Big Ramsey Degrees and Divisibility in Classes of Ultrametric Spaces

L. Nguyen Van Thé

Abstract. Given a countable set S of positive reals, we study finite-dimensional Ramsey-theoretic properties of the countable ultrametric Urysohn space $\mathbf{Q}_S$ with distances in S.

1 Introduction

This note is the continuation of our paper [8] where Ramsey-type properties of several classes of finite ultrametric spaces are studied. Recall that a metric space $\mathbf{X} = (X, d^{\mathbf{X}})$ is *ultrametric* when given any x, y, z in $\mathbf{X}$,

$$d^{\mathbf{X}}(x, z) \leqslant \max(d^{\mathbf{X}}(x, y), d^{\mathbf{X}}(y, z)).$$

Given $S \subset \,]0, +\infty[$, the class $\mathcal{U}_S$ is defined as the class of all finite ultrametric spaces with strictly positive distances in S. It turns out that when S is at most countable, there is, up to isometry, a unique countable metric space $\mathbf{Q}_S$ such that the family of finite metric subspaces of $\mathbf{Q}_S$ is exactly $\mathcal{U}_S$ and $\mathbf{Q}_S$ is *ultrahomogeneous*, that is, any isometry between any two finite subspaces of $\mathbf{Q}_S$ can be extended to an isometry of $\mathbf{Q}_S$ onto itself. This $\mathbf{Q}_S$ is called the *Urysohn space associated with* $\mathcal{U}_S$. It is a variation of the rational Urysohn space $\mathbf{U}_0$ constructed by Urysohn [9]. The difference between $\mathbf{Q}_S$ and $\mathbf{U}_0$ is that whereas $\mathbf{Q}_S$ is the countable universal ultrahomogeneous space attached to $\mathcal{U}_S$, $\mathbf{U}_0$ is related to the class $\mathcal{M}_{\mathbb{Q}}$ of all finite metric spaces with rational distances. Unlike $\mathbf{U}_0$, $\mathbf{Q}_S$ can be represented quite simply. Namely, $\mathbf{Q}_S$ can be seen as the set of all finitely supported elements of ω^S equipped with the distance $d^{\mathbf{Q}_S}$ defined by $d^{\mathbf{Q}_S}(x, y) = \max\{s \in S : x(s) \neq y(s)\}$.

The roles that $\mathbf{U}_0$ and $\mathbf{Q}_S$ play with respect to $\mathcal{M}_{\mathbb{Q}}$ and $\mathcal{U}_S$ respectively are exactly the same as the role that the Rado graph $\mathcal{R}$ plays for the class of finite graphs. Our concern here is to obtain for $\mathbf{Q}_S$ the analogs of well-known results of the form

$$\mathcal{R} \longrightarrow (\mathcal{R})^{\mathbf{G}}_{k,l}.$$

More precisely, for metric spaces spaces $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{Z}$, write $\mathbf{X} \cong \mathbf{Y}$ when there is an isometry from $\mathbf{X}$ onto $\mathbf{Y}$ and define the set $\binom{\mathbf{Z}}{\mathbf{X}}$ as

$$\binom{\mathbf{Z}}{\mathbf{X}} = \{\widetilde{\mathbf{X}} \subset \mathbf{Z} : \widetilde{\mathbf{X}} \cong \mathbf{X}\}.$$

413

Then given positive integers k, l, the symbol $\mathbf{Z} \rightarrow (\mathbf{Y})^{\mathbf{X}}_{k,l}$ abbreviates the fact that for any $\chi \colon \binom{\mathbf{Z}}{\mathbf{X}} \rightarrow k$ there is $\widetilde{\mathbf{Y}} \in \binom{\mathbf{Z}}{\mathbf{Y}}$ such that χ does not take more than l values on $\binom{\widetilde{\mathbf{Y}}}{\mathbf{X}}$.

In [8], it was proved that any element $\mathbf{X}$ has a *finite Ramsey degree in* $\mathcal{U}_S$, meaning that there is an integer l (depending on $\mathbf{X}$) for which, given any $\mathbf{Y} \in \mathcal{U}_S$ and any $k \in \omega \setminus \{0\}$, there exists $\mathbf{Z} \in \mathcal{U}_S$ such that $\mathbf{Z} \rightarrow (\mathbf{Y})^{\mathbf{X}}_{k,l}$.

Consequently, $\forall k \in \omega \setminus \{0\},\ \mathbf{Q}_S \rightarrow (\mathbf{Y})^{\mathbf{X}}_{k,l}$.

The purpose of this paper is to present the conditions on $S \subset \,]0, +\infty[$ and $\mathbf{X} \in \mathcal{U}_S$ under which this latter result remains valid when $\mathbf{Y}$ is replaced by $\mathbf{Q}_S$ and if so to compute the least integer l such that $\forall k \in \omega \setminus \{0\},\ \mathbf{Q}_S \rightarrow (\mathbf{Q}_S)^{\mathbf{X}}_{k,l}$.

When defined, this least l is called the *big Ramsey degree of X in* $\mathcal{U}_S$ and is in fact part of the more general notion of big Ramsey degree for an arbitrary class of finite structures, a concept defined by Kechris, Pestov and Todorcevic in the general setting of oscillation stability of topological groups [6].

Theorem 1.1 *Let S be a finite subset of* $]0, +\infty[$. *Then every element of* $\mathcal{U}_S$ *has a big Ramsey degree in* $\mathcal{U}_S$.

Theorem 1.2 *Let S be an infinite countable subset of* $]0, +\infty[$ *and let X be in* $\mathcal{U}_S$ *such that* $|X| \geqslant 2$. *Then X does not have a big Ramsey degree in* $\mathcal{U}_S$.

Theorem 1.2 does not cover the case $|X| = 1$, which is related to the *divisibility* properties of $\mathbf{Q}_S$. A metric space $\mathbf{X}$ is *indivisible* when, given any $k \in \omega \setminus \{0\}$ and any map $\chi \colon \mathbf{X} \rightarrow k$, there is an isometric copy $\widetilde{\mathbf{X}}$ of $\mathbf{X}$ included in $\mathbf{X}$ on which χ is constant. Otherwise, $\mathbf{X}$ is *divisible*. Our results read as follows.

Theorem 1.3 *Let S be an infinite countable subset of* $]0, +\infty[$ *and assume that the reverse linear ordering* $>$ *on* $\mathbb{R}$ *does not induce a well-ordering on S. Then there is a map* $\chi \colon \mathbf{Q}_S \rightarrow \omega$ *whose restriction on any isometric copy X of* $\mathbf{Q}_S$ *inside* $\mathbf{Q}_S$ *has range* ω.

In particular, in this case, $\mathbf{Q}_S$ is divisible.

Theorem 1.4 *Let S be an infinite countable subset of* $]0, +\infty[$ *and assume that the reverse linear ordering* $>$ *on* $\mathbb{R}$ *induces a well-ordering on S. Then* $\mathbf{Q}_S$ *is indivisible.*

It should be mentioned at this point that we are now aware of the fact that Theorem 1.3 and Theorem 1.4 were first obtained completely independently by Delhommé, Laflamme, Pouzet and Sauer [1] where a precise analysis of divisibility is carried out in the realm of countable metric spaces. In particular, [1] provides a necessary condition for indivisibility and solves the indivisibility problem for several countable Urysohn spaces. For example, every sphere of $\mathbf{U}_0$ is divisible.

We finish with a consequence of Theorem 1.4.

Theorem 1.5 *Let S be an infinite countable subset of* $]0, +\infty[$ *and assume that the reverse linear ordering* $>$ *on* $\mathbb{R}$ *induces a well-ordering on S. Then given any map* $f \colon \mathbf{Q}_S \rightarrow \omega$, *there is an isometric copy X of* $\mathbf{Q}_S$ *inside* $\mathbf{Q}_S$ *such that* f *is continuous or injective on X.*

The paper is organized as follows. In Section 2, we recall the connection between the notions of trees and ultrametric spaces, and use classical Ramsey's theorem to prove Theorem 1.1 and Theorem 1.2. In Section 3, we concentrate on the divisibility properties of $\mathbf{Q}_S$, and prove Theorem 1.3 and Theorem 1.4. Finally, we close in Section 4 with the proof of Theorem 1.5.

2 Big Ramsey Degrees in $\mathcal{U}_S$

The purpose of this section is to provide the proofs for Theorem 1.1 and Theorem 1.2. The ideas we use to reach this goal are not new. The way we met them is through some unpublished work of Galvin, but Milner [7] writes that they were also known to and exploited by several other authors, including Hajnal, who apparently realized first the equivalent of Lemma 2.1 and stated it explicitly [2], and Haddad and Sabbagh [3–5].

Since everything here is connected to the notion of tree, let us start with some general facts about these objects.

A *tree* $\mathbf{T} = (T, <^{\mathbf{T}})$ is a partially ordered set such that given any element $t \in T$, the set $\{s \in T : s <^{\mathbf{T}} t\}$ is $<^{\mathbf{T}}$-well-ordered. Let $e(\mathbf{T})$ denote the set of all linear orderings on T which extend $<^{\mathbf{T}}$. When $\mathbf{U}$ is also a tree, $\mathbf{T}$ and $\mathbf{U}$ are *isomorphic* when there is an order-preserving bijection from $\mathbf{T}$ to $\mathbf{U}$ with order-preserving inverse (in symbols, $\mathbf{T} \cong \mathbf{U}$). When every element of T has finitely many $<^{\mathbf{T}}$-predecessors, the *height of* $t \in \mathbf{T}$ is

$$\mathrm{ht}(t) = |\{s \in T : s <^{\mathbf{T}} t\}|.$$

When S is finite and given by elements $s_0 > s_1 \cdots > s_{|S|-1} > 0$, it will be convenient to see the space $\mathbf{Q}_S$ as the set $\omega^{|S|}$ of maximal nodes of the tree $\omega^{\leqslant |S|} = \bigcup_{i \leqslant n} \omega^i$ ordered by set-theoretic inclusion and equipped with the metric defined for $x \neq y$ by $d(x, y) = s_{\Delta(x,y)}$, where $\Delta(x, y)$ is the height of the largest common predecessor of x and y in $\omega^{\leqslant |S|}$. For $A \subset \omega^{|S|}$, set $A^{\downarrow} = \{a \upharpoonright k : a \in A \wedge k \leqslant n\}$.

It should be clear that when $A, B \subset \omega^{|S|}$, then A and B are isometric if and only if $A^{\downarrow} \cong B^{\downarrow}$. Consequently, when $\mathbf{X} \in \mathcal{U}_S$, one can define the natural tree associated to $\mathbf{X}$ in $\mathcal{U}_S$ to be the unique (up to isomorphism) subtree $\mathbf{T_X}$ of $\omega^{\leqslant |S|}$ such that for any copy $\widetilde{\mathbf{X}}$ of $\mathbf{X}$ in $\mathbf{Q}_S$, $\widetilde{\mathbf{X}}^{\downarrow} \cong \mathbf{T_X}$.

We now introduce some notations for the partition calculus on trees. Given a subtree $\mathbf{T}$ of $\omega^{|S|}$, set

$$\binom{\omega^{\leqslant |S|}}{\mathbf{T}} = \{\widetilde{\mathbf{T}} : \widetilde{\mathbf{T}} \subset \omega^{\leqslant |S|} \wedge \widetilde{\mathbf{T}} \cong \mathbf{T}\}.$$

When $k, l \in \omega \setminus \{0\}$ and for any $\chi \colon \binom{\omega^{\leqslant |S|}}{\mathbf{T}} \to k$ there is $\mathbf{U} \in \binom{\omega^{\leqslant |S|}}{\omega^{\leqslant |S|}}$ such that χ takes at most l values on $\binom{\mathbf{U}}{\mathbf{T}}$, we write $\omega^{\leqslant |S|} \to (\omega^{\leqslant |S|})^{\mathbf{T}}_{k,l}$.

If there is $l \in \omega \setminus \{0\}$ such that for any $k \in \omega \setminus \{0\}$, $\omega^{\leqslant |S|} \to (\omega^{\leqslant |S|})^{\mathbf{T}}_{k,l}$, the least such l is called the *Ramsey degree* of $\mathbf{T}$ in $\omega^{\leqslant |S|}$. The purpose of the following lemma is precisely to compute this Ramsey degree. (Recall that $e(\mathbf{T})$ denotes the set of all linear orderings on T which extend $<^{\mathbf{T}}$.)

Lemma 2.1 *Let $X \subset \omega^{|S|}$ and let $\mathbf{T} = X^{\downarrow}$. Then $\mathbf{T}$ has a Ramsey degree in $\omega^{\leqslant |S|}$ equal to $|e(\mathbf{T})|$.*

Proof Say that a subtree $\mathbf{U}$ of $\omega^{\leqslant |S|}$ is *expanded* when:

(i)   elements of $\mathbf{U}$ are strictly increasing;

(ii)   for every $u, v \in \mathbf{U}$ and every $k \in |S|$,

$$u(k) \neq v(k) \rightarrow (\forall j \geqslant k, \ u(j) \neq v(j)).$$

Note that every expanded $\widetilde{\mathbf{T}} \in \binom{\omega^{\leqslant |S|}}{\mathbf{T}}$ is linearly ordered by $\prec^{\widetilde{\mathbf{T}}}$, where $s \prec^{\widetilde{\mathbf{T}}} t$ if and only if $(s = \varnothing$ or $s(|s|) < t(|t|))$, and that then $\prec^{\widetilde{\mathbf{T}}}$ is a linear extension of the tree ordering on $\widetilde{\mathbf{T}}$.

Now given $\prec \in e(\mathbf{T})$, let $\binom{\omega^{\leqslant |S|}}{\mathbf{T}, \prec}$ denote the set of all expanded $\widetilde{\mathbf{T}} \in \binom{\omega^{\leqslant |S|}}{\mathbf{T}}$ *with type $\prec$*, that is, such that the order-preserving bijection between the linear orderings $(\widetilde{\mathbf{T}}, \prec^{\widetilde{\mathbf{T}}})$ and $(\mathbf{T}, \prec)$ induces an isomorphism between the trees $\widetilde{\mathbf{T}}$ and $\mathbf{T}$. Define the map $\psi_{\prec} \colon \binom{\omega^{\leqslant |S|}}{\mathbf{T}, \prec} \rightarrow [\omega]^{|\mathbf{T}| - 1}$ by $\psi_{\prec}(\widetilde{\mathbf{T}}) = \{t(|t|) : t \in \widetilde{\mathbf{T}} \setminus \{\varnothing\}\}$. Then $\psi_{\prec}$ is a bijection. Call $\varphi_{\prec}$ its inverse map.

Now let $k \in \omega \setminus \{0\}$ and $\chi \colon \binom{\omega^{\leqslant |S|}}{\mathbf{T}} \rightarrow k$. Define $\Lambda \colon [\omega]^{|T| - 1} \rightarrow k^{e(\mathbf{T})}$ by

$$\Lambda(M) = (\chi(\varphi_{\prec}(M)))_{\prec \in e(\mathbf{T})}.$$

By Ramsey's theorem, find an infinite $N \subset \omega$ such that Λ is constant on $[N]^{|\mathbf{T}| - 1}$. Then on the subtree $N^{\leqslant |S|}$ of $\omega^{\leqslant |S|}$, any two expanded elements of $\binom{\omega^{\leqslant |S|}}{\mathbf{T}}$ with same type have the same χ-value. Now let $\mathbf{U}$ be an expanded everywhere infinitely branching subtree of $N^{\leqslant |S|}$. Then $\mathbf{U}$ is isomorphic to $\omega^{\leqslant |S|}$ and χ does not take more than $|e(\mathbf{T})|$ values on $\binom{\mathbf{U}}{\mathbf{T}}$.

To finish the proof, it remains to show that $|e(\mathbf{T})|$ is the best possible bound. To do that, simply observe that for any $\mathbf{U} \in \binom{\omega^{\leqslant |S|}}{\omega^{\leqslant |S|}}$, every possible type appears on $\binom{\mathbf{U}}{\mathbf{T}}$. ∎

This lemma has two direct consequences concerning the existence of big Ramsey degrees in $\mathcal{U}_S$. Indeed, it should be clear that when $\mathbf{X} \in \mathcal{U}_S$, $\mathbf{X}$ has a big Ramsey degree in $\mathcal{U}_S$ if and only if $\mathbf{T_X}$ has a Ramsey degree in $\omega^{\leqslant |S|}$ and that these degrees are equal. So on the one hand, we have the following.

Corollary 2.2 (**Theorem 1.1**)   *Let S be a finite subset of $]0, +\infty[$. Then every element of $\mathcal{U}_S$ has a big Ramsey degree in $\mathcal{U}_S$.*

On the other hand, observe that if $S \subsetneq S'$ are finite and $\mathbf{X} \in \mathcal{U}_S$ has size at least 2, then the big Ramsey degree $T_{\mathcal{U}_{S'}}(\mathbf{X})$ of $\mathbf{X}$ in $\mathcal{U}_{S'}$ is strictly larger than the big Ramsey degree of $\mathbf{X}$ in $\mathcal{U}_S$. In particular, $T_{\mathcal{U}_{S'}}(\mathbf{X})$ tends to infinity when $|S'|$ tends to infinity. This fact has the following consequence.

Corollary 2.3 (**Theorem 1.2**)   *Let S be an infinite countable subset of $]0, +\infty[$ and let $\mathbf{X}$ be in $\mathcal{U}_S$ such that $|\mathbf{X}| \geqslant 2$. Then $\mathbf{X}$ does not have a big Ramsey degree in $\mathcal{U}_S$.*

Proof It suffices to show that for every $k \in \omega \setminus \{0\}$, there is $k' > k$ and a coloring $\chi \colon \binom{\mathbf{Q}_S}{\mathbf{X}} \to k'$ such that for every $Q \in \binom{\mathbf{Q}_S}{\mathbf{Q}_S}$, the restriction of χ on $\binom{Q}{\mathbf{X}}$ has range k'.

Thanks to the previous remark, we can fix $S' \subset S$ finite such that $X \in \mathcal{U}_{S'}$ and the big Ramsey degree k' of $\mathbf{X}$ in $\mathcal{U}_{S'}$ is larger than k. Recall that $\mathbf{Q}_S \subset \omega^S$ so if $\mathbf{1}_{S'} \colon S \to 2$ is the characteristic function of S', it makes sense to define $f \colon \mathbf{Q}_S \to \mathbf{Q}_{S'}$ by $f(x) = \mathbf{1}_{S'}x$.

Observe that $d(f(x), f(y)) = d(x, y)$ whenever $d(x, y) \in S'$. Thus, given any $Q \in \binom{\mathbf{Q}_S}{\mathbf{Q}_S}$, the direct image $f''Q$ of Q under f is in $\binom{\mathbf{Q}_{S'}}{\mathbf{Q}_{S'}}$. Now let $\chi' \colon \binom{\mathbf{Q}_{S'}}{\mathbf{X}} \to k'$ be such that for every $Q' \in \binom{\mathbf{Q}_{S'}}{\mathbf{Q}_{S'}}$, the restriction of χ' to $\binom{Q'}{\mathbf{X}}$ has range k'. Then $\chi = \chi' \circ f$ is as required. ∎

3 Divisibility Properties of $\mathbf{Q}_S$

In this section, we study the divisibility properties of $\mathbf{Q}_S$ and provide the proofs for Theorem 1.3 and Theorem 1.4. Recall that a metric space $\mathbf{X}$ is *indivisible* when given any $k \in \omega$ and any map $\chi \colon \mathbf{X} \to k$, there is an isometric copy $\widetilde{\mathbf{X}}$ of $\mathbf{X}$ on which χ is constant. Otherwise, $\mathbf{X}$ is *divisible*.

Unlike Theorem 1.1 and Theorem 1.2, the proofs here do not use any classical partition result via a particular coding of the objects, but rather take place on the geometrical level.

For notational convenience, we will often simply write d instead of $d^{\mathbf{Q}_S}$.

3.1 Proof of Theorem 1.3

Fix an infinite and countable subset S of $]0, +\infty[$ such that the reverse linear ordering $>$ on $\mathbb{R}$ does not induce a well-ordering on S. The idea to prove that $\mathbf{Q}_S$ is divisible is to use a coloring which is constant on some particular spheres.

More precisely, observe that $(S, >)$ not being well-ordered, there is a strictly increasing sequence $(s_i)_{i \in \omega}$ of reals such that $s_0 = 0$ and $s_i \in S$ for every $i > 0$. Observe that we can construct a subset E of $\mathbf{Q}_S$ such that given any $y \in \mathbf{Q}_S$, there is exactly one x in E such that for some $i < \omega$, $d(x, y) < s_i$. Indeed, if $\sup_{i < \omega} s_i = \infty$, simply take E to be any singleton. Otherwise, let $\rho = \sup_{i < \omega} s_i$ and choose $E \subset \mathbf{Q}_S$ maximal such that $\forall x, y \in E$, $d(x, y) \geqslant \rho$.

To define $\chi \colon \mathbf{Q}_S \to \omega$, let $(A_j)_{j \in \omega}$ be a family of infinite pairwise disjoint subsets of ω whose union is ω. Then for $y \in \mathbf{Q}_S$, let $e(y)$ and $i(y)$ be the unique elements of E and ω respectively such that $d(e(y), y) \in [s_{i(y)}, s_{i(y)+1}[$, and set $\chi(y) = j$ if and only if $i(y) \in A_j$.

Claim 3.1 *χ is as required.*

Proof Let $Y \subset \mathbf{Q}_S$ be isometric to $\mathbf{Q}_S$. Fix $y \in Y$. For every $j \in \omega$, pick $i_j > i(y) + 1$ such that $i_j \in A_j$. Since Y is isometric to $\mathbf{Q}_S$, we can find an element y_j in Y such that $d(y, y_j) = s_{i_j}$. We claim that $\chi(y_j) = j$, or equivalently $i(y_j) \in A_j$. Indeed, consider the triangle $\{e(y), y, y_j\}$. Observe that in an ultrametric space every triangle is isosceles with short base and that here, $d(e(y), y) < s_{i_j} = d(y, y_j)$. Thus,

$d(e(y), y_j) = d(y, y_j) \in [s_{i_j}, s_{i_j+1}[$. And therefore $e(y_j) = e(y)$ and $i(y_j) = i_j \in A_j$. ∎

3.2 Proof of Theorem 1.4

Fix an infinite countable subset S of $]0, +\infty[$ such that the reverse linear ordering $>$ on $\mathbb{R}$ induces a well-ordering on S. Our goal here is to show that the space $\mathbf{Q}_S$ is indivisible.

Observe first that the collection $\mathcal{B}_S$ of metric balls of $\mathbf{Q}_S$ is a tree when ordered by reverse set-theoretic inclusion. When $x \in \mathbf{Q}_S$ and $r \in S$, $B(x, r)$ denotes the set $\{y \in \mathbf{Q}_S : d^{\mathbf{Q}_S}(x, y) \leqslant r\}$, and x is called a *center* of the ball and r a *radius*. Note that in $\mathbf{Q}_S$, non empty balls have a unique radius but admit all of their elements as centers. Note also that when $s > 0$ is in S, the fact that $(S, >)$ is well ordered allows defining $s^- = \max\{t \in S : t < s\}$.

The main ingredients are contained in the following definition and lemma.

Definition 3.2 Let $A \subset \mathbf{Q}_S$ and $b \in \mathcal{B}_S$ with radius $r \in S \cup \{0\}$. Say that *A is small in b* when $r = 0$ and $A \cap b = \varnothing$ or $r > 0$ and $A \cap b$ can be covered by finitely many balls of radius r^-.

We start with an observation. Assume that $\{x_n : n \in \omega\}$ is an enumeration of $\mathbf{Q}_S$, and that we are trying to build inductively a copy $\{a_n : n \in \omega\}$ of $\mathbf{Q}_S$ in A such that for every $n, m \in \omega$, $d(a_n, a_m) = d(x_n, x_m)$. Then the fact that we may be blocked at some finite stage exactly means that at that stage, a particular metric ball b with $A \cap b \neq \varnothing$ is such that A is small in b. This idea is expressed in the following lemma.

Lemma 3.3 *Let $X \subset \mathbf{Q}_S$. The following are equivalent.*

(i) $\binom{X}{\mathbf{Q}_S} \neq \varnothing$.

(ii)   *There is $Y \subset X$ such that Y is not small in b whenever $b \in \mathcal{B}_S$ and $Y \cap b \neq \varnothing$.*

Proof Assume that (i) holds and let Y be a copy of $\mathbf{Q}_S$ in X. Fix $b \in \mathcal{B}_S$ with radius r and such that $Y \cap b \neq \varnothing$. Pick $x \in Y \cap b$ and let $E \subset \mathbf{Q}_S$ be an infinite subset where all the distances are equal to r. Since Y is isometric to $\mathbf{Q}_S$, Y includes a copy $\tilde{E}$ of E such that $x \in \tilde{E}$. Then $\tilde{E} \subset Y \cap b$ and cannot be covered by finitely many balls of radius r^-, so (ii) holds.

Conversely, assume that (ii) holds. Let $\{x_n : n \in \omega\}$ be an enumeration of the elements of $\mathbf{Q}_S$. We are going to construct inductively a sequence $(y_n)_{n \in \omega}$ of elements of Y such that $\forall m, n \in \omega, \ d(y_m, y_n) = d(x_m, x_n)$.

For y_0, take any element in Y. In general, if $(y_n)_{n \leqslant k}$ is built, construct y_{k+1} as follows. Consider the set E defined as $E = \{y \in \mathbf{Q}_S : \forall n \leqslant k, \ d(y, y_n) = d(x_{k+1}, x_n)\}$.

Also let $r = \min\{d(x_{k+1}, x_n) : n \leqslant k\}$ and $M = \{n \leqslant k : d(x_{k+1}, x_n) = r\}$.

We want to show that $E \cap Y \neq \varnothing$. Observe first that for every $m, n \in M$, $d(y_m, y_n) \leqslant r$. Indeed, $d(y_m, y_n) = d(x_m, x_n) \leqslant \max(d(x_m, x_{k+1}), d(x_{k+1}, x_n)) = r$. So in particular, all the elements of $\{y_m : m \in M\}$ are contained in the same ball b of radius r.

Claim 3.4 $E = b \setminus \bigcup_{m \in M} B(y_m, r^-)$.

Proof It should be clear that $E \subset b \setminus \bigcup_{m \in M} B(y_m, r^-)$. On the other hand, let $y \in b \setminus \bigcup_{m \in M} B(y_m, r^-)$. Then for every $m \in M$, $d(y, y_m) = r = d(x_{k+1}, x_m)$. So it remains to show that $d(y, y_n) = d(x_{k+1}, x_n)$ whenever $n \notin M$. To do that, we use again the fact that every triangle is isosceles with short base. Let $m \in M$. In the triangle $\{x_m, x_n, x_{k+1}\}$, we have $d(x_{k+1}, x_n) > r$ so

$$d(x_m, x_{k+1}) = r < d(x_n, x_m) = d(x_n, x_{k+1})$$

Now, in the triangle $\{y_m, y_n, y\}$, $d(y, y_m) = r$ and $d(y_m, y_n) = d(x_m, x_n) > r$. Therefore, $d(y, y_n) = d(y_m, y_n) = d(x_m, x_n) = d(x_{k+1}, x_n)$. ∎

We consequently need to show that $(b \setminus \bigcup_{m \in M} B(y_m, r^-)) \cap Y \neq \varnothing$. To achieve that, simply observe that when $m \in M$, we have $y_m \in Y \cap b$. Thus, $Y \cap b \neq \varnothing$ and by property (ii), Y is not small in b. In particular, $Y \cap b$ is not included in $\bigcup_{m \in M} B(y_m, r^-)$. ∎

We are now ready to prove Theorem 1.4. However, before we do so, let us make another observation concerning the notion smallness. Let $\mathbf{Q}_S = A \cup B$.

Note that if A is small in $b \in \mathcal{B}_S$, then $A \cap b$ cannot contribute to building a copy of $\mathbf{Q}_S$ in A and $B \cap (\mathbf{Q}_S \setminus b)$ is isometric to b. So intuitively, everything happens as if b were completely included in B. So the idea is to remove from A all those parts which are not essential and to see what is left at the end. More precisely, define a sequence $(A_\alpha)_{\alpha \in \omega_1}$ recursively as follows:

- $A_0 = A$.
- $A_{\alpha+1} = A_\alpha \setminus \bigcup \{b : A_\alpha \text{ is small in } b\}$.
- For $\alpha < \omega_1$ limit, $A_\alpha = \bigcap_{\eta < \alpha} A_\eta$.

Since $\mathbf{Q}_S$ is countable, the sequence is eventually constant. Set

$$\beta = \min\{\alpha < \omega_1 : A_{\alpha+1} = A_\alpha\}.$$

Observe that if A_β is non-empty, then A_β is not small in any metric ball it intersects. Indeed, suppose that $b \in \mathcal{B}_S$ is such that A_β is small in b. Then $A_{\beta+1} \cap b = \varnothing$. But $A_{\beta+1} = A_\beta$ so $A_\beta \cap b = \varnothing$. Therefore, since $A_\beta \subset A$, A satisfies condition (ii) of Lemma 3.3 and $\binom{A}{\mathbf{Q}_S} \neq \varnothing$.

It remains to consider the case where $A_\beta = \varnothing$. According to our second observation, the intuition is that A is then unable to carry any copy of $\mathbf{Q}_S$ and is only composed of parts which do not affect the metric structure of B. Thus, B should include an isometric copy of $\mathbf{Q}_S$. For $\alpha < \omega_1$, let $\mathcal{C}_\alpha$ be the set of all minimal elements (in the sense of the tree structure on $\mathcal{B}_S$) of the collection $\{b \in \mathcal{B}_S : A_\alpha \text{ is small in } b\}$. Note that since all points of B can be seen as balls of radius 0 in which A is small, we have $B \subset \bigcup \mathcal{C}_0$. Note also that $(\mathcal{C}_\alpha)_{\alpha < \omega_1}$ is increasing. It follows that

$$(*) \qquad \forall 0 < \alpha < \omega_1, \quad A_\alpha = \mathbf{Q}_S \setminus \bigcup_{\eta < \alpha} \bigcup \mathcal{C}_\eta.$$

Claim 3.5 *Let $\alpha < \omega_1$ and $b \in \mathcal{C}_\alpha$ with radius $r \in S$. Then there are $c_0 \cdots c_{n-1}$ in $\mathcal{B}_S$ with radius r^- and included in b such that $b = \bigcup_{i<n} c_i \cup \bigcup_{\eta < \alpha} \bigcup \{c \in \mathcal{C}_\eta : c \subset b\}$.*

Proof A_α is small in b, so find $c_0 \cdots c_{n-1} \in \mathcal{B}_S$ with radius r^- and included in b such that $A_\alpha \cap b \subset \bigcup_{i<n} c_i$. Then thanks to $(*)$ $b \setminus \bigcup_{i<n} c_i \subset \bigcup_{\eta<\alpha} \bigcup \mathcal{C}_\eta$.

Note that by minimality of b, if $\eta < \alpha$, then $b \subsetneq c$ cannot happen for any element of $\mathcal{C}_\eta$. It follows that either $c \cap b = \varnothing$ or $c \subset b$. Therefore,

$$b \setminus \bigcup_{i<n} c_i \subset \bigcup_{\eta<\alpha} \bigcup \{c \in \mathcal{C}_\eta : c \subset b\}. \qquad \blacksquare$$

Claim 3.6 *Let $\alpha < \omega_1$ and $b \in \mathcal{C}_\alpha$. Then $\binom{B \cap b}{b} \neq \varnothing$.*

Proof We proceed by induction on $\alpha < \omega_1$. For $\alpha = 0$, let $b \in \mathcal{C}_0$. Without loss of generality, we may assume that the radius r of b is strictly positive and hence in S. Then $A_0 = A$ is small in b so we can find $c_0, \ldots, c_{n-1}$ with radius r^- such that $A \cap b \subset \bigcup_{i<n} c_i$. Then $b \setminus \bigcup_{i<n} c_i$ is isometric to b and is included in $B \cap b$.

Suppose now that the claim is true for every $\eta < \alpha$. Let $b \in \mathcal{C}_\alpha$ with radius $r \in S$. Thanks to the previous claim, we can find $c_0 \cdots c_{n-1} \in \mathcal{B}_S$ with radius r^- and included in b such that $b = \bigcup_{i<n} c_i \cup \bigcup_{\eta<\alpha} \bigcup \{c \in \mathcal{C}_\eta : c \subset b\}$,

Let $\mathcal{D}_\alpha$ be defined as the set of all minimal elements (still in the sense of the tree structure on $\mathcal{B}_S$) of the collection $\{c \in \bigcup_{\eta<\alpha} \mathcal{C}_\eta : c \subset b \wedge \forall i < n, \ c \cap c_i = \varnothing\}$. Then $\{c_i : i < n\} \cup \mathcal{D}_\alpha$ is a collection of pairwise disjoint balls and $\bigcup \mathcal{D}_\alpha$ is isometric to b. By induction hypothesis, $\binom{B \cap c}{c} \neq \varnothing$ whenever $c \in \mathcal{D}_\alpha$ and there is an isometry $\varphi_c \colon c \to B \cap c$. Now, let $\varphi \colon \bigcup \mathcal{D}_\alpha \to B \cap b$ be defined as $\varphi = \bigcup_{c \in \mathcal{D}_\alpha} \varphi_c$.

We claim that φ is an isometry. Indeed, let $x, x' \in \bigcup \mathcal{D}_\alpha$. If there is $c \in \mathcal{D}_\alpha$ such that $x, x' \in c$, then $d(\varphi(x), \varphi(x')) = d(\varphi_c(x), \varphi_c(x')) = d(x, x')$. Otherwise, find $c \neq c' \in \mathcal{D}_\alpha$ with $x \in c$ and $x' \in c'$. Observe that since we are in an ultrametric space, we have $\forall y, z \in c \, \forall y', z' \in c', \ d(y, y') = d(z, z')$. Thus, since $x, \varphi(x) \in c$ and $x', \varphi(x') \in c'$, we get $d(\varphi(x), \varphi(x')) = d(x, x')$. $\qquad \blacksquare$

To finish the proof of the theorem, it suffices to notice that as a metric ball (the unique ball of radius $\max S$), $\mathbf{Q}_S$ is in $\mathcal{C}_\beta$. So according to the previous claim, $\binom{B}{\mathbf{Q}_S} \neq \varnothing$, which finishes the proof of Theorem 1.4.

4 An Application of Theorem 1.4

Let $S \subset \,]0, +\infty[$ be infinite and countable such that the reverse linear ordering $>$ on $\mathbb{R}$ induces a well-ordering on S. We saw that $\mathbf{Q}_S$ is then indivisible but that there is no big Ramsey degree for any $\mathbf{X} \in \mathcal{U}_S$ as soon as $|\mathbf{X}| \geqslant 2$. In other words, in the present context, the analogue of infinite Ramsey's theorem holds in dimension 1 but fails for higher dimensions. Still, one may ask if some partition result fitting in between holds. For example, given any $f \colon \mathbf{Q}_S \to \omega$, is there an isometric copy of $\mathbf{Q}_S$ inside $\mathbf{Q}_S$ on which f is constant or injective ? It turns out that the answer is no. To see that, consider a family $(b_n)_{n \in \omega}$ of disjoint balls covering $\mathbf{Q}_S$ whose sequence of corresponding radii $(r_n)_{n \in \omega}$ decreases strictly to 0 and define $f \colon \mathbf{Q}_S \to \omega$ by $f(x) = n$ if and only if $x \in b_n$. Then f is not constant or injective on any isometric copy of $\mathbf{Q}_S$. Observe in fact that f is neither uniformly continuous nor injective on any isometric copy of $\mathbf{Q}_S$. However, if "uniformly continuous" is replaced by "continuous", then the result becomes true. The purpose of this section is to provide a proof of that fact in the general case. The reader will notice the similarities with the proof of Theorem 1.4.

Definition 4.1 Let $f\colon \mathbf{Q}_S \to \omega$, $Y \subset \mathbf{Q}_S$ and $b \in \mathcal{B}_S$ with radius $r > 0$. Say that *f has almost finite range on b with respect to Y* when there is a finite family $(c_i)_{i<n}$ of elements of $\mathcal{B}_S$ with radius r^- such that f has finite range on $Y \cap (b \setminus \bigcup_{i<n} c_i)$.

Lemma 4.2 *Let $f\colon \mathbf{Q}_S \to \omega$ and $Y \subset \mathbf{Q}_S$ such that for every $b \in \mathcal{B}_S$ meeting Y, f does not have almost finite range on b with respect to Y. Then there is an isometric copy of $\mathbf{Q}_S$ included in Y on which f is injective.*

Proof Let $\{x_n : n \in \omega\}$ be an enumeration of the elements of $\mathbf{Q}_S$. Our goal is to construct inductively a sequence $(y_n)_{n\in\omega}$ of elements of Y on which f is injective and such that $\forall m, n \in \omega, \ d(y_m, y_n) = d(x_m, x_n)$.

For y_0, take any element in Y. In general, if $(y_n)_{n\leqslant k}$ is built, construct y_{k+1} as follows. Consider the set E defined as

$$E = \{y \in \mathbf{Q}_S : \forall n \leqslant k, \ d(y, y_n) = d(x_{k+1}, x_n)\}.$$

As in Lemma 3.3, there is $b \in \mathcal{B}_S$ with radius $r > 0$ intersecting Y and a set M such that $E = b \setminus \bigcup_{m\in M} B(y_m, r^-)$.

Since f does not have almost finite range on b with respect to Y, f takes infinitely many values on E and we can choose $y_{k+1} \in E$ such that $\forall n \leqslant k, \ f(y_n) \neq f(y_{k+1})$. ∎

We now turn to a proof of Theorem 1.5. Our strategy here is to define recursively a sequence $(Q_\alpha)_{\alpha\in\omega_1}$ whose purpose is to get rid of all those parts of $\mathbf{Q}_S$ on which f is essentially of finite range:

- $Q_0 = \mathbf{Q}_S$.
- $Q_{\alpha+1} = Q_\alpha \setminus \bigcup\{b : f$ has almost finite range on b with respect to $Q_\alpha\}$.
- For $\alpha < \omega_1$ limit, $Q_\alpha = \bigcap_{\eta<\alpha} Q_\eta$.

  $\mathbf{Q}_S$ being countable, the sequence is eventually constant. Set

$$\beta = \min\{\alpha < \omega_1 : Q_{\alpha+1} = Q_\alpha\}.$$

If Q_β is non-empty, then f and Q_β satisfy the hypotheses of Lemma 4.2. Indeed, suppose that $b \in \mathcal{B}_S$ is such that f has almost finite range on b with respect to Q_β. Then $Q_{\beta+1} \cap b = \varnothing$. But $Q_{\beta+1} = Q_\beta$ so $Q_\beta \cap b = \varnothing$.

Consequently, suppose that $Q_\beta = \varnothing$. The intuition is that on any ball b, f is essentially of finite range. Consequently, we should be able to show that there is $X \in \binom{\mathbf{Q}_S}{\mathbf{Q}_S}$ on which f is continuous.

For $\alpha < \omega_1$, let $\mathcal{C}_\alpha$ be the set of all minimal elements of the collection $\{b : f$ has almost finite range on b with respect to $Q_\alpha\}$. Then

$$(**) \qquad \forall 0 < \alpha < \omega_1, \ Q_\alpha = \mathbf{Q}_S \setminus \bigcup_{\eta<\alpha} \bigcup \mathcal{C}_\eta$$

Claim 4.3 *Let $\alpha < \omega_1$ and $b \in \mathcal{C}_\alpha$. Then there is $\tilde{b} \in \binom{b}{b}$ on which f is continuous.*

Proof We proceed by induction on $\alpha < \omega_1$. For $\alpha = 0$, let $b \in \mathcal{C}_0$; f has almost finite range on b with respect to $Q_0 = \mathbf{Q}_S$ so find $c_0, \ldots, c_{n-1}$ with radius r^- such that f has finite range on $b \setminus \bigcup_{i<n} c_i$. Then $b \setminus \bigcup_{i<n} c_i$ is isometric to b. Now by Theorem 1.4, b is indivisible. Therefore, there is $\tilde{b} \in \binom{b}{b}$ on which f is constant, hence continuous.

Suppose now that the claim is true for every $\eta < \alpha$. Let $b \in \mathcal{C}_\alpha$ with radius $r \in S$. Find $c_0 \cdots c_{n-1} \in \mathcal{B}_S$ with radius r^- and included in b such that f has finite range on $Q_\alpha \cap (b \setminus \bigcup_{i<n} c_i)$. Then $b' := b \setminus \bigcup_{i<n} c_i$ is isometric to b and thanks to $(\ast\ast)$,

$$b' = (b' \cap Q_\alpha) \cup \big(b' \cap \bigcup_{\eta < \alpha} \bigcup \mathcal{C}_\eta \big).$$

For the same reason as in Section 3, if $\mathcal{D}_\alpha$ is defined as the set of all minimal elements of the collection $\{c \in \bigcup_{\eta<\alpha} \mathcal{C}_\eta : c \subset b \wedge \forall i < n, \, c \cap c_i = \varnothing\}$, then we have $b' = (b' \cap Q_\alpha) \cup \bigcup \mathcal{D}_\alpha$.

Thanks to Theorem 1.4, $b' \cap Q_\alpha$ or $\bigcup \mathcal{D}_\alpha$ includes an isometric copy $\tilde{b}$ of b. If $b' \cap Q_\alpha$ does, then for every $i < n$, $c_i \cap \tilde{b}$ is a metric ball of $\tilde{b}$ of same radius as c_i. Thus, $\tilde{b} \setminus \bigcup_{i<n} c_i$ is an isometric copy of b on which f takes only finitely many values and Theorem 1.4 allows us to conclude. Otherwise, suppose that $\bigcup \mathcal{D}_\alpha$ includes an isometric copy of b. Note that $\bigcup \mathcal{D}_\alpha$ includes an isometric copy of itself on which f is continuous. Indeed, by induction hypothesis, for every $c \in \mathcal{D}_\alpha$, there is an isometry $\varphi_c \colon c \to c$ such that f is continuous on the range $\varphi_c'' c$ of φ_c. As in the previous section, $\varphi := \bigcup \mathcal{D}_\alpha \to \bigcup \mathcal{D}_\alpha$ defined as $\varphi = \bigcup_{c \in \mathcal{D}_\alpha} \varphi_c$ is an isometry. Thus, its range $\varphi'' \bigcup \mathcal{D}_\alpha$ is an isometric copy of $\bigcup \mathcal{D}_\alpha$ on which f is continuous. Now since $\bigcup \mathcal{D}_\alpha$ includes an isometric copy of b, so does $\varphi'' \bigcup \mathcal{D}_\alpha$, and we are done. ∎

We conclude with the same argument we used at the end of Theorem 1.4. As a metric ball, $\mathbf{Q}_S$ is in $\mathcal{C}_\beta$. Thus, there is an isometric copy X of $\mathbf{Q}_S$ inside $\mathbf{Q}_S$ on which f is continuous.

5   Concluding Remarks

The Ramsey theory of classes of finite metric spaces and their corresponding ultra-homogeneous objects is far from being fully developed so there is still a lot to investigate in that field. However, the deep connection between ultrametric spaces and trees which is exploited in this article considerably simplifies the combinatorial core of the problem and brings no help out of this specific context. Consequently, new results in the area, for example concerning Euclidean metric spaces, will presumably require the introduction of new techniques. So far the situation is not clear. We hope it will become so soon.

References

[1] C. Delhommé, C. Laflamme, M. Pouzet, and N. Sauer, *Divisibility of countable metric spaces.* European J. Combin. **28**(2007), no. 6, 1746–1769.

[2] P. Erdős and A. Hajnal, *Unsolved problems in set theory.* In: Axiomatic Set Theory. Proc. Sympos. Pure Math.13, American Mathematical Society, Providence, RI, 1971, pp. 17–48.

[3] L. Haddad and G. Sabbagh, *Sur une extension des nombres de Ramsey aux ordinaux.* C. R. Acad. Sci. Paris Sér. A-B **268**(1969), A1165–A1167.

[4] ———, *Calcul de certains nombres de Ramsey généralisés.* C. R. Acad. Sci. Paris Sér. A-B **268**(1969), A1233–A1234.

[5] ———, *Nouveaux résultats sur les nombres de Ramsey généralisés.* C. R. Acad. Sci. Paris Sér. A-B **268**(1969), A1516-A-1518.

[6] A. S. Kechris, V. Pestov, and S. Todorcevic, *Fraïssé limits, Ramsey theory, and topological dynamics of automorphism groups.* Geom. Funct. Anal. **15**(2005), no. 1, 106–189.

[7] E. C. Milner, *A finite algorithm for the partition calculus.* In: Proceedings of the Twenty-Fifth Summer Meeting of the Canadian Mathematical Congress. Lakehead University, Thunder Bay, ON, 1971, pp. 117–128.

[8] L. Nguyen Van Thé, *Ramsey degrees of finite ultrametric spaces, ultrametric Urysohn spaces and dynamics of their isometry groups.* European J. Combin., to appear.

[9] P. Urysohn, *Sur un espace métrique universel.* Bull. Sci. Math. **51**(1927). 43–64, 74–90.

Equipe de Logique Mathématique, UFR de Mathématiques, Université Denis Diderot Paris 7, 75251 Paris Cedex 05, France
e-mail: nguyenl@logique.jussieu.fr