Glasgow Math. J. **49** (2007) 423–429. © 2007 Glasgow Mathematical Journal Trust.
doi:10.1017/S0017089507003850. Printed in the United Kingdom

MULTIPLICITY RESULTS FOR A PERTURBED NONLINEAR SCHRÖDINGER EQUATION

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(Received 2 December, 2005; revised 22 May, 2007; accepted 24 May, 2007)

Abstract. In this paper, using a recent critical point theorem of Ricceri, we establish two multiplicity results for the Schrödinger equation of the form

$$-\Delta u + a(x)u = \lambda f(x, u) + \mu g(x, u), \quad x \in \mathbb{R}^n,\ u \in W^{1,2}(\mathbb{R}^n),$$

where $f,\ g : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ ($n \geq 3$) are Carathéodory functions, λ and μ two positive parameters.

2000 *Mathematics Subject Classification.* 35Q55.

1. Introduction. In the last few years, several authors have studied the following Schrödinger equation

$$-\Delta u + a(x)u = f(x, u), \quad x \in \mathbb{R}^n,\ u \in W^{1,2}(\mathbb{R}^n) \tag{S}$$

establishing, under suitable assumptions, existence or multiplicity of solutions. We refer the reader to [**1**], [**2**], [**6**]. Very recently, in [**4**], Kristaly obtained two results concerning three weak solutions for the Schrödinger equation of the form

$$-\Delta u + a(x)u = \lambda b(x)f(u), \quad x \in \mathbb{R}^n,\ u \in W^{1,2}(\mathbb{R}^n) \tag{P$_\lambda$}$$

under the following conditions:

(a_0) $a \in L^\infty_{loc}(\mathbb{R}^n)$ with $\operatorname{ess\,inf}_{\mathbb{R}^n} a > 0$ and

$$\mathrm{m}(\{x \in B(y, r) \,:\, a(x) \leq M\}) \to 0 \quad \text{as } |y| \to \infty,$$

for each $M > 0$, $r > 0$, where m stands for the Lebesgue measure.

(b_0) $b \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n),\ b \geq 0,$ and $\sup_{R>0} \operatorname{ess\,inf}_{|x|\leq R} b(x) > 0$.

(1) $f \in C(\mathbb{R}, \mathbb{R})$, and there exist $c > 0$ and $q \in]0, 1[$, such that

$$|f(s)| \leq c|s|^q \quad \text{for } s \in \mathbb{R}.$$

(2) $\displaystyle\lim_{s\to 0} \frac{f(s)}{|s|} = 0.$

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(3) $\sup_{s\in\mathbf{R}} F(s) > 0$, where $F(s) = \int_0^s f(t)\,dt$.

In particular, under the above assumptions, he proved the existence of an open interval of positive parameters λ and a number ν for which (P_λ) admits at least two distinct nonzero weak solutions, whose norms are less than ν.

Motivated by this fact, we obtain the same multiplicity results for the following more general nonlinear Schrödinger equation

$$-\Delta u + a(x)u = \lambda f(x,u) + \mu g(x,u), \quad x \in \mathbb{R}^n,\ u \in W^{1,2}(\mathbb{R}^n), \qquad (\mathrm{P}_{\lambda,\mu})$$

where $f,\ g : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ $(n \geq 3)$ are Carathéodory functions, λ and μ being two positive parameters. The proofs of our theorems are all based on a recent two local minima result of Ricceri (see [**8**]), while in [**4**] the aim is achieved using a three critical points theorem of Bonanno (see [**3**]).

We shall use in this paper the following conditions on the nonlinearity f:

(f_0) there exist a nonnegative function $b \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and a constant $q \in]0,1[$, such that

$$|f(x,t)| \leq b(x)|t|^q \quad \text{for } t \in \mathbb{R},\ \text{a.e. } x \in \mathbb{R}^n,$$

(f_1) $\lim_{t\to 0} \operatorname{ess\,sup}_{x\in\mathbf{R}^n} \left|\frac{f(x,t)}{t}\right| = 0$,

(f_2) there exists a constant $d \in \mathbb{R}$ such that $\sup_{R>0} \inf_{|x|\leq R} F(x,d) > 0$, where $F(x,t) = \int_0^t f(x,s)\,ds$.

A *weak solution of* $(\mathrm{P}_{\lambda,\mu})$ is any function $u \in W^{1,2}(\mathbb{R}^n)$ satysfying $(\mathrm{P}_{\lambda,\mu})$ in the weak sense. We shall consider $W^{1,2}(\mathbb{R}^n)$ endowed with the norm

$$\|u\| = \left(\int_{\mathbf{R}^n} (|\nabla u|^2 + u^2)\,dx\right)^{1/2},$$

and the subspace of $W^{1,2}(\mathbb{R}^n)$ defined by

$$E := \left\{u \in W^{1,2}(\mathbb{R}^n) \ :\ \int_{\mathbf{R}^n} a(x)u^2 < +\infty\right\}.$$

The space E, endowed with the inner product

$$\langle u, v\rangle_E = \int_{\mathbf{R}^n} (\nabla u \nabla v + a(x)uv)\,dx$$

and the corresponding norm

$$\|u\|_E = \langle u, u\rangle_E^{1/2},$$

is a Hilbert space.

It is known (see [**1**]) that (a_0) implies that E can be continuously embedded into $L^p(\mathbb{R}^n)$ whenever $p \in [2, 2^*]$, and the embedding is compact when $p \in [2, 2^*[\,,\ 2^* = \frac{2n}{n-2}$. In the sequel, we denote by k_p the Sobolev embedding constant.

The main tool is a recent critical point result by Ricceri [**8**]. We state it below in a form which is enough for our purposes.

THEOREM 1.1. ([**8**], Theorem 4) *Let X be a reflexive real Banach space, $I \subseteq \mathbb{R}$ an interval, and $\Psi : X \times I \to \mathbb{R}$ a function such that $\Psi(x, \cdot)$ is concave in I for all $x \in X$, while $\Psi(\cdot, \lambda)$ is continuous, coercive and sequentially weakly lower semicontinuous in X for all $\lambda \in I$. Further, assume that*

$$\sup_{\lambda \in I} \inf_{x \in X} \Psi(x, \lambda) < \inf_{x \in X} \sup_{\lambda \in I} \Psi(x, \lambda).$$

Then, for each $\rho > \sup_I \inf_X \Psi(x, \lambda)$ there exist a non-empty open set $A \subseteq I$ with the following property: for every $\lambda \in A$ and every sequentially weakly lower semicontinuous functional $\Phi : X \to \mathbb{R}$, there exists $\delta > 0$ such that, for each $\mu \in]0, \delta[$, the functional $\Psi(\cdot, \lambda) + \mu\Phi(\cdot)$ has at least two local minima lying in the set $\{x \in X \ : \ \Psi(x, \lambda) < \rho\}$.

Moreover, the application of Theorem 1.1 in proving our main result is made possible by the following proposition.

PROPOSITION 1.1. ([**7**], Proposition 3.1) *Let X be a nonempty set and Φ, J two real functions on X. Assume that there exist $\sigma > 0$, $u_0, \bar{u} \in X$, such that*

$$\Phi(u_0) = J(u_0) = 0, \qquad \Phi(\bar{u}) > \sigma, \qquad \sup_{\Phi(u) \le \sigma} J(u) < \sigma \frac{J(\bar{u})}{\Phi(\bar{u})}.$$

Then, for each ρ satisfying

$$\sup_{\Phi(u) \le \sigma} J(u) < \rho < \sigma \frac{J(\bar{u})}{\Phi(\bar{u})},$$

one has

$$\sup_{\lambda \ge 0} \inf_{u \in X} (\Phi(u) - \lambda J(u) + \lambda\rho) < \inf_{u \in X} \sup_{\lambda \ge 0} (\Phi(u) - \lambda J(u) + \lambda\rho).$$

2. Main results. The following theorems guarantee the existence of one and two nontrivial solutions in which the perturbation term g satisfies conditions of the types

(g_0) there exist two positive constants c, s with $s \in]1, \frac{n+2}{n-2}[$, such that

$$|g(x, t)| \le c|t|^s \quad \text{for } t \in \mathbb{R}, \text{ a.e. } x \in \mathbb{R}^n.$$

(g_1) there exist a nonnegative function $c \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and a constant $r \in]0, 1[$, such that

$$|g(x, t)| \le c(x)|t|^r \quad \text{for } t \in \mathbb{R}, \text{ a.e. } x \in \mathbb{R}^n.$$

THEOREM 2.1. *If the assumptions* (a_0) *and* (f_0)-(f_2) *hold, then there exist a number r and a non-degenerate compact interval $C \subseteq [0, +\infty[$ such that, for every $\lambda \in C$ and every Carathéodory function $g : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ satisfying the condition* (g_0) *there exists $\delta > 0$ such that, for each $\mu \in]0, \delta[$, the problem* ($P_{\lambda,\mu}$) *has at least one nonzero weak solution whose norm is less than r.*

Proof. Put $X = E$ and define the following functionals:

$$\Phi(u) = \frac{1}{2}\|u\|_E^2, \quad J(u) = \int_{\mathbf{R}^n} F(x, u(x))\, dx$$

for each $u \in X$.

It is well known that assumptions (a_0) and (f_0) and compact embedding, imply that the functional J is well defined and of class C^1 on E.

In particular we have

$$J'(u)(v) = \int_{\mathbf{R}^n} f(x, u(x))v(x)\, dx,$$

for all $u, v \in E$.

By (f_2) there exists $R_0 > 0$ such that $\rho_0 := \inf_{|x|\le R_0} F(x, d) > 0$. Let $0 < \epsilon < 1$, and define $u_\epsilon \in E$ such that $u_\epsilon(x) = 0$ for any $x \in \mathbb{R}^n \setminus B(0, R_0)$, $u_\epsilon(x) = d$ for any $x \in B(0, \epsilon R_0)$, and $\|\bar{u}\|_{L^\infty} \le |d|$. One has

$$\begin{aligned} J(u_\epsilon) &= \int_{B(0,\epsilon R_0)} F(x, d)\, dx + \int_{B(0,R_0)\setminus B(0,\epsilon R_0)} F(x, u_\epsilon(x))\, dx \\ &\ge \rho_0\epsilon^n \mathrm{m}\,(B(0, R_0)) - \|b\|_{L^\infty} d^{q+1}\mathrm{m}(B(0, R_0)). \end{aligned}$$

Now, for some ϵ close to 1, the expression above will be strictly positive. Denote $\bar{u} = u_\epsilon$ for such a value.

Fixing p with $2 < p < 2^*$ and using the hypotheses (f_0) and (f_1), we find, for each $\varepsilon > 0$ a constant $c_\varepsilon > 0$ with

$$|F(x, t)| \le \varepsilon|t|^2 + c_\varepsilon|t|^p \quad \text{for every } t \in \mathbb{R} \text{ and a.e. } x \in \mathbb{R}^n. \tag{1}$$

Applying inequality (1) with $\varepsilon = \dfrac{J(\bar{u})}{\Phi(\bar{u})}$ we get

$$|F(x, t)| \le \frac{\varepsilon}{4k_2^2}|t|^2 + c_\varepsilon|t|^p \quad \text{for every } t \in \mathbb{R} \text{ and a.e. } x \in \mathbb{R}^n. \tag{2}$$

At this point, in order to apply Proposition 1.1, choose

$$0 < \sigma < \min\left\{\Phi(\bar{u}), \left(\frac{\varepsilon}{2^{1+p/2}c_\varepsilon k_p^p}\right)^{2/(p-2)}\right\}.$$

For every $u \in E$ with $\Phi(u) \le \sigma$ we have

$$\begin{aligned} J(u) &\le \frac{\varepsilon}{4k_2^2}\int_{\mathbf{R}^n} |u(x)|^2\, dx + c_\varepsilon \int_{\mathbf{R}^n} |u(x)|^p\, dx \\ &\le \frac{\varepsilon}{4k_2^2}\|u\|_{L^2}^2 + c_\varepsilon\|u\|_{L^p}^p \le \frac{\varepsilon}{4}\|u\|_E^2 + c_\varepsilon k_p^p\|u\|_E^p \le \frac{\varepsilon}{2}\sigma + c_\varepsilon k_p^p(2\sigma)^{p/2}. \end{aligned}$$

Thus

$$\frac{\sup_{\Phi(u)\le\sigma} J(u)}{\sigma} \le \frac{\varepsilon}{2} + c_\varepsilon k_p^p 2^{p/2}\sigma^{(p/2-1)} < \frac{J(\bar{u})}{\Phi(\bar{u})}.$$

Then, choosing

$$\sup_{\Phi(u)\leq\sigma} J(u) < \rho < \sigma\frac{J(\bar{u})}{\Phi(\bar{u})},$$

Proposition 1.1 ensures that

$$\sup_{\lambda\geq 0}\inf_{u\in E}\Psi(u,\lambda) < \inf_{u\in E}\sup_{\lambda\geq 0}\Psi(u,\lambda),$$

where

$$\Psi(u,\lambda) = \Phi(u) - \lambda J(u) + \lambda\rho \quad \forall u\in E,\ \forall\lambda\geq 0.$$

Now, we can apply Theorem 1.1. Clearly, $\Psi(u,\cdot)$ is concave in $I = [0,+\infty[$ for every $u\in E$. By (a_0), (f_0) and the compact embedding, the functional J' is compact and so sequentially weakly continuous, (see Corollary 41.9 of [**9**]). Then, we have that $\Psi(\cdot,\lambda)$ is sequentially weakly lower semicontinuous.

Now, we prove the coercivity of $\Psi(\cdot,\lambda)$ for each $\lambda\in I$. For fixed $\lambda\in I$, by (f_0) one has

$$\Psi(u,\lambda) = \frac{1}{2}\|u\|_E^2 - \lambda J(u) + \lambda\rho \geq \frac{1}{2}\|u\|_E^2 - \lambda k_2^{q+1}\|b\|_{L^{2/(1-q)}}\|u\|_E^{q+1} + \lambda\rho.$$

Since $q < 1$, $\Psi(u,\lambda)\to+\infty$ as $\|u\|_E\to+\infty$.

Now, for fixed $\alpha > \sup_{\lambda\in I}\inf_{u\in E}\Psi(u,\lambda)$, Theorem 1.1 ensures that there exists a non-empty open set $A\subseteq I$ with the following property: for every $\lambda\in A$ and every Carathéodory function $g:\mathbb{R}^n\times\mathbb{R}\to\mathbb{R}$ satisfying condition (g_0), there exists $\delta > 0$ such that, for each $\mu\in]0,\delta[$, the functional

$$\mathcal{E}_{\lambda,\mu}(u,v) = \Psi(u,\lambda) - \mu\mathcal{G}(u)$$

has at least two local minima lying in the set $\{u\in E:\ \Psi(u,\lambda) < \alpha\}$, where $\mathcal{G}$ is the sequentially weakly continuous functional defined by

$$\mathcal{G}(u) = \int_{\mathbf{R}^n}\left(\int_0^{u(x)} g(x,t)\,dt\right)dx.$$

These minima are also the critical points of $\mathcal{E}_{\lambda,\mu}$ and hence weak solutions of the equation ($P_{\lambda,\mu}$).

Finally, let $[a,b]\subset A$ be any non-degenerate compact interval. Observe that

$$\bigcup_{\lambda\in[a,b]}\{u\in E:\ \Psi(u,\lambda)\leq\alpha\} \subseteq \{u\in E:\ \Psi(u,a)\leq\alpha\}\cup\{u\in E:\ \Psi(u,b)\leq\alpha\}.$$

This implies that the set $S := \bigcup_{\lambda\in[a,b]}\{u\in E:\ \Psi(u,\lambda)\leq\alpha\}$ is bounded. Hence, the two local minima of $\mathcal{E}_{\lambda,\mu}$ have norm less than or equal to r, taking $r = \sup_{u\in S}\|u\|$.

Finally, since one of them may be the trivial one, we shall have a nonzero weak solution. □

Through the same arguments made in the proof of Theorem 2.1, but applying also the Palais-Smale properties, we obtain the following result.

THEOREM 2.2. *Let us assume the same hypotheses of Theorem 2.1. Then, there exists a non-empty open set* $A \subseteq [0, +\infty[$ *such that, for every* $\lambda \in A$ *and every Carathéodory function* $g : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ *satisfying the condition* (g_1) *there exists* $\delta > 0$ *such that, for each* $\mu \in]0, \delta[$*, the problem* $(P_{\lambda,\mu})$ *has at least two distinct nontrivial weak solutions.*

Proof. Reasoning as in the first part of proof of Theorem 2.1, there exists a non-empty open set A with certain properties. In particular, fix a Carathéodory function $g : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ satisfying the condition (g_1), for each $\lambda \in A$. There exists $\delta > 0$ such that, for each $\mu \in]0, \delta[$, the problem $(P_{\lambda,\mu})$ has at least two solutions which are critical points of the functional $\mathcal{E}_{\lambda,\mu}(u) = \Psi(u, \lambda) - \mu\mathcal{G}(u)$, where $\mathcal{G}(u)$ is the weakly sequential continuous function defined by

$$\mathcal{G}(u) = \int_{\mathbf{R}^n} \left(\int_0^{u(x)} g(x, t)\, dt \right) dx.$$

From (g_1) we have

$$\mathcal{G}(u) \leq k_2^{r+1} \|c\|_{L^{2/(1-r)}} \|u\|_E^{r+1}$$

for each $u \in E$ and so the functional $\mathcal{E}_{\lambda,\mu}$ is coercive for each $\lambda \in A$ and $\mu \in]0, \delta[$.

Now, by Example 38.25 of [**9**], the functional $\mathcal{E}_{\lambda,\mu}$ has the Palais-Smale property.

Since this functional is also C^1 in E, Corollary 1 of [**5**] ensures that there exists a third critical point of the functional $\mathcal{E}_{\lambda,\mu}$ that is a solution of equation $(P_{\lambda,\mu})$. Since one of the solutions may be the trivial one, we conclude that the equation $(P_{\lambda,\mu})$ has at least two distinct, nontrivial weak solutions. □

EXAMPLE 1.1. As an example of nonlinearity of f satisfying (f_0)-(f_2), g satisfying (g_0) (resp. (g_1)) of Theorem 2.1 (resp. Theorem 2.2), let $0 < q < 1$, and consider the functions defined by

$$f(x, t) = \frac{1}{(1 + |x|^n)^2} |t|^q \sin t,$$

$$g(x, t) = \cos |x| \, |\sin t|^s \quad \text{with} \quad s \in \left]1, \frac{n+2}{n-2}\right[,$$

$$\left(g(x, t) = \frac{1}{(1 + |x|^n)^2} |\sin t|^r \quad \text{with} \quad r \in]0, 1[\right).$$

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