

THE DISTRIBUTION OF ZEROS OF SOLUTIONS OF FIRST ORDER FUNCTIONAL DIFFERENTIAL EQUATIONS

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In this paper, we establish some estimates for the distance between adjacent zeros of the solutions of the first order delay differential equation

$$x'(t) + P(t)x(t - \tau) = 0$$

and the first order advanced differential equation

$$x'(t) - P(t)x(t + \tau) = 0$$

where $P \in C([t_0, \infty), [0, \infty))$ and $\tau \in \mathbb{R}^+$. Our results improve recent results in the literature.

1. INTRODUCTION

The oscillation theory of delay differential equations has been extensively developed during the past several years. We refer to the monographs by Erbe, Kong and Zhang [1], Gopalsamy [2], Györi and Ladas [3] and the references cited therein. But results dealing with the distribution of zeros of the oscillatory solutions of delay differential equations are relatively scarce. Erbe et al. [3], Liang [4], and Zhou [5] established estimates for the distance between adjacent zeros of the solutions of first order delay differential equations. Recently, Zhou [6, 7, 8] and Wang [8] studied the distribution of zeros of the oscillatory solutions of neutral differential equations.

In this paper, we consider first order delay differential equations of the form

$$(1) \quad x'(t) + P(t)x(t - \tau) = 0,$$

and first order advanced differential equations of the form

$$(2) \quad x'(t) - P(t)x(t + \tau) = 0,$$

where

$$(3) \quad P \in C([t_0, \infty), [0, \infty)), \tau > 0.$$

By using a new technique, we obtain a better estimate for the distance between adjacent zeros of the solutions of (1) which improve the results in [1, 4, 5]. We also extend this result to (2).

Received 2nd September, 1998

This research was supported by the National Natural Science Foundation of the People's Republic of China.

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2. LEMMAS

First we define a sequence $\{f_n(\rho)\}$ by

$$(4) \quad f_1(\rho) = e^\rho, \quad f_{n+1}(\rho) = e^{\rho f_n(\rho)}, \quad n = 1, 2, \dots$$

where ρ is a positive constant.

It is easily seen that for $\rho > 0$,

$$f_{n+1}(\rho) > f_n(\rho), \quad n = 1, 2, \dots$$

Observe that when $\rho > 1/e$ then

$$\lim_{n \rightarrow \infty} f_n(\rho) = +\infty,$$

because otherwise the sequence $\{f_n(\rho)\}$ would have a finite limit f_0 , such that

$$f_0 = e^{\rho f_0}.$$

Using the known inequality $e^x \geq ex$, we have

$$f_0 = e^{\rho f_0} \geq e\rho f_0 > f_0$$

which is a contradiction.

When $1/e < \rho < 1$, we also define a sequence $\{g_m(\rho)\}$ by

$$(5) \quad g_1(\rho) = \frac{2(1-\rho)}{\rho^2}, \quad g_{m+1}(\rho) = \frac{2(1-\rho)}{\rho^2 + \frac{2}{g_m^2(\rho)}}, \quad m = 1, 2, \dots$$

It is easily seen that for $1/e < \rho < 1$

$$g_{m+1}(\rho) < g_m(\rho), \quad m = 1, 2, \dots$$

We now consider the first order delay differential inequality

$$(6) \quad x'(t) + P(t)x(t - \tau) \leq 0$$

and the first order advanced differential inequality

$$(7) \quad x'(t) - P(t)x(t + \tau) \geq 0.$$

In order to prove our main results, we present several useful lemmas.

LEMMA 1. Suppose that (3) holds and let $x(t)$ be a solution of inequality (6) on $[t_x, \infty)$, where $t_x \geq t_0$. Further assume that there exist t_1 ($t_1 \geq t_x$) and positive constant ρ such that

$$(8) \quad \int_{t-\tau}^t P(s) ds \geq \rho > \frac{1}{e}, \quad \text{for } t \geq t_1$$

and that there exist $T_0 \geq t_1$ and $T \geq T_0 + 3\tau$ such that $x(t)$ is positive on $T_0 \leq t \leq T$. Then for some $n \geq 1$

$$(9) \quad \frac{x(t-\tau)}{x(t)} > f_n(\rho), \quad \text{for } t \in [T_0 + (2+n)\tau, T]$$

where $f_n(\rho)$ is defined by (4).

PROOF: Let $x(t)$ be a positive solution of inequality (6) on $[T_0, T]$. Then by (6) we get

$$(10) \quad x'(t) \leq -P(t)x(t-\tau) \leq 0, \quad \text{for } t \in [T_0 + \tau, T]$$

which implies that $x(t)$ is nonincreasing on $[T_0 + \tau, T]$. It follows that

$$(11) \quad \frac{x(t-\tau)}{x(t)} \geq 1, \quad \text{for } t \in [T_0 + 2\tau, T].$$

When $T_0 + 3\tau \leq t \leq T$, dividing (6) by $x(t)$, and integrating from $t-\tau$ to t , we get

$$\ln \left(\frac{x(t)}{x(t-\tau)} \right) + \int_{t-\tau}^t P(s) \frac{x(s-\tau)}{x(s)} ds \leq 0.$$

By using (8) and (11), we have

$$\ln \left(\frac{x(t-\tau)}{x(t)} \right) \geq \int_{t-\tau}^t P(s) \frac{x(s-\tau)}{x(s)} ds \geq \rho.$$

It follows that

$$(12) \quad \frac{x(t-\tau)}{x(t)} \geq e^\rho = f_1(\rho), \quad \text{for } t \in [T_0 + 3\tau, T].$$

Repeating the above procedure, we get

$$(13) \quad \frac{x(t-\tau)}{x(t)} \geq e^{\rho f_{n-1}(\rho)} = f_n(\rho), \quad \text{for } t \in [T_0 + (2+n)\tau, T].$$

The proof of Lemma 1 is completed. □

LEMMA 2. *Suppose that (3) holds and let $x(t)$ be a solution of inequality (6) on $[t_x, \infty)$, where $t_x \geq t_0$. Further assume that there exist t_1 ($t_1 \geq t_x$) and positive constant $\rho < 1$ such that*

$$(14) \quad \int_{t-\tau}^t P(s) ds \geq \rho > \frac{1}{e}, \quad \text{for } t \geq t_1$$

and that there exist $T_0 \geq t_1$ and a positive integer $N \geq 2$ such that $x(t)$ is positive on $T_0 \leq t \leq T_0 + (2 + N)\tau$. Then for some $m \leq N - 1$

$$(15) \quad \frac{x(t - \tau)}{x(t)} < g_m(\rho), \quad \text{for } t \in [T_0 + 3\tau, T_0 + (2 + N - m)\tau]$$

where $g_m(\rho)$ is defined by (5).

PROOF: Let $x(t)$ is a positive solution of inequality (6) on $[T_0, T_0 + (2 + N)\tau]$. From (14), when $t \geq T_x$ we know that

$$\int_{t-\tau}^t P(s) ds \geq \rho \quad \text{and} \quad \int_t^{t+\tau} P(s) ds \geq \rho.$$

Observe that $f(\lambda) = \int_t^\lambda P(s) ds$ is a continuous function, $f(t) = 0$ and $f(t + \tau) \geq \rho$ and there exists λ_t such that $\int_t^{\lambda_t} q(s) ds = \rho$, where $t < \lambda_t < t + \tau$.

When $T_0 + 3\tau \leq t \leq T_0 + (2 + N - 1)\tau$, integrating both sides of (6) from t to λ_t , we obtain

$$(16) \quad x(t) - x(\lambda_t) \geq \int_t^{\lambda_t} P(s)x(s - \tau) ds.$$

Since $t \leq s \leq t + \tau$, we easily see that $T_0 + 2\tau \leq t - \tau \leq s - \tau \leq t$. Integrating both side of (6) from $s - \tau$ to t , we get

$$x(s - \tau) - x(t) \geq \int_{s-\tau}^t P(u)x(u - \tau) du.$$

From (10), it is clear that $x(u - \tau)$ is nonincreasing on $T_0 + 2\tau \leq s - \tau \leq u \leq t$, thus, we have

$$(17) \quad \begin{aligned} x(s - \tau) &\geq x(t) + x(t - \tau) \int_{s-\tau}^t P(u) du \\ &= x(t) + x(t - \tau) \left\{ \int_{s-\tau}^s P(u) du - \int_t^s P(u) du \right\} \\ &\geq x(t) + x(t - \tau) \left\{ \rho - \int_t^s P(u) du \right\}. \end{aligned}$$

From (16) and (17), we have

$$\begin{aligned}
 (18) \quad x(t) &\geq x(\lambda_t) + \int_t^{\lambda_t} P(s)x(s - \tau) ds \\
 &\geq x(\lambda_t) + \int_t^{\lambda_t} P(s) \left\{ x(t) + x(t - \tau) \left(\rho - \int_t^s P(u) du \right) \right\} ds \\
 &= x(\lambda_t) + \rho x(t) + \rho^2 x(t - \tau) - x(t - \tau) \int_t^{\lambda_t} ds \int_t^s P(s)P(u) du.
 \end{aligned}$$

As is well known, the identical relation

$$\int_t^{\lambda_t} ds \int_t^s P(s)P(u) du = \int_t^{\lambda_t} du \int_u^{\lambda_t} P(s)P(u) ds$$

holds. On the right hand side we exchange the variables of integration s and u . The above equality becomes

$$\int_t^{\lambda_t} ds \int_t^s P(s)P(u) du = \int_t^{\lambda_t} ds \int_s^{\lambda_t} P(u)P(s) du$$

which implies

$$\begin{aligned}
 \int_t^{\lambda_t} ds \int_t^s P(s)P(u) du &= \frac{1}{2} \int_t^{\lambda_t} ds \int_t^{\lambda_t} P(u)P(s) du \\
 &= \frac{1}{2} \left(\int_t^{\lambda_t} P(s) ds \right)^2 = \frac{\rho^2}{2}.
 \end{aligned}$$

Substituting this into (18), we have

$$(19) \quad x(t) \geq x(\lambda_t) + \rho x(t) + \frac{\rho^2}{2} x(t - \tau).$$

Noting that

$$x(\lambda_t) > 0, \quad \text{for } t \in [T_0 + 3\tau, T_0 + (2 + N - 1)\tau],$$

by (19), we get

$$(20) \quad \frac{x(t - \tau)}{x(t)} < \frac{2(1 - \rho)}{\rho^2} = g_1(\rho), \quad \text{for } t \in [T_0 + 3\tau, T_0 + (2 + N - 1)\tau].$$

When $T_0 + 3\tau \leq t \leq T_0 + (2 + N - 2)\tau$, we easily see that $T_0 + 3\tau \leq t \leq \lambda_t \leq t + \tau \leq T_0 + (2 + N - 1)\tau$.

Thus, by (20), we have

$$(21) \quad x(\lambda_t) > \frac{1}{g_1(\rho)}x(\lambda_t - \tau), \quad \text{for } t \in [T_0 + 3\tau, T_0 + (2 + N - 2)\tau].$$

Since $x(t)$ is nonincreasing on $[T_0 + \tau, T_0 + (2 + N)\tau]$ and $T_0 + 2\tau \leq \lambda_t - \tau < t < \lambda_t < T_0 + (2 + N - 1)\tau$, we get

$$x(\lambda_t) > \frac{1}{g_1(\rho)}x(\lambda_t - \tau) \geq \frac{1}{g_1(\rho)}x(t) > \frac{1}{g_1^2(\rho)}x(t - \tau).$$

Substituting this into (19), we have

$$x(t) > \frac{1}{g_1^2(\rho)}x(t - \tau) + \rho x(t) + \frac{\rho^2}{2}x(t - \tau), \quad \text{for } t \in [T_0 + 3\tau, T_0 + (2 + N - 2)\tau].$$

Therefore

$$\frac{x(t - \tau)}{x(t)} < \frac{2(1 - \rho)}{\rho^2 + \frac{2}{g_1^2(\rho)}} = g_2(\rho), \quad \text{for } t \in [T_0 + 3\tau, T_0 + (2 + N - 2)\tau].$$

Repeating the above procedure, we obtain

$$(22) \quad \frac{x(t - \tau)}{x(t)} < \frac{2(1 - \rho)}{\rho^2 + \frac{2}{g_{m-1}^2(\rho)}} = g_m(\rho), \quad \text{for } t \in [T_0 + 3\tau, T_0 + (2 + N - m)\tau].$$

The proof of Lemma 2 is completed. □

The above lemmas can be extended to the inequality (7). Since the proofs are similar we omit them and we formulate only the corresponding results.

LEMMA 3. *Suppose that (3) holds and let $x(t)$ be a solution of inequality (7) on $[t_x, \infty)$, where $t_x \geq t_0$. Further assume that there exist t_1 ($t_1 \geq t_x$) and positive constant ρ such that*

$$(23) \quad \int_t^{t+\tau} P(s) ds \geq \rho > \frac{1}{e}, \quad \text{for } t \geq t_1$$

and that there exist $T_0 \geq t_1$ and a positive integer $N \geq 2$ such that $x(t)$ is positive on $T_0 \leq t \leq T_0 + (2 + N)\tau$. Then for some $n \leq N$

$$(24) \quad \frac{x(t + \tau)}{x(t)} \geq f_n(\rho), \quad \text{for } t \in [T_0, T_0 + (N - n)\tau]$$

where $f_n(\rho)$ is defined by (4).

LEMMA 4. *Suppose that (3) holds and let $x(t)$ be a solution of inequality (7) on $[t_x, \infty)$, where $t_x \geq t_0$. Further assume that there exist t_1 ($t_1 \geq t_x$) and positive constant $\rho < 1$ such that*

$$(25) \quad \int_t^{t+\tau} P(s) ds \geq \rho > \frac{1}{e}, \quad \text{for } t \geq t_1$$

and that there exist $T_0 \geq t_1$ and $T \geq T_0 + 4\tau$ such that $x(t)$ is positive on $T_0 \leq t \leq T$. Then for $t \in [T_0 + m\tau, T - 3\tau]$

$$(26) \quad \frac{x(t + \tau)}{x(t)} < g_m(\rho)$$

where m is a positive integer and $g_m(\rho)$ is defined by (5).

3. MAIN RESULTS

In this section we shall apply the lemmas in section 2 to establish estimates for the distance between adjacent zeros of the solutions of equations (1) and (2).

In the following, $d(x)$ denotes the distance between adjacent zeros of the solution $x(t)$ of (1) or (2).

THEOREM 1. *Suppose that (3) holds and let $x(t)$ be a solution of (1) on $[t_x, \infty)$, where $t_x \geq t_0$. Further assume that there exist t_1 ($t_1 \geq t_x$) and positive constant ρ such that*

$$(27) \quad \int_{t-\tau}^t P(s) ds \geq \rho > \frac{1}{e}, \quad \text{for } t \geq t_1.$$

Then $x(t)$ on $[t_1, \infty)$ has arbitrarily large zeros and $d(x) < (2 + k_\rho)\tau$, where

$$(28) \quad k_\rho = \begin{cases} 1, & \text{when } \rho \geq 1, \\ \min_{n \geq 1, m \geq 1} \{n + m \mid f_n(\rho) \geq g_m(\rho)\}, & \text{when } 1/e < \rho < 1, \end{cases}$$

and $f_n(\rho)$, $g_m(\rho)$ are defined by (4) and (5).

PROOF: It suffices to prove that for any $T_0 \geq t_1$ the solution $x(t)$ of (1) has zeros on $[T_0, T_0 + k_\rho\tau]$. Otherwise, without loss of generality, we assume that $x(t)$ is positive on $[T_0, T_0 + k_\rho\tau]$. We consider the following two cases:

CASE 1. $\rho \geq 1$

In this case, $k_\rho = 1$. From (1) and (3) we get

$$x'(t) = -P(t)x(t - \tau) \leq 0, \quad \text{for } t \in [T_0 + \tau, T_0 + 3\tau]$$

which implies that $x(t)$ is nonincreasing on $[T_0 + \tau, T_0 + 3\tau]$, and thus

$$x(t) \geq x(T_0 + 2\tau), \quad \text{for } t \in [T_0 + \tau, T_0 + 2\tau].$$

Integrating both sides of (1) from $T_0 + 2\tau$ to $T_0 + 3\tau$, we obtain

$$(29) \quad \begin{aligned} x(T_0 + 3\tau) &= x(T_0 + 2\tau) - \int_{T_0 + 2\tau}^{T_0 + 3\tau} P(s)x(s - \tau) ds \\ &\leq x(T_0 + 2\tau) \left\{ 1 - \int_{T_0 + 2\tau}^{T_0 + 3\tau} P(s) ds \right\}. \end{aligned}$$

By (27) and (29), we have

$$x(T_0 + 3\tau) \leq x(T_0 + 2\tau)(1 - \rho) \leq 0$$

which is a contradiction.

CASE 2. $1/e < \rho < 1$

Without loss of generality, we assume that $x(t)$ is positive on $[T_0, T_0 + (2 + k_\rho)\tau]$.

Let $k_\rho = n^* + m^*$ where

$$(30) \quad f_{n^*}(\rho) \geq g_{m^*}(\rho).$$

By Lemma 1, we have

$$(31) \quad \frac{x(t - \tau)}{x(t)} \geq f_{n^*}(\rho), \quad \text{for } t \in [T_0 + (2 + n^*)\tau, T_0 + (2 + k_\rho)\tau].$$

On the other hand, by Lemma 2 we get

$$(32) \quad \frac{x(t - \tau)}{x(t)} < g_{m^*}(\rho), \quad \text{for } t \in [T_0 + 3\tau, T_0 + (2 + k_\rho - m^*)\tau].$$

Setting $t = T_0 + (2 + n^*)\tau$ in (30) and (31), we obtain

$$f_{n^*}(\rho) \leq \frac{x(T_0 + (1 + n^*)\tau)}{x(T_0 + (2 + n^*)\tau)} < g_{m^*}(\rho)$$

which contradicts (30) and completes the proof. \square

REMARK 1. Theorem 1 improves [1, Theorem 2.2.1 and 2.2.2] [4, Theorem 1 and 2] and [5, Theorem 1].

COROLLARY 1. When $P(t)$ in (1) is a positive constant p , $p\tau = \rho > 1/e$, and $x(t)$ is a solution of (1) on $[t_x, \infty)$, then $x(t)$ on $[t_x, \infty)$ has arbitrarily large zeros and $d(x) < (2 + k_\rho)\tau$, where k_ρ is defined by (28).

Using a similar method to the proof of Theorem 1, by Lemmas 3 and 4, we obtain:

THEOREM 2. Assume that (3) holds and $x(t)$ is a solution of (2) on $[t_x, \infty)$, where $t_x \geq t_0$. Assume that there exist t_1 ($t_1 \geq t_x$) and positive constant ρ such that

$$(33) \quad \int_t^{t+\tau} P(s) ds \geq \rho > \frac{1}{e}, \quad \text{for } t \geq t_1.$$

Then $x(t)$ on $[t_1, \infty)$ has arbitrarily large zeros and $d(x) < (2 + k_\rho)\tau$, where k_ρ is defined by (28).

COROLLARY 2. When $P(t)$ in (2) is a positive constant p , $p\tau = \rho > 1/e$, and $x(t)$ is a solution of (2) on $[t_x, \infty)$, then $x(t)$ on $[t_x, \infty)$ has arbitrarily large zeros and $d(x) < (2 + k_\rho)\tau$, where k_ρ is defined by (28).

EXAMPLE 1. Consider the delay differential equation

$$x'(t) + x(t - 0.4) = 0$$

where $p = 1$. We have $\rho = \tau = 0.4$ and $f_1(\rho) = 1.491\dots, f_2(\rho) = 1.816\dots, f_3(\rho) = 2.067\dots, \dots, f_{10}(\rho) = 4.387\dots, f_{11}(\rho) = 5.784\dots, f_{12}(\rho) = 10.111\dots, \dots; g_1(\rho) = 7.500\dots, g_2(\rho) = 6.136\dots, g_3(\rho) = 5.631\dots, g_4(\rho) = 5.379\dots, \dots$

Thus, we find $f_n(\rho) < 5 < g_m(\rho), 1 \leq n \leq 10, m \geq 1; f_{11}(\rho) > g_m(\rho), m \geq 3; f_{12}(\rho) > g_m(\rho), m \geq 1$.

Hence, by Corollary 1, we have $k_\rho = 12 + 1 = 13$ and $d(x) < 15 \times 0.4$. This improves the result in [4] : $d(x) < 28 \times 0.4$.

EXAMPLE 2. Consider the differential equation

$$x'(t) + x(t - 0.55) = 0$$

where $p = 1$. We have $\rho = \tau = 0.55$ and $f_1(\rho) = 1.733\dots, f_2(\rho) = 2.594\dots, f_3(\rho) = 4.165\dots, f_4(\rho) = 9.884\dots, \dots, g_1(\rho) = 2.975, g_2(\rho) = 1.703\dots, g_3(\rho) = 0.907\dots, \dots$

Thus, we find $f_1(\rho) > g_m(\rho), m \geq 2; f_2(\rho) > g_m(\rho), m \geq 2; f_3(\rho) > g_m(\rho), m \geq 1$. Hence, by Corollary 3.1, we have $k_\rho = 1 + 2 = 3$ and $d(x) < 5 \times 0.55$.

REMARK 2. We can also apply our results to neutral differential equations

$$\frac{d}{dt} [x(t) + C(t)x(t - \tau)] + P(t)x(t - \sigma) = 0$$

where $C, P \in C([t_0, \infty), R^+)$ and $\tau, \sigma \in R^+$. We can improve the results in [6, 7, 8].

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