$

Similarly

$$b_n = -\frac{n}{\pi}\int_{-\pi}^{\pi} G(x) \cos nx\, dx \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$$

Thus $-\dfrac{b_n}{n}, \dfrac{a_n}{n}$ are the Fourier coefficients of the continuous function $G(x)$. Again, the condition is sufficient. For if it is satisfied (6), (7) are true and on integrating by parts, reversing the previous procedure, we obtain (3).

[1] *cf.* Hobson. *loc. cit.*, vol. I., p. 506.

§ 3. *A trigonometric integral for the cardinal function.*

(4) will certainly be the Fourier series of a continuous function if

$$\sum_{n=1}^{\infty} \frac{|a_n| + |b_n|}{n} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (8)$$

converges, in particular, if (1) is satisfied with $p > 1$, since

$$\Sigma \left| \frac{a_n}{n} \right| \leqslant \left\{ \Sigma \, |a_n|^p \right\}^{\frac{1}{p}} \left\{ \Sigma \, \frac{1}{n^q} \right\}^{\frac{1}{q}}, \; q = \frac{p}{p-1} > 1.$$

Changing to the notation of the cardinal function there results part of the following theorem.

THEOREM 2. *If*

$$\sum_{n=1}^{\infty} \frac{|a_n| + |a_{-n}|}{n} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (9)$$

converges, the cardinal series

$$C(x) = \frac{\sin \pi x}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-)^n a_n}{x - n} \qquad \ldots\ldots\ldots\ldots\ldots (10)$$

is absolutely convergent, and its sum is of the form

$$\int_0^1 \left\{ \cos \pi xt \, d\Phi(t) + \sin \pi xt \, d\Psi(t) \right\}, \; \Phi, \Psi \textit{ continuous functions} \quad \ldots. (11)$$

Given any function $f(x)$ of the form (11), *the series*

$$\frac{\sin \pi x}{\pi} \left[\frac{f(0)}{x} + \sum_{n=1}^{\infty} (-)^n \left\{ \frac{f(n)}{x-n} + \frac{f(-n)}{x+n} \right\} \right] \quad \ldots\ldots (12)$$

is summable $(C, 1)$ *to* $f(x)$.

In fact, in this notation Theorem 1 states that (9) implies the existence of a function $f(x)$ of the form (11) such that

$$f(n) = a_n, \; n \textit{ an integer or zero},$$

and it is clear that (9) implies the absolute convergence of the series (10). To prove the remainder of the theorem, consider the Fourier series for $\cos \pi xt$, $\sin \pi xt$, namely

$$\frac{\sin \pi x}{\pi} \left[\frac{1}{x} + \sum_{n=1}^{\infty} (-)^n \cos \pi nt \left(\frac{1}{x-n} + \frac{1}{x+n} \right) \right],$$

$$\frac{\sin \pi x}{\pi} \sum_{n=1}^{\infty} (-)^n \sin \pi nt \left(\frac{1}{x-n} - \frac{1}{x+n} \right) \qquad \ldots\ldots\ldots (13)$$

https://doi.org/10.1017/S0013091500013511 Published online by Cambridge University Press

It has been shown by Hahn[1] that if

(i) *$G(t)$ is continuous in* (a, b)

(ii) $V_a{}^b f_n + |f_n(b)| < k$ *(all n)*

(iii) $f_n(t) \to f(t)$, *as* $n \to \infty$

then

$$\int_a^b f_n(t)\, dG(t) \to \int_a^b f(t)\, dG(t).$$

$V_a{}^b f_n$ denotes the total variation of $f_n(t)$ in (a, b). Since the functions $f(t) = \cos \pi xt$, $\sin \pi xt$ are bounded and of bounded variation in $a \leqslant t \leqslant b$, the condition (ii) is satisfied[2] when $f_n(t)$ is the nth partial Cesàro sum of either of the series (13). Again, since the functions are continuous, $f_n(t)$ converges uniformly to $\cos \pi xt$, $\sin \pi xt$ and the condition (iii) is satisfied. The second part of Theorem 2 is now an immediate consequence of Hahn's theorem.

§ 4. *The "consistency" of the cardinal function.*

In the last of the papers cited above Ferrar has given the following theorem.

If $\sum\limits_{n=-\infty}^{\infty} |a_n|^p$ *is convergent,* $(p > 1)$ ....(14)

and $C(x)$ *is defined by* (10), *then*

$$C(x) = \frac{\sin \pi (x - b)}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-)^n C(b + n)}{x - b - n} \qquad \text{........(15)}$$

the condition (14) *implying the convergence of* (10), (15).

With the restriction that the series (15) must be bracketed like the series (12) this can be proved very easily by means of Theorem 2. For (14) implies (9),[3] so that the cardinal series (10) converges to an integral of the form (11). Thus

$$C(x) = \int_0^1 \left\{ \cos \pi xt\, d\Phi(t) + \sin \pi xt\, d\Psi(t) \right\} \qquad \text{..................(16)}$$

$$\begin{aligned} C(b + x) &= \int_0^1 \Big[\cos \pi xt \left\{ \cos \pi bt\, d\Phi(t) + \sin \pi bt\, d\Psi(t) \right\} \\ &\qquad + \sin \pi xt \left\{ - \sin \pi bt\, d\Phi(t) + \cos \pi bt\, d\Psi(t) \right\} \Big] \\ &= \int_0^1 \left\{ \cos \pi xt\, d\Phi_1(t) + \sin \pi xt\, d\Psi_1(t) \right\} \end{aligned}$$

---

[1] *Monatshefte für Math. u. Physik*, **32** (1922), 84.

[2] Hobson. *loc. cit.*, vol. II., pp. 560, 580.

[3] *cf.* sec. 3.

where[1]

$$\Phi_1(x) = \int_0^x \{\cos \pi bt \, d\Phi(t) + \sin \pi bt \, d\Psi(t)\},$$

$$\Psi_1(x) = \int_0^x \{-\sin \pi bt \, d\Phi(t) + \cos \pi bt \, d\Psi(t)\}.$$

$C(b+x)$ is therefore a function of the form (11). Thus the series

$$\frac{\sin \pi x}{\pi}\left[\frac{C(b)}{x} + \sum_{n=1}^{\infty}(-)^n\left\{\frac{C(b+n)}{x-n} + \frac{C(b-n)}{x+n}\right\}\right]$$

is summable $(C, 1)$ to sum $C(b+x)$; or, changing x to $x-b$ the series

$$\frac{\sin \pi (x-b)}{\pi}\left[\frac{C(b)}{x-b} + \sum_{n=1}^{\infty}(-)^n\left\{\frac{C(b+n)}{x-b-n} + \frac{C(b-n)}{x-b-n}\right\}\right] \quad \ldots (17)$$

is summable $(C, 1)$ to sum $C(x)$. To prove that it is convergent it is enough to show that $C(b+n)$ is bounded for all integral values of n. For the nth term of (17) will then be $O\left(\frac{1}{n}\right)$ and Hardy's convergence theorem can be applied. This is so, for

$$|C(b+n)| \leqslant \left|\frac{\sin \pi b}{\pi}\right| \sum_{r=-\infty}^{\infty}\left|\frac{a_r}{b+n-r}\right|$$

$$\leqslant \left\{\sum_{r=-\infty}^{\infty}|a_r|^p\right\}^{\frac{1}{p}}\left\{\sum_{r=-\infty}^{\infty}\left|\frac{1}{b+n-r}\right|^q\right\}^{\frac{1}{q}}, \; q = \frac{p}{p-1} > 1$$

by Hölder's inequality. The first factor is finite by (14); and the second is less than a constant independent of n, since

$$\sum_{r=-\infty}^{\infty}\left|\frac{1}{b+n-r}\right|^q \leqslant 2\sum_{r=0}^{\infty}\left|\frac{1}{b+r}\right|^q.$$

§ 5. *The cardinal function and Fourier integrals of finite type.*

Let $a_0, a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ be two sequences of real numbers. Define $a_{-n} = a_n$, $b_{-n} = -b_n$ and then interpolate by means of the cardinal function. Thus

$$a(x) = \frac{\sin \pi x}{\pi}\left[\frac{a_0}{x} + \sum_{n=1}^{\infty}(-)^n a_n\left(\frac{1}{x-n} + \frac{1}{x+n}\right)\right] \quad \ldots (18)$$

$$b(x) = \frac{\sin \pi x}{\pi}\sum_{n=1}^{\infty}(-)^n b_n\left(\frac{1}{x-n} - \frac{1}{x+n}\right) \quad \ldots\ldots\ldots\ldots (19)$$

[1] By a theorem due to Hyslop, *Proc. Edin. Math. Soc.*, **44** (1926), 79.

If the a's and b's are the Fourier coefficients of a function $f(x)$ these series can be expressed as trigonometric integrals. For on multiplying the Fourier series for $\cos xt$, $\sin xt$ by $f(t)$ and integrating term by term with respect to t,[1] it is found that

$$a(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) \cos xt\, dt,\ b(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) \sin xt\, dt \quad \ldots\ldots (20)$$

These are the "Fourier integrals of finite type" discussed by Pollard.[2] The latter has shown that the integral

$$\int_0^{\infty} \{a(t) \cos xt + b(t) \sin xt\}\, dt$$

behaves in very much the same way as the series (2). The cardinal series expansions of $a(x)$, $b(x)$, if convergent or summable in some sense, afford a means of defining the trigonometric integral associated with any trigonometric series, whether the latter is a Fourier series or not. They will certainly be convergent if the series is a Fourier series and will be summable $(C, 1)$ if it is a "Fourier-Stieltjes" series.

§ 6. *The interpolation formula of de la Vallée Poussin.*

The interpolation function

$$F(x) = \frac{\sin mx}{m} \sum_a^b (-)^k \frac{f(a_k)}{x-a_k},\ a_k = \frac{k\pi}{m} \quad \ldots\ldots\ldots\ldots (21)$$

has been studied by de la Vallée Poussin[3] as a means of approximating to a given function $f(x)$. m is an integer and the interval (a, b) in which $f(x)$ is defined is assumed to be finite.

$F(x)$ is evidently a particular case of the cardinal function, and de la Vallée Poussin's principal result is included in the following theorem.

Theorem 3. *Let $f(x)$ be a real function defined in the interval $(-\infty, \infty)$ and let $C_m(x)$ be the cardinal function formed by interpolating from the values of $f(x)$ at the points a_k. Thus*

$$C_m(x) = \frac{\sin mx}{m} \sum_{k=-\infty}^{\infty} (-)^k \frac{f(a_k)}{x - a_k} \quad \ldots\ldots\ldots\ldots (22)$$

---

[1] This process is legitimate. *cf.* Hobson, *loc. cit.*, p. 582.

[2] *Proc. Lond. Math. Soc.*, **2, 26** (1926), 12. *Proc. Camb. Phil. Soc.*, **23** (1926), 373 . See also Miss M. E. Grimshaw, "A case of distinction between Fourier integrals and Fourier series," *loc. cit.*, p. 755.

[3] *Bull. de l'Acad. Roy. de Belg. (Classe de Sciences)* (1908), pp. 319-410 esp. p. 341.

Then if

(i) $f(x)$ *is bounded and integrable* (R) *in any finite interval*

(ii) $\dfrac{f(x)}{x}$ *is of bounded variation in* (N, ∞), $(-\infty, -N)$ *for some* N.

(iii) $f(x)$ *is continuous at* ξ *and of bounded variation in a neighbourhood of* ξ.

$$C_m(\xi) \to f(\xi), \text{ as } m \to \infty.$$

The conditions (i), (iii) are required to prove de la Vallée Poussin's theorem.

The convergence of the cardinal series follows from (ii). For, find M so that

$$V_M^{\infty}\left\{\frac{f(x)}{x}\right\} < \epsilon.$$

Then if $P, Q > M > 0$ and x is (say) positive

$$\left|\frac{\sin mx}{m} \sum_P^Q (-)^k \frac{f(a_k)}{x - a_k}\right| \leqslant \frac{1}{m}\left|\sum_P^Q (-)^k \frac{f(a_k)}{a_k} \cdot \frac{a_k}{a_k - x}\right|$$
$$\leqslant \frac{1}{m} \text{Max.}\left[\left|\sum_{k_1}^{k_2} (-)^k \frac{f(a_k)}{a_k}\right|\right]$$

since $\left\{\dfrac{a_k}{a_k - x}\right\}$ is an increasing sequence. k_1, k_2 are any two integers such that a_{k_1}, a_{k_2} lie in the interval (P, Q).

$$\therefore \quad \left|\frac{\sin mx}{m} \sum_P^Q (-)^k \frac{f(a_k)}{x - a_k}\right|$$

$$\leqslant \frac{1}{m} \text{Max.}\left[\left|\frac{f(a_{k_1})}{a_{k_1}} - \frac{f(a_{k_1+1})}{a_{k_1+1}}\right| + \left|\frac{f(a_{k_1+1})}{a_{k_1+1}} - \frac{f(a_{k_1+2})}{a_{k_1+2}}\right| + \ldots + \left|\frac{f(a_{k_2-1})}{a_{k_2-1}} - \frac{f(a_{k_2})}{a_{k_2}}\right|\right]$$

$$\leqslant \frac{1}{m} V_P^Q\left\{\frac{f(x)}{x}\right\} \leqslant \frac{1}{m}\epsilon < \epsilon.$$

Thus (22) converges. Now, let $(-L, L)$, $(L > N)$, be an interval containing ξ. Then by de la Vallée Poussin's Theorem III,

$$\frac{\sin mx}{m} \sum_{-L}^{L} (-)^k \frac{f(a_k)}{x - a_k} \to f(\xi), \text{ as } m \to \infty.$$

It remains to prove that

$$\frac{\sin mx}{m} \sum_{L}^{\infty} (-)^k \frac{f(a_k)}{x - a_k}, \quad \frac{\sin mx}{m} \sum_{-\infty}^{-L} (-)^k \frac{f(a_k)}{x - a_k}$$

tend to zero as m tends to ∞. As above we find

$$\left| \frac{\sin mx}{m} \sum_{L}^{\infty} (-)^k \frac{f(a_k)}{x - a_k} \right| \leqslant \frac{1}{m} V_L{}^{\infty} \left\{ \frac{f(x)}{x} \right\} \to 0, \text{ as } m \to \infty$$

and similarly the second expression tends to zero.

Added 9*th May* 1928. Another consistency theorem is suggested by the recent work of E. C. Titchmarsh[1] on reciprocal series.

THEOREM 4. *If*

$$\sum_{n=1}^{\infty} (-)^n \frac{a_n}{n}, \quad \sum_{n=1}^{\infty} (-)^n \frac{a_{-n}}{n}$$

converge, the series (10) *defining* $C(x)$ *is convergent, and the series* (17) *is summable* $(C, 1)$ *to* $C(x)$.

This includes Titchmarsh's Theorem 1, and is proved in much the same way. The proof is long but fairly straightforward, and as it does not contain any points of particular interest it may be omitted. Titchmarsh's other theorems can be extended in the same way.

[1] *Proc. Lond. Math. Soc.* (2), **26** (1926), 1.