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Introduction

The topic of this book is stochastic differential equations (SDEs). As their name suggests, they really are differential equations that produce a different "answer" or solution trajectory each time they are solved. This peculiar behaviour gives them properties that are useful in modeling of uncertainties in a wide range of applications, but at the same time it complicates the rigorous mathematical treatment of SDEs.

The emphasis of the book is on applied rather than theoretical aspects of SDEs and, therefore, we have chosen to structure the book in a way that we believe supports learning SDEs from an applied point of view. In the following, we briefly outline the purposes of each of the remaining chapters and explain how the chapters are connected to each other. In the chapters, we have attempted to provide a wide selection of examples of the practical application of theoretical and methodological results. Each chapter (except for the Introduction and Epilogue) also contains a representative set of analytic and hands-on exercises that can be used for testing and deepening understanding of the topics.

Chapter 2 is a brief outline of concepts and solutions methods for deterministic ordinary differential equations (ODEs). We especially emphasize solution methods for linear ODEs, because the methods translate quite easily to SDEs. We also examine commonly used numerical methods such as the Euler method and Runge–Kutta methods, which we extend to SDEs in the later chapters.

Chapter 3 starts with a number of motivating examples of SDEs found in physics, engineering, finance, and other applications. It turns out that in a modeling sense, SDEs can be regarded as noise-driven ODEs, but this notion should not be taken too far. The aim of the rest of the chapter is to show where things start to go wrong. Roughly speaking, with linear SDEs we are quite safe with this kind of thinking, but anything beyond them will not work.

https://doi.org/10.1017/9781108186735.002 Published online by Cambridge University Press

In **Chapter 4**, we reformulate SDEs properly as stochastic integral equations where one of the terms contains a new kind of integral called the Itô integral. We then derive the change of variable formula, that is, the Itô formula for the integral, and use it to find complete solutions to linear SDEs. We also discuss some methods to solve nonlinear SDEs and look briefly at Stratonovich integrals.

The aim of **Chapter 5** is to analyze the statistics of SDEs as stochastic processes. We discuss and derive their generators, the Fokker–Planck–Kolmogorov equations, as well as Markov properties and transition densities of SDEs. We also derive the formal equations of the moments, such as the mean and covariance, for the SDE solutions. It turns out, however, that these equations cannot easily be solved for other than linear SDEs. This challenge will be tackled later in the numerical methods chapters.

As linear SDEs are very important in applications, we have dedicated **Chapter 6** to solution methods for their statistics. Although explicit solutions to linear SDEs and general moment equations for SDEs were already given in Chapters 4 and 5, here we also discuss and derive explicit mean and covariance equations, transition densities, and matrix fraction methods for the numerical treatment of linear SDEs. We also discuss steady-state solutions and Fourier analysis of linear time-invariant (LTI) SDEs as well as temporal covariance functions of general linear SDEs.

In **Chapter 7**, we discuss some useful theorems, formulas, and results that are typically required in more advanced analysis of SDEs as well as in their numerical methods. In addition to the Lamperti transform, Girsanov theorem, and Doob's h-transform, we also show how to find solutions to partial differential equations with Feynman–Kac formulas and discuss some connections to path integrals in physics. This chapter is not strictly necessary for understanding the rest of the chapters and can be skipped during a first reading.

Although the Itô stochastic calculus that is derivable from the Itô formula is theoretically enough for defining SDEs, it does not help much in practical solution of nonlinear SDEs. In **Chapter 8**, we present numerical simulation-based solution methods for SDEs. The methods are based primarily on Itô–Taylor series and stochastic Runge–Kutta methods, but we also discuss the Verlet and exact algorithm methods.

In many applications we are interested in the statistics of SDEs rather than their trajectories per se. In **Chapter 9**, we develop methods for approximate computation of statistics such as means and covariances or probability densities of SDEs – however, many of the methods are suitable for

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numerical simulation of SDEs as well. We start with classical and modern Gaussian "assumed density" approximations and then proceed to other linearization methods. We also discuss Taylor and Hermite series approximations of transition densities and their moments, numerical solutions of Fokker–Planck–Kolmogorov equations, simulation-based approximations, and finally pathwise Wong–Zakai approximations of SDEs.

An important and historically one of the first applications of SDEs is the filtering and smoothing theory. In **Chapter 10**, we describe the basic ideas of filtering and smoothing and then proceed to the classical Kushner–Stratonovich and Zakai equations. We also present the linear and nonlinear Kalman–Bucy and Kalman filters and discuss their modern variants. Finally, we present formal equations and approximation methods for the corresponding smoothing problems.

The aim of **Chapter 11** is to give an overview of parameter estimation methods for SDEs. The emphasis is on statistical likelihood-based methods that aim at computing maximum likelihood (ML) or maximum a posteriori (MAP) estimates or are targeted to full Bayesian inference on the parameters. We start with brief descriptions of the ideas of ML and MAP estimates as well as Markov chain Monte Carlo (MCMC) methods. Parameter estimation in linear SDEs is then discussed, and finally we give approximate likelihood methods for parameter estimation in nonlinear SDEs. We also discuss some parameter estimation methods for indirectly observed SDEs.

Chapter 12 addresses the somewhat less traditional topic of connections between machine learning and SDEs. The aim is to discuss links between Gaussian process regression, Kalman filtering, and SDEs, along with applications of the methods across the fields of signal processing and machine learning.

Finally, **Chapter 13** concludes the book with an overview and gives some hints where to go next. We also discuss additional topics such as fractional Brownian motions, Lévy process driven SDEs, and stochastic control problems.

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