Canad. Math. Bull. Vol. **18** (2), 1975

DIRECT PRODUCT DECOMPOSITIONS OF TWISTED WREATH PRODUCTS

BY
JEFFREY M. BROWN

The twisted wreath product of two groups was first defined by B. H. Neumann ([1]) who used this construction to present a group-theoretic proof of a theorem due to Auslander and Lyndon. In this paper we present a complete characterization of the direct product decompositions of a restricted twisted wreath product of two groups A and B provided this product is not simply a semi-direct product of A by B.

1. Preliminary results. To recall the definition let A and B be two groups, let $S \leq B$ and T a right transversal for S in B. Let $\rho: S \rightarrow \mathrm{Aut}(A)$ be a homomorphism. If $\tau: B \rightarrow T$ denotes the obvious right coset representative map we denote by $1-\tau: B \rightarrow S$ the corresponding projection of B onto S, and accordingly

$$b^{\tau-1} = (b^{1-\tau})^{-1}, \qquad b \in B.$$

Clearly B acts as a transitive group of permutations on the set T:

$$t^b = (tb)^\tau, \qquad t \in T, \qquad b \in B.$$

Likewise for the action of S on A we write:

$$a^s = (\rho(s))(a), \qquad a \in A, \qquad s \in S.$$

Also, convenience would have us adopt the notation

$$s(b, t) = (tb^{-1})^{\tau-1}, \qquad b \in B, \qquad t \in T.$$

The base group F is defined to be the restricted direct product of $|B:S|$ copies of A indexed by the elements of T:

$$F = \prod\nolimits_{t \in T}^{(\times)} A[t], \qquad A[t] \simeq A.$$

Neumann ([1]) established a homomorphism mapping B into $\mathrm{Aut}(F)$, the element $b \in B$ being mapped to the automorphism which sends an element $f \in F$ to the element f^b where

$$f^b(t) = (f(t^{b^{-1}}))^{s(b,t)}, \qquad t \in T.$$

The restricted twisted wreath product $W = W(A, B, S, \rho)$ is defined to be the semi-direct product of F by B, i.e., $B \cdot F$, with respect to the above mentioned

---

Received by the editors January 26, 1973 and, in revised form, March 12, 1973.

This material has been taken in part from the authors Ph.D. dissertation submitted to Stevens Institute of Technology, Hoboken, N.J. and supervised by Prof. J. Ledlie.

homomorphism. (If one allows F to be the cartesian product of the copies of A then, using the same mapping, one obtains the unrestricted twisted wreath product which is not considered in this paper). Neumann ([1]) also established that, up to isomorphism, W is independent of the choice of the transversal T.

Several special cases are of interest. If $S=1$ then W is the restricted standard wreath product of A by B, while if $S=B$ then W is the semi-direct product of A by B with respect to ρ. We call W a proper twisted wreath product if $S\neq B$. Finally, if ρ is trivial and S contains no non-trivial normal subgroups of B, then W is a restricted (not necessarily standard) wreath product.

We now present several facts and establish some relevant notation. The largest normal subgroup of B contained in S is denoted $\mathrm{Core}_B(S)$: the intersection of the conjugates of S in B. The centralizer $C_S(A)$ of A in S is seen to be the kernel of ρ. The centralizer $C_A(S)$ of S in A, on the other hand, consists of those elements which are fixed points of every automorphism in $\rho(S)$. We also adopt the notation $\eta: W\to B$ for the obvious natural projection of W onto B; and $\gamma(a)$ for the inner automorphism of A corresponding to conjugation by a.

For any subgroup $A_0\leq A$ we define

$$D(A_0) = \{f\in F \mid f(t) = f(t')\in A_0,\ t,\ t'\in T\}.$$

Clearly $D(A_0)\simeq A_0$ provided $|B:S|<\infty$, and is trivial otherwise. Using Lemma 3.3 of [2] it is easily verified that

$$C_W(B) = \zeta(B)\times D(C_A(S)), \tag{1.1}$$

and

$$N_W(B) = B\times D(C_A(S)), \tag{1.2}$$

from which

$$C_W(F) = \{bf \mid b\in \mathrm{Core}_B(S),\ \rho(s(b,t)) = \gamma(f(t))\} \tag{1.3}$$

is easily deduced.

Another useful notation is to let $a[t]$ denote that element of F defined by

$$(a[t])(t') = \begin{cases} 1 & \text{if} \quad t\neq t', \\ a & \text{if} \quad t = t', \end{cases}$$

so that

$$(a[t])^b = (a^{s(b,t)^{-1}})[t^b], \qquad a\in A, \qquad b\in B, \qquad t\in T. \tag{1.4}$$

Observe that for any element $bf\in W$ and any $t\in T$ such that $t^b\neq t$, the element

$$bf_0 = (bf)^{(f(t)^{-1})[t]} \tag{1.5}$$

is conjugate to bf in W and has the property that $f_0(t)=1$, a fact useful in the sequel.

For any $f \in F$, let

$$\sigma(f) = \{t \in T \mid f(t) \neq 1\}.$$

Then there exists a homomorphism $\pi: F \to A/A'$ defined by

$$\pi(f) = f(t_1) \cdots f(t_k)A',$$

where $\{t_1, \ldots, t_k\}$ is any ordering of $\sigma(f)$. Now, for $A_0 \leq A$, define

$$M(A_0) = \{f \in F \mid \pi(f) \in A_0A'/A'\} \leq F.$$

As the kernel of π is obviously $M(A')$, it follows by a straightforward argument that:

(1.6) If M^+ is any subgroup of F containing $M(A')$ and if $\pi(M^+) = A^+/A'$, then

(i) $M^+ = M(A^+)$,

(ii) $A^+[t] = M^+ \cap A[t], \quad t \in T$,

and

(iii) A^+ is $\rho(S)$-invariant whenever $M^+ \lhd W$.

Finally, generalizing the proofs of the corresponding results for restricted standard wreath products ([2], Theorem 4.1 and Corollary 4.3), we have

(1.7)

(i) $[B, F] = M([A, S])$,

(ii) $W' = B' \cdot A' \cdot [A, S], \quad \text{if } |B:S| = 1$

(iii) $W' = B' \cdot F' \cdot M([A, S]), \quad \text{if } |B:S| \lhd 1$,

where

$$[A, S] = gp_A\{a^{-1}a^s \mid a \in A, s \in S\}.$$

2. Active and fully-active subgroups. In this section we establish results concerning two classes of normal subgroups. Recall that $\eta: W \to B$ is the obvious natural projection.

Definition 2.1. A subgroup $G \leq W$ is said to be active if for every $t \in T$ there exists $b \in \eta(G)$ such that $t^b \neq t$.

Clearly, if $\eta(G) \leq \text{Core}_B(S)$ then G is not active. On the other hand suppose G is normal and not active. Then there exists some $t \in T$ such that $t = t^b$ for every $b \in \eta(G)$. If c is any element of B, then

$$(t^c)^b = t^{cb} = t^{(cbc^{-1})c} = t^c$$

for every $b \in \eta(G)$. But B acts transitively on T, hence b fixes every element of T, that is, $b \in \text{Core}_B(S)$. Thus we conclude that a normal subgroup G is active if and only if $\eta(G) \not< \mathrm{Core}_B(S)$. As a consequence we have:

LEMMA 2.1. *If N is an active normal subgroup of W, where W is proper ($S \neq B$), then*

(i) *for every $t \in T$ and $a \in A$ there exists an element $f \in N \cap F$ such that $f(t) = a$,*

(ii) $F' \leq N \cap F$.

Proof. As N is active there exists, for every $t \in T$, an element $bg \in N$ such that $t \neq t^b$. Letting

$$f = [bg, a[t]] = g^{-1} \cdot (a^{-1}[t])^b \cdot g \cdot a[t],$$

we have, using (1.4),

$$f(t) = g^{-1}(t)g(t)a = a,$$

and as a is arbitrary the proof of (i) is complete.

Clearly (ii) follows if it can be shown that for every $t \in T$, $A'[t] \leq N \cap F$. Hence let $t \in T$ and $a_1, a_2 \in A$. Since N is active, there exists $bf \in N$ such that $t^b \neq t$. Moreover, by (1.5), we may assume that $f(t) = 1$. However, any commutator of the form $[bf, g]$ must be an element of N for any $g \in F$; in particular each of

$$f_1 = [bf, a_1[t]],$$
$$f_2 = [bf, a_2[t]],$$
$$f_3 = [bf, (a_1^{-1}a_2^{-1})[t]]$$

is in N. Upon expansion,

$$f_1 \cdot f_2 \cdot f_3 = ([a_1, a_2])[t] \in N.$$

Since the set of such elements generate $A'[t]$ the proof is complete.

The second, more restricted class of subgroups, is defined as follows.

DEFINITION 2.2. A subgroup $G \leq W$ is said to be fully-active if, for every $a \in A$, $t \in T$ and $\beta \in B$, there exists an element $bf \in G$ such that

$$(a[t])^\beta = (a[t])^{bf}.$$

The most important property of fully-active subgroups is described in

LEMMA 2.2. *If N is a fully-active normal subgroup of a proper restricted twisted wreath product W, then*

$$M([A, S]) \leq N \cap F.$$

Proof. It is sufficient, by (1.7)(i), to prove that

$$[B, F] \leq N \cap F.$$

Consider the element $[\beta, a[t]]$ for an arbitrary $\beta \in B$, $a \in A$ and $t \in T$. Clearly

$$[\beta, a[t]] = (a^{-1}[t])^\beta a[t].$$

From our definition there exists an element $bf \in N$ such that

$$(a^{-1}[t])^{\beta} = (a^{-1}[t])^{bf},$$

and hence

$$[\beta, a[t]] = [bf, a[t]] \in N \cap F.$$

Since such elements generate $[B, F]$ this proof is complete.

3. Direct product decompositions. It will be shown in this section that the direct product decompositions of a proper restricted twisted wreath product are of two types. The first type, called the regular decompositions, are characterized by the fact that at most one of the factors is active. The second type of decomposition, the irregular decomposition, occurs only under special restrictions on A, B, S and ρ, as will be clear from the statement of Theorem 3.1.

In order to avoid a more cumbersome notational change at a later date we let $B_P = \eta(P)$, $B_Q = \eta(Q)$, $B_{P\cap B} = B \cap P$, and $B_{Q\cap B} = B \cap Q$, where $\eta: W \to B$ is the natural projection of W onto B.

We begin by considering some of the conditions necessary for a regular decomposition. If $W = P \times Q$ and Q is not active then, since $S \neq B$ and $B = B_P \cdot B_Q$, clearly P is active; hence, by Lemma 2.1, for every $a \in A$ and every $t \in T$ there exists an element $f \in P \cap F$ such that $f(t) = a$. As $[P, Q] = 1$, it follows that for every $a \in A$ and every $t \in T$

$$(a[t])^{bg} = a[t], \qquad bg \in Q,$$

and hence, in fact, $Q \leq C_W(F)$. From (1.3) we have

$$s(b, t) = \gamma(f(t)), \qquad bf \in Q, \qquad t \in T,$$

and

$$\rho(Q \cap B) = 1.$$

Now, since $Q \leq C_W(F)$, it is clear that P is fully-active, and by Lemma 2.2

$$M[A, S] \leq P \cap F.$$

Moreover, since $F' \leq P \cap F$ (Lemma 2.1), it follows that

$$M(A') \leq P \cap F.$$

Thus, by (1.6), there exists a $\rho(S)$-invariant normal subgroup $A_{P\cap F} \leq A$ such that

$$P \cap F = M(A_{P\cap F}).$$

At the same time, since P is normal the set

$$P_F = \{f \in F \mid bf \in F, b \in B_P = \eta(P)\}$$

is a subgroup of F containing $P \cap F$, and again by (1.6) there exists a normal subgroup $A_P \leq A$ such that

$$P_F = M(A_P).$$

https://doi.org/10.4153/CMB-1975-033-x Published online by Cambridge University Press

Returning to Q we see that

$$Q \cap F \leq C_W(B),$$

since P is fully-active and $[P, Q]=1$. Hence, by 1.1, there exists a subgroup $A_{Q\cap F} \leq C_A(S)$ such that

$$Q \cap F = D(A_{Q\cap F}).$$

Indeed, applying Lemma 2.1(i) to P and using the fact that $[P, Q]=1$, it is clear that

$$A_{Q\cap F} = \zeta(A),$$

where $\zeta(A)$ denotes the center of A. If $|B:S|=n$ and $(A_{P\cap F})^{1/n}$ denotes the n-isolator of $A_{P\cap F}$, then

$$A_{Q\cap F} \cap (A_{P\cap F})^{1/n}$$

is trivial; for if a is in this set, then the element $f \in Q \cap F$ for which $f(1)=a$ has the property that

$$\pi(f) \in A_{P\cap F}/A',$$

contradicting the obvious restriction that

$$M(A_{P\cap F}) \cap D(A_{Q\cap F}) = 1.$$

The set

$$Q_F = \{f \in F \mid bf \in Q,\ b \in B_Q = \eta(Q)\}$$

also can be seen to be a subgroup of F, since Q is normal. In fact, we can now prove that

$$Q_F \leq C_W(B_P);$$

for if $bf \in Q$ and $b'f' \in P$ then

$$bf = (bf)^{b'f'} = (bf)^{f^{b'^{-1}}b'} = b^{b'}f^{b'} = bf^{b'},$$

since $Q \leq C_W(F)$ and $[B_P, B_Q]=1$. To determine the structure of Q_F we sidestep for a moment.

Since P is fully-active there exists, for each $t \in T$, at least one element $bf \in P$ such that

$$(a[1])^{t^{-1}} = (a[1])^{bf}, \qquad a \in A.$$

If we select one such element $b_t f_t$ for each $t \in T$ we have, by (1.4),

$$(a[t])^{f_t(t)^{-1}} = (a^{s(b_t,t)})[t], \qquad a \in A.$$

Now for each $t \in T$ define

$$\tilde{a}(t) = f_t(t)^{-1}, \qquad t \in T.$$

Then the set of elements $\{\tilde{a}(t) \mid t \in T\}$ has the property that for every $t \in T$ there exists an element $b \in B_P$ (namely b_t) such that

$$1^b = t$$

and

$$\rho(s(b,1)^{-1}) = \gamma(\tilde{a}(t)).$$

Coming back to Q_F we have that

$$Q_F \leq \{f \in F \mid f(1) = a, f(t) = a^{\tilde{a}(t)}, t \in T, a \in A\},$$

since $Q_F \leq C_W(B_P)$. Letting

$$A_Q = \{a \in A \mid a = f(1), f \in Q_F\},$$

we observe that

$$(a^{\tilde{a}(t)}[t])^b = (a^{\tilde{a}(t)^b})[t^b], \qquad a \in A_Q, \qquad b \in B_P, \qquad t \in T. \tag{3.1}$$

Based on this development we are led to make the following definition. Caution should be taken concerning the notation of this definition as it is perhaps suggestive. The subscripts which appear, though very useful for visualizing what is happening, must at all times be considered only as labels, especially when attempting a proof of the sufficiency of the conditions set forth in Theorem 3.1.

DEFINITION 3.1. Let A and B be two groups, S a proper subgroup of B, T a fixed right transversal for S in B, and $\rho: S \to \mathrm{Aut}(A)$ a homomorphism. A system consisting of four normal subgroups

$$A_P, \qquad A_{P\cap F}, \qquad A_Q, \qquad A_{Q\cap F}$$

of A, and four normal subgroups

$$B_P, \qquad B_{P\cap B}, \qquad B_Q, \qquad B_{Q\cap B}$$

of B, will be called a $\rho(S)$-fracture of A and B provided that:

(1) (i) $A_P \cdot A_Q = A$,

(ii) $A_P \geq A_{P\cap F} \geq [A, S] \cdot A'$,

(iii) $A_{P\cap F}$ is $\rho(S)$-invariant,

(iv) $A_{Q\cap F} \leq A_Q \cap \zeta(A) \cap C_A(S)$,

(note that $A_Q = 1$ if $|B:S| = \infty$)

(v) $A_{Q\cap F} \cap (A_{P\cap F})^{1/n} = 1$, $n = |B:S|$;

(2) (i) $B_P \cdot B_Q = B$,

(ii) $[B_P \cdot B_Q] = 1$,

(iii) $B_Q \leq \mathrm{Core}_B(S)$,

(iv) $B_{Q\cap B} \leq B_Q$,

(v) $B_{Q\cap B} \leq C_S(A) = \mathrm{Ker}\, \rho$,

(vi) $B_{P\cap B} \leq B_P$,

(vii) $B_{P\cap B} \cap B_{Q\cap B} = 1$;

(3) at least one of A_P, B_P and one of A_Q, B_Q must be non-trivial.

The substance of the previous discussion is that the appropriate subgroups do form a $\rho(S)$-fracture of A and B. The existence of a $\rho(S)$-fracture unfortunately is not sufficient to ensure that the corresponding restricted twisted wreath product will have a direct product decomposition; indeed we need additional compatibility conditions to ensure that the pieces of our fracture will fit together.

DEFINITION 3.2. A $\rho(S)$-fracture is said to be mendable provided there exist two isomorphisms,

$$\pi_P: B_P/B_{P\cap B} \to A_P/A_{P\cap F},$$

and

$$\pi_Q: B_Q/B_{Q\cap B} \to A_Q/A_{Q\cap F},$$

and a set of elements

$$\{\tilde{a}(t) \in A \mid t \in T\},$$

such that

(i) $(a^{\tilde{a}(t)}[t])^b = (a^{\tilde{a}(t^b)})[t^b]$, $a \in A_Q$, $b \in B_P$, $t \in T$,

(ii) $\rho(s(b, t)^{-1}) = \gamma(a^{\tilde{a}(t)})$, $b \in B_Q$, $a \in \pi_Q(bB_{Q\cap B})$,

(iii) the map $a \to \pi_P^{-1}(aA_P) \cdot \pi_Q^{-1}(a^{-1}A_Q)$ is an isomorphism of $(A_P)^{1/n} \cap A_Q$ to $B/(B_{P\cap B} \cdot B_{Q\cap B})$,

(iv) the map $b \to \pi_P(bB_P) \cdot \pi_Q(b^{-1}B_Q)^n$ is an isomorphism of $B_P \cap B_Q$ to $A/(A_{P\cap F} \cdot (A_{Q\cap F})^n)$.

The fact that the subgroups of our discussion form a mendable $\rho(S)$-fracture is also quite direct. Indeed, condition (i) has already been verified (see (3.1)). The isomorphisms π_P and π_Q arise in the following natural way. To each element $b \in B_P$ there corresponds a set of elements $f \in F$ such that $bf \in P$. This correspondence is in fact a homomorphism from B_P onto $M(A_P)/M(A_{P\cap F})$ which in turn is isomorphic to $A_P/A_{P\cap F}$. The kernel of this homomorphism is simply $B_{P\cap B}$, and we let π_P then be the appropriate induced isomorphism. If to every $b \in B_Q$ one assigns the set of all $f \in F$ such that $bf^{-1} \in Q$, it is easily shown that this defines a homomorphism from B_Q onto $Q_F/Q \cap F$ while $Q_F/Q \cap F \simeq A_Q/A_{QF}$. The kernel of this homomorphism is clearly $B_{Q\cap B}$, and π_Q can be taken to be the corresponding induced isomorphism.

To verify condition (ii) it is necessary to investigate the product of two elements of Q, for (ii) follows from the fact that this product must be in Q.

The last two conditions follow from the definitions of π_P and π_Q and arise in the following natural way. For every $b \in B$ there exists $b_1 f_1 \in P$ and $b_2 f_2 \in Q$ such that

$$b = b_1 f_1 b_2 f_2.$$

Since $Q \le C_W(F)$, it is clear that $f_2 = f_1^{-1}$. If we assign to each $b \in B$ this unique element $f_1 \in P_F \cap Q_F$, we obtain a homomorphism having kernel $B_{P\cap B} \cdot B_{Q\cap B}$. Hence

$$B/B_{P\cap B} \cdot B_{Q\cap B} \simeq P_F \cap Q_F.$$

It is not hard to show that

$$P_F \cap Q_F \simeq (A_P)^{1/n} \cap A_Q,$$

and condition (iii) is then easily verified. Condition (iv) arises in like fashion by considering an arbitrary element $f \in F$.

Conversely, proving the existence of a regular direct product decomposition of W, given a mendable $\rho(S)$-fracture of A and B is straightforward, though lengthy. As candidates for the factors we let

$$P = \{bf \mid b \in B_P, f \in M(\pi_P(bB_{P\cap B}) \cdot A_{P\cap F})\}$$

and

$$Q = \{bf \mid b \in B_Q, f(t)^{\tilde{a}(t)^{-1}} = f(1) \in \pi_Q(b^{-1}B_{Q\cap B}) \cdot A_{Q\cap F}\}.$$

This proof will not be presented here.

Our theorem then reads:

THEOREM 3.1. *A proper restricted twisted wreath product $W(A, B, S, \rho)$ has a non-trivial direct product decomposition if and only if either:*

1. *A and B have a mendable $\rho(S)$-fracture, or*
2. (i) *A is abelian, with a unique square root for every element;*
 (ii) *$|B:S|=2$, with right transversal t_1, t_2;*
 (iii) *$\rho(S)$ consists of the identity and the automorphism (denoted $(\ )^{-1}$) which sends every element of A to its inverse;*
 (iv) *$B=B_P\times B_Q$*

where (a) $B_P \not< \mathrm{Core}_B(S)=S$, $B_Q \not< S$,
 (b) for $b \in B_Q$, $\rho(s(b, t_i))=(\ )^{-1}$ if and only if $t_i^b \neq t_i$, $i=1, 2$,
 (c) *for* $b \in B_P$, $\rho(s(b, t_i))=1$, $i=1, 2$.

The proof consists of showing that W has an irregular decomposition if and only if W satisfies condition 2 of Theorem 3.1. We begin by showing these conditions to be necessary. Let $B_P=\eta(P)$ and $B_Q=\eta(Q)$ as before. The fact that A is abelian follows from Lemma 2.1 since both P and Q are assumed to be active. Then, since F is abelian,

$$[B_P, F] = [P, F] \leq P,$$

and

$$[B_Q, F] = [Q, F] \leq Q.$$

For each $t \in T$ let $B_P[t]=\{b \in B_P \mid t^b \neq t\}$. Since P is active, $B_P[t]\neq \varnothing$ for each $t \in T$. Then for each $b[t] \in B_P[t]$

$$[b[t], a[t]] = (a^{-1}[t])^{b[t]} \cdot a[t] \in P, \qquad a \in A.$$

Since this element commutes with every $\beta \in B_Q$, we have

$$(a^{-1}[t])^{b[t]} \cdot a[t] = (a^{-1}[t]^{b[t]})^{\beta} \cdot (a[t])^{\beta}$$

https://doi.org/10.4153/CMB-1975-033-x Published online by Cambridge University Press

for every $\beta \in B$, $t \in T$, $a \in A$ and $b[t] \in B_P[t]$. Since Q also is active there exists some triple

$$(t_0, b_0, \beta_0), \qquad t_0 \in T, \qquad b_0 \in B_P[t], \qquad \beta_0 \in B_Q,$$

such that

$$a[t_0] = (a^{-1}[t_0])^{b_0 \cdot \beta_0}, \qquad t_0^{b_0} \neq t_0, \qquad a \in A,$$

and

$$(a^{-1}[t_0])^{\beta_0} = (a[t_0])^{b_0}, \qquad t_0^{b_0} \neq t_0, \qquad a \in A.$$

It now follows that $s(b, t_0)=1$ for every $b \in B_P$ such that $t_0^b = t_0$, and likewise for any $b \in B_Q$ for which $t_0^b = t_0$. It is also clear that A has no element of order 2, and that no element of A is a fixed point for every automorphism in $\rho(S)$, i.e., $C_A(S)=1$.

A short argument also proves that if b is an element of B_P not centralizing F, then b acts like b_0 at t_0, that is

$$(a[t_0])^b = (a[t_0])^{b_0}, \qquad a \in A, \qquad b \in B_P, \qquad b \notin C_W(F).$$

Likewise

$$(a[t_0])^b = (a[t_0])^{\beta_0}, \qquad a \in A, \qquad b \in B_Q, \qquad b \notin C_W(F).$$

But $B_P \cdot B_Q = B$, hence $|B:S|=2$. Moreover, either $\rho(s(b_0, t_0)) = (\ )^{-1}$ or $\rho(s(\beta_0, t_0)) = (\ )^{-1}$ but not both. For definiteness we assume the latter. It follows that

$$\rho(s(b, t)) = 1, \qquad b \in B_P, \qquad t \in T,$$

while

$$\rho(s(b, t)) = (\ \ )^{-1}, \qquad b \in B_P, \qquad t^b \neq t \in T.$$

In particular

$$C_F(B_P) = [B_Q, F],$$
$$C_F(B_Q) = [B_P, F],$$

and in fact

$$F = (P \cap F) \times (Q \cap F);$$

hence

$$B = B_P \times B_Q.$$

The fact that A has a unique square root for every element now follows easily from the direct decomposition of F above, and the necessity of condition 2 is established.

The proof that condition 2 is sufficient will again be omitted since it is lengthy but not difficult provided one takes as candidates for the direct factors

$$P = B_P \cdot [B_P, A[1]],$$

and

$$Q = B_Q \cdot [B_Q, A[1]].$$

REFERENCES

1. B. H. Neumann, *Twisted wreath products of groups*, Archiv der Math., **14** (1963), 1–6.

2. P. M. Neumann, *On the structure of standard wreath products of groups*, Math. Z., **84** (1964), 343–373.

MARIANOPOLIS COLLEGE
MONTREAL, CANADA

https://doi.org/10.4153/CMB-1975-033-x Published online by Cambridge University Press