A CLASS OF TRANSLATION LATTICES

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1. Introduction. Let E be a completely regular space, let C^* denote the Banach algebra of continuous bounded real-valued functions on E, and let A^* denote a Banach subalgebra which contains the constant function 1. Let

$$L(A^*) = \{f|(f \wedge \lambda) \vee (-\lambda) \in A^* \text{ for all } \lambda \geqslant 0\}.$$

The purpose of this note is to discuss some properties of the subsets $L(A^*)$ of the ring C of continuous real-valued functions on E. They arose out of an unsuccessful attempt to distinguish C^* from its Banach subalgebras.

The set $L(C^*)$ equals C, which is an algebra closed under inversion and uniform convergence, and hence under composition **(2)**. The initial hope was that this was the only one of these sets possessing these properties. This turned out to be false.

The subsets of C of the form $L(A^*)$ are completely characterized and it is shown that $L(A^*)$ is closed under addition (or multiplication) if and only if it is closed under composition.

The sets $L(A^*)$ are closely related to certain rings of quotients $R(A^*)$. The homomorphism spaces of $L(A^*)$ and $R(A^*)$ are shown to be homeomorphic, even though in general $L(A^*) \neq R(A^*)$. It is shown that $L(A^*) = R(A^*)$ if and only if $L(A^*)$ is closed under composition.

The relation between $L(A^*)$ and a certain closed equivalence relation $r(\bar{A}^*)$ on βE is discussed and used to relate closure under composition to a property of $r(\bar{A}^*)$.

2. Translation lattices. Shirota **(3)** defined a translation lattice to be a distributive lattice L, together with an action: $L \times \mathbf{R} \to L$ (denoted by $+$) such that:

(TL_1) $x + 0 = x$ for all $x \in L$;

(TL_2) $(x + \lambda) + \mu = x + (\lambda + \mu)$ for all $x \in L$ and $\lambda, \mu \in \mathbf{R}$;

(TL_3) $\lambda > 0$ implies $x + \lambda > x$ for all $x \in L$; and

(TL_4) $x > y$ implies $x + \lambda > y + \lambda$ for all $\lambda \in \mathbf{R}$.

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Let E be an arbitrary set, and let S denote a subset of $\mathbf{R}^E$ which is closed under the usual lattice operations $\wedge$ and $\vee$, and which in addition is closed under the addition of constant functions. Then S is a translation lattice.

Assume further that S contains the constant functions. Let $\mathfrak{L}_0(S)$ denote the set of lattice homomorphisms $l: S \to \mathbf{R}$ for which $l(f + \lambda) = l(f) + \lambda$ and $l(0) = 0$ (where 0 and λ both denote constant functions and real numbers).

Given any set S of real-valued functions on E, a map (the evaluation map) $e_S: E \to \mathbf{R}^S$ is defined by setting $(e_S(x))_f = f(x)$. When S is a translation lattice, the usual argument with a product topology shows that $\overline{e_S E}$ is contained in $\mathfrak{L}_0(S)$.

PROPOSITION 1. *Let S be closed under multiplication by (-1). Then $l \in \mathfrak{L}_0(S)$ is a point of $\overline{e_S E}$ if and only if $l(-f) = -l(f)$ for all $f \in S$.*

Proof. The homomorphism $e_S(x)$ evaluates functions at x. It commutes with multiplication by (-1). The usual argument with a product topology then shows that if l is a point of $\overline{e_S E}$, $l(-f) = -l(f)$ for all $f \in S$.

Assume $l(-f) = -l(f)$ for all $f \in S$ and that l is not in $\overline{e_S E}$. Then there are n functions $f_1, \ldots, f_n$ in S and $\epsilon > 0$ such that, if $x \in E$, then for some value of i, $|f_i x - l(f_i)| \geqslant \epsilon$.

Let $g_i = f_i - l(f_i)$ and set

$$g_i^1 = -[g_i \wedge \epsilon] + \epsilon, \qquad g_i^2 = [g_i \vee (-\epsilon)] + \epsilon.$$

Then

$$l(g_i^1) = l(g_i^2) = \epsilon, \qquad i = 1, \ldots, n.$$

Let

$$g = \bigwedge_{i=1}^{n} [g_i^1 \wedge g_i^2].$$

If $x \in E$ and $|f_i x - l(f_i)| \geqslant \epsilon$, then

$$(g_i^1 \wedge g_i^2)x = 0.$$

Since all the g_i^j are positive, this implies that $g = 0$. Hence $l(g) = 0$. However, from the definition of g, $l(g) = \epsilon$.

Remark. If S is a Banach subalgebra of the Banach algebra of bounded real-valued functions on E, then the translation lattice homomorphisms that commute with multiplication by (-1) are just those algebra homomorphisms h for which $h(1) = 1$.

Let S be a translation lattice of functions on a set E which contains the constant functions and is closed under multiplication by (-1). Let $\mathfrak{L}(S)$ denote the subspace $\overline{e_S E}$ of $\mathbf{R}^S$. It will be called the space of translation lattice homomorphisms of S. The evaluation map $e_S: E \to \mathbf{R}^S$ can be considered as a function which maps E into $\mathfrak{L}(S)$.

Given any translation lattice of functions S, the set of bounded functions in S is also a translation lattice. Let it be denoted by S^*. It contains the constant functions and is closed under multiplication by (-1) whenever S has these properties.

Let S_1 and S_2 be two translation lattices of functions which contain the constant functions and are closed under multiplication by (-1). Assume that $S_1 \subseteq S_2$.

Let t be the restriction to $\mathfrak{L}(S_2)$ of the projection map of $\mathbf{R}^{S_2}$ onto $\mathbf{R}^{S_1}$. Then $t\mathfrak{L}(S_2) \subseteq \mathfrak{L}(S_1)$ and $t \circ e_{S_2} = e_{S_1}$.

PROPOSITION 2. *Let $S_1 \subseteq S_2$ be as above. If $S_1^* = S_2^*$, then t is an embedding.*

Proof. Let l_1 and l_2 be elements of $\mathfrak{L}(S_2)$ for which $t(l_1) = t(l_2)$. Then, if $f \in S_1^*$, $l_1(f) = l_2(f)$.

Let $g \in S_2$. Then $(g \wedge n) \vee (-n)$ is in $S_2^* = S_1^*$. Hence

$$(l_1(g) \wedge n) \vee (-n) = l_1((g \wedge n) \vee (-n)) = l_2((g \wedge n) \vee (-n)) = (l_2(g) \wedge n) \vee (-n).$$

Since this is true for all natural numbers n, $l_1(g) = l_2(g)$. Hence $l_1 = l_2$.

The topology for $\mathfrak{L}(S)$ is the weak topology defined by $\{\dot{f} | f \in S\}$, where $\dot{f}(l) = l(f)$. This is the same as the weak topology defined by $\{\dot{f} | f \in S^*\}$. Hence, the fact that t is an embedding now follows from the observation that if $f \in S_1$, then $\dot{f} \circ t$ is the function on $\mathfrak{L}(S_2)$ corresponding to f (considered as an element of S_2).

3. The translation lattices $L(A^*)$. Let E be a completely regular space, and let C denote the algebra of continuous real-valued functions on E. Then C is also a translation lattice, as is C^*, the Banach algebra of bounded continuous real-valued functions on E.

Denote by A^* a Banach subalgebra of C^* which contains the constant function 1. Let $L(A^*) = \{f | (f \wedge \lambda) \vee (-\lambda) \in A^*, \text{ for all } \lambda \geqslant 0\}$.

THEOREM 1. *For any A^*, $L(A^*)$ has the following properties:*

(L_1) *It is a translation lattice containing the constants and is closed under multiplication by real numbers.*

(L_2) *It is uniformly closed.*

(L_3) *It is closed under positive inversion (i.e. if $f \in L(A^*)$ and $fx > 0$ for all $x \in E$, then $1/f \in L(A^*)$).*

(L_4) *$f \in L(A^*)$ if and only if $f^+, f^- \in L(A^*)$.*

Conversely, if $S \subseteq C$ satisfies (L_1), (L_2), (L_3), and (L_4), then S^* is a Banach subalgebra of C^* and $S = L(S^*)$.

Proof. The distributivity of the lattice $\mathbf{R}$ and the fact that A^* is a sublattice of C^* imply that $L(A^*)$ is a sublattice of C. Clearly, it contains the constant functions.

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If $\alpha > 0$ is a real number, then

$$(\alpha f \wedge \lambda) \vee (-\lambda) = \alpha[(f \wedge \lambda/\alpha) \vee (-\lambda/\alpha)].$$

Hence $\alpha f \in L(A^*)$ if $f \in L(A^*)$. Since

$$((-f) \wedge \lambda) \vee (-\lambda) = -[(f \wedge \lambda) \vee (-\lambda)],$$

it then follows that $L(A^*)$ is closed under multiplication by real numbers.

Let $f \in L(A^*)$ and let $\alpha \in \mathbf{R}$. Then $(f + \alpha) \wedge \lambda = [f \wedge (\lambda - \alpha)] + \alpha$ and $(f + \alpha) \vee \mu = [f \vee (\mu - \alpha)] + \alpha$. Therefore

$$[(f + \alpha) \wedge \lambda] \vee (-\lambda) = [f \wedge (\lambda - \alpha)] \vee (-\lambda - \alpha) + \alpha.$$

Since this last function is in $L(A^*)$ and bounded, it is in A^*. Hence $f + \alpha \in L(A^*)$.

Let (f_n) be a sequence of functions in $L(A^*)$ that converge uniformly to f. Then for any $\lambda \geqslant 0$, $((f_n \wedge \lambda) \vee (-\lambda))$ converges uniformly to $(f \wedge \lambda) \vee (-\lambda)$. Hence $f \in L(A^*)$.

Assume $f \in L(A^*)$ is such that $fx > 0$ for all $x \in E$. For any two real numbers $\alpha, \beta > 0$, the function $(f \wedge \alpha) \vee \beta$ is in A^*. Since it is bounded below by β, its inverse is in A^*. This is the function $((1/f) \vee (1/\alpha)) \wedge 1/\beta$. Consequently, $(1/f) \vee (1/\alpha)$ is in $L(A^*)$. The sequence $((1/f) \vee (1/n))$ converges uniformly to $1/f$. Hence $1/f \in L(A^*)$.

If f is in $L(A^*)$, then $f^+ = f \vee 0$ and $f^- = (-f) \vee 0$ are both in $L(A^*)$. Conversely, if f^+ and f^- are in $L(A^*)$, then $f \in L(A^*)$, since

$$[(f^+ - f^-) \wedge \lambda] \vee (-\lambda) = (f^+ \wedge \lambda) + (-f^-) \vee (-\lambda).$$

To prove the converse, first note that, when extended to βE, S^* (since it satisfies (L_1) and (L_2)) corresponds to a Banach subalgebra of the algebra corresponding to C^*. Hence S^* itself is a Banach subalgebra of C^*.

Clearly, $S \subseteq L(S^*)$. In view of (L_4), it suffices to show that every positive f in $L(S^*)$ is also in S. Consider $f \vee 1/n$. It is in $L(S^*)$ and its inverse is in S^*. Since $S^* \subseteq S$, taking inverses once again shows that $f \vee 1/n \in S$. S uniformly closed implies that $f \in S$.

4. Closed equivalence relations on βE and the lattices $L(A^*)$. Given any compact space K, there is a 1–1 correspondence between the Banach subalgebras A^* of $C(K)$ and the closed equivalence relations r on K. To each algebra A^* corresponds the relation $r = r(A^*)$ defined by: $x\, r\, y$ if $fx = fy$ for all f in A^*. It is well known that $f \in A^*$ if and only if f is in C^* and compatible with $r(A^*)$.

If $f \in C$, let $\bar{f}$ denote its unique extension to βE as a continuous function valued in $\bar{\mathbf{R}}$ (the two-point compactification of $\mathbf{R}$). If A^* is a Banach subalgebra of C^*, then $\bar{A}^* = \{\bar{f} | f \in A^*\}$ is a Banach subalgebra of $\bar{C}^* = C(\beta E)$. The correspondence which associates $r(\bar{A}^*)$ with A^* is a 1–1 correspondence between the Banach subalgebras of C^* and the closed equivalence relations on βE.

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PROPOSITION 3. *$f \in L(A^*)$ if and only if $xr(\bar{A}^*)y$ implies $\bar{f}x = \bar{f}y$.*

Proof. Let $f \in L(A^*)$ and let $f_n = (f \wedge n) \vee (-n)$. Since $f_n \in A^*$, $xr(\bar{A}^*)y$ implies $\bar{f}_n x = \bar{f}_n y$. Now

$$\bar{f}x = \lim_{n\to\infty} \bar{f}_n x,$$

and so $\bar{f}x = \bar{f}y$.

Assume that $f \notin L(A^*)$. Then for sufficiently large n, say $n \geqslant n_0$, $f_n \notin A^*$. This means that there exist $x, y \in E$ with $xr(\bar{A}^*)y$ and $\bar{f}_{n_0} x \neq \bar{f}_{n_0} y$. Since $(f_{n+1} \wedge n) \vee (-n) = f_n$, it follows that $\bar{f}_n x \neq \bar{f}_n y$ for all $n \geqslant n_0$. This implies that $\bar{f}x \neq \bar{f}y$.

COROLLARY. *Let $f, g \in L(A^*)$. Then $f + g \in L(A^*)$, if either $f, g \geqslant 0$ or $g \in A^*$.*

Proof. If $f \geqslant 0$, then $\bar{f}x \geqslant 0$ for all $x \in \beta E$. Hence if $f, g \geqslant 0$, $\bar{f} + \bar{g}$ is defined. It is compatible with $r(\bar{A}^*)$. Since $\bar{f} + \bar{g} = \overline{f+g}$, $f + g \in L(A^*)$.

If g is bounded, then $\bar{f} + \bar{g}$ is defined and compatible with $r(\bar{A}^*)$. As before, this implies $f + g \in L(A^*)$.

Remark. In view of the proposition, the functions $\bar{f}$ on βE corresponding to the functions f in $L(A^*)$ can be identified with functions $\tilde{f}$ on the quotient space $\beta E/r(\bar{A}^*)$. It can be seen that $\{\tilde{f} | f \in L(A^*)\}$ is precisely the set of continuous $\bar{R}$-valued functions on $\beta E/r(\overline{A^*})$ whose restriction to the image of E under $\pi \circ i$ is real-valued (where π is the canonical map of βE on $\beta E/r(\overline{A^*})$ and i is the inclusion of E in βE).

5. The space of translation lattice homomorphisms of $L(A^*)$. Let S be a translation lattice of functions on E closed under multiplication by (-1) and for which $A^* \subseteq S \subseteq L(A^*)$. Then since $S^* = L(A^*)^* = A^*$, it follows from Proposition 2 that: $\mathfrak{L}(L(A^*))$ can be embedded in $\mathfrak{L}(S)$ by a map t_1; $\mathfrak{L}(S)$ can in turn be embedded in $\mathfrak{L}(A^*)$ by a map t_2; and, further, that $\mathfrak{L}(L(A^*))$ can be embedded in $\mathfrak{L}(A^*)$ by a map t.

Now $t \circ e_{L(A^*)} = e_{A^*} = t_2 \circ e_S = t_2 \circ t_1 \circ e_{L(A^*)}$. Since $e_{L(A^*)}E$ is dense in $\mathfrak{L}(L(A^*))$, it follows that $t = t_2 \circ t_1$.

THEOREM 2. *The subset X of $\mathfrak{L}(A^*)$ corresponding to $\mathfrak{L}(S)$ is the complement of a union of G_δ-sets disjoint from $e_{A^*} E$. Conversely, if Y is a union of G_δ-subsets of $\mathfrak{L}(A^*)$ which are disjoint from $e_{A^*} E$, then there exists a translation lattice S for which:*

(1) $A^* \subseteq S \subseteq L(A^*)$; *and*

(2) *the corresponding subset X of $\mathfrak{L}(A^*)$ is the complement of Y.*

Hence, the subset of $\mathfrak{L}(A^*)$ corresponding to $\mathfrak{L}(L(A^*))$ is the complement of the union of all the G_δ-subsets of $\mathfrak{L}(A^*)$ disjoint from $e_{A^*} E$.

Proof. It is well known that there is a homeomorphism

$$h: \beta E/r(\overline{A^*}) \to \mathfrak{L}(A^*), \text{ such that } h \circ \pi \circ i = e_{A^*},$$

where π and i are as in the last remark.

To prove the first assertion, it suffices to show that the complement Z of $(h \circ \pi)^{-1} X$ is a union of G_δ-sets, each of which is a countable intersection of $r(\bar{A}^*)$-saturated open sets.

By considering $\{\bar{f} | f \in S\}$, it can be seen that $Z = \{u \in \beta E |$ for some $f \in S$, $|\bar{f}u| = \infty\}$. If $f \in S$ and $|\bar{f}u| = \infty$, then

$$u \in \bigcap_{n=1}^{\infty} \{v \in \beta E | |\bar{f}_n v| > (n-1)\} \subseteq Z.$$

Clearly, $\{v \in \beta E | |\bar{f}_n v| > (n-1)\}$ is an $r(\overline{A^*})$-saturated open set.

To prove the converse, let $Z = (h \circ \pi)^{-1} Y$ and let S be the set of functions $f \in L(A^*)$ for which $\bar{f} | (\beta E - Z)$ is real-valued. Then S is a translation lattice which contains A^* and is closed under multiplication by (-1).

Since Y is a union of G_δ-sets,

$$\beta E - Z = \{u \in \beta E | |\bar{f}u| < \infty, \text{ for all } f \in S\}.$$

As a result, it can be seen that the subset X of $\mathfrak{L}(A^*)$ corresponding to $\mathfrak{L}(S)$ is $\pi(\beta E - Z) = \mathfrak{L}(A^*) - Y$.

The last assertion follows automatically, since the subset of $\mathfrak{L}(A^*)$ corresponding to $\mathfrak{L}(L(A^*))$ is a subset of the subsets corresponding to the spaces $\mathfrak{L}(S)$, $A^* \subseteq S \subseteq L(A^*)$.

Remark. The translation lattice S determines the subset Z of βE, which in turn determines a translation lattice. This new lattice contains S and is in general not equal to S.

6. The ring of quotients $R(A^*)$. Let A^* be a Banach subalgebra of C^*. Then the set

$$M(A^*) = \{f \in A^* | fx > 0 \text{ for all } x \in E\}$$

is multiplicatively closed. None of these functions is a zero divisor. Therefore A^* can be embedded in its ring of quotients with respect to $M(A^*)$. Let $R(A^*)$ denote this ring. Then every element of $R(A^*)$ is a continuous function h on E which can be written as $h = f/g, f \in A^*$ and $g \in M(A^*)$.

The ring $R(A^*)$ is a sublattice of C since for $h \in R(A^*)$ and $h = f/g$, $|h| = |f|/g$.

$R(A^*)$ is closed under positive inversion. If $h = f/g$, and $hx > 0$ for all $x \in E$, then $f \in M(A^*)$.

Consequently, $R(A^*)$ satisfies (L_1), (L_2), and (L_4). It is related to $L(A^*)$ as shown by

PROPOSITION 4. $A^* \subseteq L(A^*) \subseteq R(A^*)$. *$L(A^*)$ is a set of multiplicative generators for $R(A^*)$.*

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Proof. Let $f \geqslant 0$ be in $L(A^*)$, and let $g = f + 1$. Then $h = 1/g$ is in $M(A^*)$, and so g is in $R(A^*)$. Consequently, $f = g - 1$ is in $R(A^*)$. As a result, $L(A^*) \subseteq R(A^*)$.

If $f \in M(A^*)$, then $1/f \in L(A^*)$, and so $L(A^*)$ generates $R(A^*)$ multiplicatively.

$R(A^*)$ is an algebra of functions. The real-valued algebra homomorphisms h on $R(A^*)$, for which $h(1) = 1$, define a subset of $\mathbf{R}^{R(A^*)}$ in the usual way. Since $R(A^*)$ is closed under positive inversion, it is also closed under bounded inversion. Isbell **(2)** showed that this implies that the set of points in $\mathbf{R}^{R(A^*)}$ which are algebra homomorphisms is $\overline{e_{R(A^*)}E}$. Let $\mathfrak{H}(R(A^*))$ denote this subspace.

Proposition 5. *There is a homeomorphism* $c\colon \mathfrak{H}(R(A^*)) \to \mathfrak{L}(L(A^*))$ *such that* $c \circ e_{R(A^*)} = e_{L(A^*)}$.

Proof. Let $p\colon \mathbf{R}^{R(A^*)} \to \mathbf{R}^{L(A^*)}$ denote the projection obtained by dropping the co-ordinates corresponding to the functions that are in $R(A^*)$ but not in $L(A^*)$. Clearly, $p \circ e_{R(A^*)} = e_{L(A^*)}$. Hence $p\mathfrak{H}(R(A^*)) \subseteq \mathfrak{L}(L(A^*))$.

Define $q\colon \mathfrak{L}(L(A^*)) \to \mathbf{R}^{R(A^*)}$ by setting $q(u) = v$, where

$$v_h = \begin{cases} u_h & \text{if} \quad h \in L(A^*), \\ u_f/u_g & \phantom{\text{if}} \quad h = f/g, f \text{ and } g \in A^*. \end{cases}$$

If $h = f/g = f'/g'$, then $fg' = gf'$. Since a translation lattice homomorphism of A^* that commutes with multiplication by (-1) is also an algebra homomorphism, it follows that $u_f u_{g'} = u_g u_{f'}$. This shows that q is well defined. It is clear that it is continuous.

Now $p \circ q \circ e_{L(A^*)} = p \circ e_{R(A^*)} = e_{L(A^*)}$ and

$$q \circ p \circ e_{R(A^*)} = q \circ e_{L(A^*)} = e_{R(A^*)}.$$

Hence $c = p|\mathfrak{H}(R(A^*))$ is a homeomorphism of $\mathfrak{H}(R(A^*))$ with $\mathfrak{L}(L(A^*))$.

Although the homomorphism spaces of $L(A^*)$ and $R(A^*)$ coincide, in general $L(A^*) \neq R(A^*)$.

Example. Let E be a non-compact real-compact space, let $x \neq y$ be two points in $\beta E - E$, and let $A^* = \{f \in C^* | \bar{f}x = \bar{f}y\}$. There is a function $f_0 \geqslant 0$ in A^*, with no zeros in E, for which $\bar{f}_0 x = \bar{f}_0 y = 0$. Consequently, $1/f_0$ is in A^*. By Proposition 3, if $g \in C^*$, then $1/f_0 + g \in L(A^*)$ (and $f_0 g \in A^*$). From this it follows that $L(A^*)$ is not closed under either addition or multiplication. In particular, $L(A^*) \neq R(A^*) = C$.

7. When is $L(A^*) = R(A^*)$?

Theorem 3. *Let E be a locally compact space which is countable at infinity, and let $A^* \subseteq C^*$ be a Banach subalgebra whose weak topology is the topology of E. The following statements are equivalent:*

(1) *$L(A^*)$ is closed under addition.*

(2) *$L(A^*)$ is closed under multiplication.*

(3) *$L(A^*) = C$, i.e. $A^* = C^*$.*

Hence, under these conditions, $L(A^*) = R(A^*)$ if and only if $A^* = C^*$.

Proof. E is locally compact if and only if $\beta E - E$ is closed. This can be seen to be equivalent to the existence of a smallest Banach subalgebra A_0^* whose weak topology is the topology of E. This algebra corresponds to the equivalence relation whose only non-trivial equivalence class is $\beta E - E$.

A locally compact space E is countable at infinity if and only if there is a function $f_0 \in C^*$, $0 \leqslant f_0 \leqslant 1$, with $Z(f_0) = \beta E - E$. Clearly, $f_0 \in A_0^* \subseteq L(A^*)$.

If $g \in C^*$, then by Proposition 3, $g + 1/f_0 \in L(A^*)$ and $gf_0 \in L(A^*)$. Since $1/f_0 \in L(A^*)$, it follows that (1) implies (3) and (2) implies (3). The converses are clear.

Remark. It is tempting to conjecture that this theorem holds if E is not countable at infinity. This is not so. Isbell pointed out to the author that the set B of Baire functions on the real line satisfy (L_1), (L_2), (L_3), and (L_4), the weak topology is the discrete topology, the set is certainly closed under addition, but $B \neq \mathbf{R}^R$. He also pointed out that Lemma 5.3 of Henriksen and Johnson **(1)** implies that for E a Lindelöf space, the theorem is still true.

Theorem 3 is very similar to Isbell's Theorem 1.13 **(2)** which he used to show that a function algebra A is closed under composition if it is uniformly closed and inversion closed. A set S of real-valued functions is said to be closed under composition, if for each n and $f_1, \ldots, f_n \in S$, g: $\mathbf{R}^n \to \mathbf{R}$ continuous implies $g(f_1, \ldots, f_n) \in S$. Isbell's argument is applied in the proof of the next theorem.

THEOREM 4. *Let $A^* \subseteq C^*$ be a Banach subalgebra. The following statements are equivalent:*

(1) *$L(A^*)$ is closed under addition.*

(2) *$L(A^*)$ is closed under multiplication.*

(3) *$L(A^*)$ is closed under composition.*

Hence $L(A^*) = R(A^*)$ if and only if $L(A^*)$ is closed under composition.

Proof. Obviously, (3) implies (1) and (2). Given the equivalence of (2) and (3), the last assertion follows immediately.

Pick $f_1, \ldots, f_n \in L(A^*)$, and let X be the closure in $\mathbf{R}^n$ of

$$\{(f_1 x, \ldots, f_n x) | x \in E\}.$$

Then, as a subspace, X is locally compact and countable at infinity.

Let $S = \{g \in C(X) | g(f_1, \ldots, f_n) \in L(A^*)\}$. The set S inherits properties (L_1), (L_2), (L_3), and (L_4) from $L(A^*)$. Therefore, by Theorem 1, $S = L(A_1^*)$, where A_1^* is a Banach subalgebra of $C^*(X)$. S contains the restrictions to X

of the co-ordinate projections, and hence the weak topology defined by A_1^* is the subspace topology.

If $L(A^*)$ satisfies (1) or (2), S inherits the corresponding property. Applying Theorem 3, it follows that (1) implies $S = C(X)$ and that (2) implies $S = C(X)$.

8. Composition and closed equivalence relations. The algebra A^* is determined by the equivalence relation $r(\bar{A}^*)$ on E. The example of Section 6 suggests that properties of the relation $r(\bar{A}^*)$ should determine when $L(A^*)$ is closed under composition.

A subset X of βE will be said to determine the algebra A^* if, for any $g \in C^*$, $\bar{g}|X$ compatible with $r(\bar{A}^*)|X$ implies that $g \in A^*$. In other words, the restrictions imposed by $r(\bar{A}^*)|X$ are severe enough to define A^*.

THEOREM 5. *Let $A^* \subseteq C^*$ be a Banach subalgebra. The following assertions are equivalent:*

(1) *$L(A^*)$ is closed under composition.*

(2) *$R(A^*)^* = A^*$.*

(3) *For all $f \in M(A^*)$, $fg \in A^*$ and $g \in C^*$ implies $g \in A^*$.*

(4) *For all $f \in M(A^*)$, $\beta E - Z(\bar{f})$ determines A^*.*

Proof. By the previous theorem, (1) implies (2). If $f \in M(A^*)$ and $fg \in A^*$, then $g = fg/f \in R(A^*)$. Hence (2) implies (3).

Let $f \in M(A^*)$ and let $g \in C^*$ be such that $\bar{g}|\beta E - Z(\bar{f})$ is compatible with $r(\bar{A}^*)|\beta E - Z$. Then $fg \in A^*$, since $\overline{fg} = \bar{f}\bar{g}$ is compatible with $r(\bar{A}^*)$. Hence (3) implies (4).

Let $f, g \in L(A^*)$. Since $f + g = (f^+ + g^+) - (f^- + g^-)$, to show that $f + g \in L(A^*)$, it suffices (in view of the corollary to Proposition 3) to consider the case where $f \geqslant 0$ and $g \leqslant 0$.

Let $f_1 = f + 1$ and let $g_1 = g - 1$. Then, $1/f_1$ and $1/g_1$ are in A^*. Hence $h = -1/f_1 g_1 \in M(A^*)$.

$$Z(h) = \{u \in \beta E|\bar{f}_1 u = +\infty\} \cup \{v \in \beta E|\bar{g}_1 v = -\infty\}.$$

Consequently, if $k = f + g$, $\bar{k}_n|\beta E - Z(\bar{h})$ is compatible with $r(\bar{A}^*)|\beta E - Z(\bar{h})$. Hence $k_n \in A^*$ for all n. This implies $k = f + g \in L(A^*)$.

Hence, Theorem 4 shows that (4) implies (1).

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