8

Gravity and cosmology of domain walls

Domain walls resulting from a symmetry breaking in the early universe could have novel and dramatic gravitational and cosmological consequences.

We first derive the gravitational effects of a planar domain wall, describing the different ways to view the system. Then we discuss spherical walls as an example of curved domain walls. To discuss the cosmological consequences, it is necessary to have a picture of domain wall formation in the cosmological context. With the background of Chapter 6 we discuss the formation of the wall network in cosmology, then the evolution and cosmological implications. We end by reviewing the cosmological constraints on domain walls and the few possible ways around the constraints.

8.1 Energy-momentum of domain walls

The energy-momentum tensor for a scalar field with potential $V(\phi)$ is given in Eq. (1.39)

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\left\{\frac{1}{2}(\partial_\alpha\phi)^2 - V(\phi)\right\} \tag{8.1}$$

In the thin-wall limit, varying the Nambu-Goto action (Eq. (7.15)) gives the energy-momentum tensor

$$T^{\mu\nu}\Big|_{\rm NG} = \frac{\sigma}{\sqrt{-g}}\int d^3\rho\sqrt{|h|}\, h^{ab}\partial_a X^\mu\partial_b X^\nu\ \delta^{(4)}(x^\mu - X^\mu) \tag{8.2}$$

where X^μ is the location of the wall. For a planar wall located at $x = 0$ in flat space-time, this gives

$$T^{\mu\nu}\Big|_{\rm NG,plane} = \sigma(1, 0, -1, -1)\delta(x) \tag{8.3}$$

https://doi.org/10.1017/9781009290456.009 Published online by Cambridge University Press

For a planar wall, including self-gravity, the energy-momentum tensor can be explicitly written once we have chosen a suitable ansatz for the metric (see Eq. (8.4) below).

8.2 Gravity: thin planar domain walls

The gravitational effects of a planar domain wall have been found in the thin-wall limit in [167, 169, 80] using the vacuum solutions found in [154]. The thin-wall limit simplifies the analysis because then there is no need to solve the field equations of motion. All the energy-momentum is localized on the thin domain wall and so only the vacuum Einstein equations need to be solved on either side of the wall. The presence of the wall shows up in matching the vacuum solutions on the two sides of the wall i.e. implementing the "junction conditions." Such a matching is facilitated by using the Gauss-Codazzi formalism [81] and this has been done in [80]. Here we derive the metric of a domain wall without going through the general Gauss-Codazzi formalism, following the derivation in [169] instead.

A planar domain wall located in the $x = 0$ plane has rotational symmetry in this plane. Further we expect space-time symmetry under $x \to -x$. Under these conditions the form of the line element can be taken to be [154]

$$\mathrm{d}s^2 = \mathrm{e}^{2u}(+\mathrm{d}t^2 - \mathrm{d}x^2) - \mathrm{e}^{2v}(\mathrm{d}y^2 + \mathrm{d}z^2) \tag{8.4}$$

where u and v are functions of t and $|x|$. Note that the possibility that the metric is time-dependent has been retained.

In the thin-wall limit, there is no energy-momentum off the wall and so the energy-momentum tensor, $T_{\mu\nu}$, vanishes everywhere except on the wall. Therefore only the vacuum Einstein equations, $R_{\mu\nu} = 0$, where $R_{\mu\nu}$ is the Ricci tensor, need be solved. The solution for $x > 0$ is

$$\mathrm{e}^{2v} = f(t + x) + g(x - t) \tag{8.5}$$

$$u = -\frac{1}{4}\ln(f + g) + h(x + t) + k(x - t) \tag{8.6}$$

where the functions f, g, h, and k satisfy

$$f'' - g'' - 2f'h' + 2g'k' = 0 \tag{8.7}$$

$$f'' + g'' - 2f'h' - 2g'k' = 0 \tag{8.8}$$

where primes denote derivatives with respect to x. The solution for $x < 0$ can be obtained by symmetry since u and v are functions of $|x|$.

Next we solve the Einstein equations, $T_{\mu\nu} = G_{\mu\nu}/8\pi G$, where $G_{\mu\nu}$ is the Einstein tensor calculated for the metric in Eq. (8.4). This leads to

$$\begin{aligned} T_0^0 &= \frac{1}{4\pi G} v_0' \mathrm{e}^{-2u_0} \delta(x) \\ T_1^1 &= 0 \\ T_2^2 &= T_3^3 = -\frac{1}{8\pi G}(u_0' + v_0')\mathrm{e}^{-2u_0}\delta(x) \end{aligned} \tag{8.9}$$

where $u_0 = u(t, x = 0+)$, $v_0 = v(t, x = 0+)$.

In general, u_0, u_0', and v_0' are time-dependent, and so these expressions for T_ν^μ are also time-dependent. However, the energy-momentum tensor for the wall should be time-independent. This gives us the constraint that the functions f, g, h, and k must be chosen so that u_0, u_0', and v_0' are time-independent. Then the only possible choice for the functions (for $x > 0$) that also satisfy Eq. (8.8) is

$$\begin{aligned} &f = 0, \qquad g = \mathrm{e}^{K(t-x)} \\ &h = -\frac{K}{4}(t + x), \qquad k = \frac{K}{2}(t - x) \end{aligned} \tag{8.10}$$

where

$$K = 4\pi G\sigma \tag{8.11}$$

The corresponding functions for $x < 0$ are

$$\begin{aligned} &f = \mathrm{e}^{K(t+x)}, \qquad g = 0 \\ &h = \frac{K}{2}(t + x), \qquad k = -\frac{K}{4}(t - x) \end{aligned} \tag{8.12}$$

Then the domain wall line element is

$$\mathrm{d}s^2 = \mathrm{e}^{-K|x|}[\mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{e}^{Kt}(\mathrm{d}y^2 + \mathrm{d}z^2)] \tag{8.13}$$

which can also be put in the commonly encountered form

$$\mathrm{d}s^2 = (1 - \kappa|X|)^2\mathrm{d}t^2 - \mathrm{d}X^2 - (1 - \kappa|X|)^2\mathrm{e}^{2\kappa t}(\mathrm{d}y^2 + \mathrm{d}z^2) \tag{8.14}$$

where $\kappa = 2\pi G\sigma$ via the coordinate transformation

$$|X| = \frac{1}{\kappa}(1 - \mathrm{e}^{-\kappa|x|}) \tag{8.15}$$

8.3 Gravitational properties of the thin planar wall

On spatial slices of constant X ($X = X_0$) the metric of Eq. (8.14) takes the form

$$\mathrm{d}s_3^2 = \mathrm{d}\bar{t}^2 - \mathrm{e}^{2\bar{\kappa}\bar{t}}(\mathrm{d}\bar{y}^2 + \mathrm{d}\bar{z}^2) \tag{8.16}$$

https://doi.org/10.1017/9781009290456.009 Published online by Cambridge University Press

where overbars denote that the coordinates have been rescaled by the factor $(1-\kappa|X_0|)$ and $\bar{\kappa}=\kappa/(1-\kappa|X_0|)$. The three-dimensional line element of Eq. (8.16) shows that space-like slices of constant X are expanding exponentially fast, just as in an inflationary space-time.

The inflationary nature of the metric can be understood from the viewpoint of an observer living on the wall who is blind to the coordinate normal to the wall. From such an observer's perspective, the space-time is filled with vacuum energy, as given by the energy-momentum tensor of Eq. (8.3), and hence is inflating.

Next we examine the metric on spatial slices obtained by setting $y=y_0$, $z=z_0$

$$\mathrm{d}s^2=(1-\kappa|X|)^2\mathrm{d}t^2-\mathrm{d}X^2 \tag{8.17}$$

This is the metric of $1+1$ dimensional Rindler space-time, which is Minkowski space-time written in the rest frame coordinates of a uniformly accelerated observer with acceleration $a=1/\kappa$ *away* from the wall which is located at $X=0$. To see this, use the coordinate transformation

$$\begin{aligned}\tau&=\frac{(1-\kappa|X|)}{2\kappa}(\mathrm{e}^{\kappa t}-\mathrm{e}^{-\kappa t})\\ \xi&=\frac{(1-\kappa|X|)}{2\kappa}(\mathrm{e}^{\kappa t}+\mathrm{e}^{-\kappa t})\end{aligned} \tag{8.18}$$

In these coordinates the Rindler line element is of Minkowski form

$$\mathrm{d}s^2=\mathrm{d}\tau^2-\mathrm{d}\xi^2 \tag{8.19}$$

Now note that

$$\xi^2-\tau^2=\left(\frac{1}{\kappa}-|X|\right)^2 \tag{8.20}$$

Therefore the world line of a particle at fixed X is a hyperboloid in Minkowski space-time, which describes a particle moving at constant acceleration. In particular, the wall located at $X=0$ has acceleration $1/\kappa$. Therefore an inertial observer sees the wall accelerating away with acceleration $1/\kappa$. From the perspective of an observer on the wall, all particles are repelled from the wall.

In the Rindler space metric there is a horizon at $|X|=1/\kappa$. It is clear from the coordinate transformation given above, this is a coordinate singularity since the space-time is equivalent to Minkowski space-time.

As discussed in [80] the full domain wall metric in Eq. (8.14) can also be brought to Minkowski form. This shows explicitly that the domain wall space-time is flat everywhere except on the wall itself. As in the reduced metric of Eq. (8.17), in the

Minkowski coordinates $(t_{\rm M}, x_{\rm M}, y_{\rm M}, z_{\rm M})$ the wall is located at (see Eq. (8.20))

$$x_{\rm M}^2 + y_{\rm M}^2 + z_{\rm M}^2 = t_{\rm M}^2 + \frac{1}{\kappa^2} \tag{8.21}$$

Hence, in the coordinates where the metric is Minkowski, the wall is spherical with time-dependent radius that decreases (for $t_{\rm M} < 0$) until it gets to $1/\kappa$ at $t_{\rm M} = 0$ and then bounces back. This behavior does not depend on which side of the wall the observer is located. Both see the wall accelerating away from them with constant acceleration $1/\kappa$.

There is an intuitive way to see that the wall's gravity must be repulsive. In the weak field approximation, the gravitational potential of the wall is proportional to $\rho + p_1 + p_2 + p_3$ where ρ is the energy density of the wall and p_i are the pressure components of the energy-momentum tensor. From the energy-momentum tensor in Eq. (8.3) we have $p_2 = p_3 = -\rho$ and $p_1 = 0$. Therefore $\rho + p_1 + p_2 + p_3 = -\rho < 0$ instead of the positive value obtained for matter without pressure. Therefore the gravitational potential is repulsive instead of being attractive.

Since the metric is Minkowski in the $(t_{\rm M}, x_{\rm M}, y_{\rm M}, z_{\rm M})$ coordinates, geodesics are given by

$$x_{\rm M}^\mu(t_{\rm M}) = x_0^\mu + u^\mu\,(t_{\rm M} - t_0) \tag{8.22}$$

where, x_0^μ is the position of the particle at time $t_{\rm M} = t_0$, and u^μ is the (constant) velocity vector.

8.4 Gravity: thick planar wall

Here we consider the gravitational field of a *thick* domain wall i.e. taking both the scalar field and Einstein equations into account.

The Einstein equations are

$$\begin{aligned} G_{\mu\nu} &= 8\pi G T_{\mu\nu} \\ &= 8\pi G\left[\partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\left\{\frac{1}{2}(\partial_\alpha\phi)^2 - V(\phi)\right\}\right] \end{aligned} \tag{8.23}$$

where we have used $T_{\mu\nu}$ from Eq. (8.1). The scalar field equation is

$$\nabla_\mu\nabla^\mu\phi + V'(\phi) = 0 \tag{8.24}$$

where ∇_μ is the covariant derivative.

These equations have been solved in [181] for the case when $16\pi G\eta^2 \ll 1$ where η is the vacuum expectation value of the field ϕ. The line element outside the thick wall is still given by Eq. (8.14) and there are no qualitative new effects.

https://doi.org/10.1017/9781009290456.009 Published online by Cambridge University Press

The case when $16\pi G\eta^2 > 1$, however, does lead to new effects as first discussed in [170, 101, 102] and as summarized in the next section.

8.5 Topological inflation

If $16\pi G\eta^2 > 1$, the gravitational forces within the wall are stronger than the forces associated with the self-interaction of the scalar field. This can be seen by the following heuristic argument [170].

Consider the Z_2 model with the quartic potential

$$V(\phi) = \frac{\lambda}{4}(\phi^2 - \eta^2)^2 \tag{8.25}$$

The thickness of the domain wall can be estimated by equating gradient and potential energies, which also agrees with the Bogomolnyi equation (see Eq. (1.31)), in the case when gravitational effects are ignored. The field ϕ gets an expectation value η and so, in the interior of the domain wall,

$$\frac{1}{2}(\nabla\phi)^2 = \frac{\eta^2}{2\delta^2} \sim V(0) = \frac{\lambda}{4}\eta^4 \tag{8.26}$$

and the thickness, δ, is

$$\delta \sim \sqrt{\frac{2}{\lambda\eta}} \tag{8.27}$$

This is an estimate of the length scale on which the scalar field interactions are working.

Next, the length scale associated with gravitational effects is found from the Friedman-Robertson-Walker equation, which relates the space-time expansion rate, H, to the energy density

$$H^2 \sim \frac{8\pi G}{3}\rho \tag{8.28}$$

which, when used inside the wall with $\rho \sim \lambda\eta^4/2$, gives

$$H^{-1} \sim \sqrt{\frac{3}{4\pi G\lambda}}\frac{1}{\eta^2} \tag{8.29}$$

Hence scalar field forces dominate over gravitational forces inside the domain wall if $H^{-1} > \delta$, or the order of magnitude condition,

$$16\pi G\eta^2 < 1 \tag{8.30}$$

Therefore when $16\pi G\eta^2 < 1$ we can expect that gravitational effects are small in the interior of the domain wall. If, however, $H^{-1} < \delta$, the field is approximately

https://doi.org/10.1017/9781009290456.009 Published online by Cambridge University Press

smooth over a region where gravitational effects are strong. The field inside the domain wall has potential energy $\sim \lambda\eta^4$ and this is what drives the gravitational effects. Therefore, we expect that the space-time inside the domain wall inflates in the direction normal to the wall, in addition to the inflation parallel to the wall that we have already seen in the thin-wall case (Eq. (8.14)). Furthermore, the field inside the wall is stuck on top of the potential owing to the topology that led to the existence of the wall. So the inflation goes on forever for topological reasons. Hence this inflation is called "topological inflation."

This picture has been confirmed by numerical solution of the coupled scalar field and Einstein equations in [131, 31, 94] with the conclusion that topological inflation inside the Z_2 domain wall occurs for $\eta > 0.33 m_{\rm P}$ where $m_{\rm P}$ is the Planck mass defined by $G = 1/m_{\rm P}^2$.

Domain walls that are undergoing topological inflation cannot however form in the usual way during a cosmological phase transition as we discuss in Section 8.9 below.

8.6 Spherical domain wall

The metric of a thin spherical domain wall has been discussed in the thin-wall limit in [80]. Inside the wall the metric is flat using Birkhoff's theorem (e.g. see [177])

$$ds^2 = dT^2 - dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad r < R(t) \tag{8.31}$$

where $R(t)$ is the radius of the spherical wall and

$$\dot{T} = (1 + \dot{R}^2)^{1/2} \tag{8.32}$$

with overdots denoting derivatives with respect to the proper time of an observer moving with the domain wall. The proper time is related to the time coordinate t via the relation

$$\left(1 - \frac{2GM}{R}\right)\dot{t} = \left(1 - \frac{2GM}{R} + \dot{R}^2\right)^{1/2} \tag{8.33}$$

Outside the sphere, the metric is Schwarzschild with mass parameter M

$$ds^2 = \left(1 - \frac{2GM}{r}\right)dt^2 - \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad r > R(t) \tag{8.34}$$

The mass is related to the maximum radius of the spherical wall, $R_{\rm m}$, by

$$M = 4\pi\sigma R_{\rm m}^2(1 - 2\pi G\sigma R_{\rm m}) \tag{8.35}$$

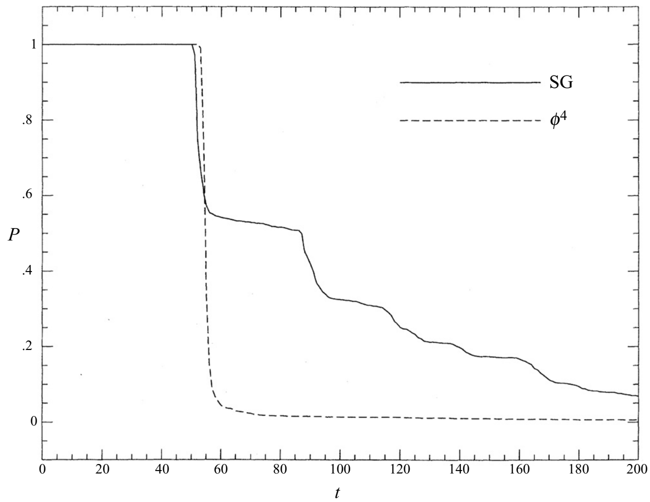

Figure 8.1 Energy in a large volume enclosing a collapsing spherical domain wall in the sine-Gordon and Z_2 models [183] as a function of time. The energy is roughly conserved until the radius becomes comparable to the wall thickness and then decreases sharply. The step-like features in the sine-Gordon models occur because the sphere bounces several times before annihilating. The spherical wall in the Z_2 model annihilates without bouncing. [Figure reprinted from [183].]

provided $R_\mathrm{m} < 1/4\pi G\sigma$. If $R_\mathrm{m} > 1/4\pi G\sigma$, it means that the spherical domain wall is a black hole even at the maximum value of its radius and the analysis breaks down.

8.7 Scalar and gravitational radiation from domain walls

A collapsing spherical domain wall emits scalar radiation and loses energy. It may be possible to extend the formalism in Section 3.5 to calculate this energy loss. However, such an analysis is not currently available. Instead the energy emission rate has been found numerically and is shown in Fig. 8.1 for spherical walls in the sine-Gordon and Z_2 models [183].

A collapsing spherical domain wall does not emit gravitational radiation since the spherical symmetry implies a vanishing quadrupole moment of the energy-momentum distribution. However, colliding domain walls can lead to gravitational [157] and scalar radiation [175]. A dimensional analysis based on the quadrupole approximation for the gravitational power emitted when two relativistic spherical walls collide gives [157]

$$P_\mathrm{g} \sim \frac{GM_\mathrm{B}^2}{R^2} \tag{8.36}$$

https://doi.org/10.1017/9781009290456.009 Published online by Cambridge University Press

where $M_B \sim 4\pi\sigma R^2$ is the mass of the bubble and R is the radius upon collision. Numerical analyses of bubble collisions (during first-order phase transitions) found that the quadrupole approximation overestimates the power radiated in gravitational radiation by about a factor of 50 [91].

8.8 Collapse into black holes

If the radius, $R(t)$ of a collapsing spherical domain wall remains larger than the Schwarzschild radius, $R_S = 2GM$, where M is the mass of the domain wall, then the domain wall does not become a black hole. As the wall collapses, it emits scalar radiation and, if this is rapid enough, M decreases sufficiently rapidly so that $R_S < R$ at all times. Whether this happens can be checked explicitly by numerical evolution of the scalar field plus Einstein equations. We expect that if the Schwarzschild radius of the spherical domain wall is smaller than the width of the wall, black holes are not formed since rapid wall annihilation and radiation precede collapse to within the Schwarzschild radius.

The converse case of black hole formation in the case when the scalar radiation is not too rapid is harder to demonstrate convincingly. The reason is that the time evolution of the fields gets slower as the black hole event horizon is about to form. By simply evolving the fields, it is impossible to see the formation of the event horizon and hence conclude that the domain wall collapses to form a black hole. However, it is hard to imagine any other outcome, especially since the scalar radiation rate only becomes significant once the spherical domain wall collapses to a size comparable to the thickness of the wall.

The collapse of a slightly perturbed spherical domain wall has been studied numerically in [182] with the result that the amplitude of perturbations stays constant during the collapse. This means that the ratio of the perturbation amplitude to the radius grows during collapse as $1/R(t)$ and the shape of the wall deviates increasingly from being spherical.

8.9 Cosmological domain walls: formation

The formation of domain walls in a phase transition in flat non-expanding spacetime has been discussed in Chapter 6. Since the universe is expanding and cooling, cosmic phase transitions can occur, just as in the laboratory, and domain walls can also form. If the phase transition proceeds quickly on cosmological time scales, the structure of these domain wall networks is similar to those formed in the laboratory and described in Chapter 6. The network is dominated by one infinite domain wall with very complicated topology. However, if the phase transition occurs slowly on cosmological time scales, the expansion can prevent the phase transition from completion. For example, in a first-order phase transition, if the bubble nucleation rate is very slow, the bubbles will not be able to percolate because the expansion increases the separation of the bubbles that have already nucleated. These considerations are important for inflationary cosmology but here we will assume that the phase transition completes since otherwise domain walls would not be formed.

In a model with $16\pi G\eta^2 > 1$ (η is the vacuum expectation value of the scalar field), domain wall formation requires some new considerations [22, 166]. The reason is that the energy density inside the domain walls is larger than that outside. Hence if such inflating domain walls (see Section 8.5) were to form, the space-time expansion rate within them would be greater than that of the ambient cosmological expansion rate in which they were created. It is possible to show that a faster expanding region within a horizon of a slower expanding region can be created only if the null energy condition[1] is violated. The formation of defects proceeds according to the classical dynamics of a scalar field with energy-momentum tensor given by Eq. (8.1). Contracting the energy-momentum tensor twice with a null vector, N^μ, and using $g_{\mu\nu}N^\mu N^\nu = 0$ gives

$$N^\mu N^\nu T_{\mu\nu} = (N^\mu \partial_\mu \phi)^2 \geq 0 \tag{8.37}$$

Hence the null energy condition is satisfied during defect formation and there is an obstruction to the formation of topologically inflating domain walls. The exception is if the faster expanding region has an extent that is larger than the cosmological horizon. In this situation, the domain wall is fatter than the horizon during the phase transition. Then the particle interaction rate is also slower than the Hubble expansion rate and the particles are not in a thermal state unless they were set up in that state as an initial condition at the Big Bang. The domain wall network that is produced will depend on the initial state of the particles.

8.10 Cosmological domain walls: evolution

If we assume that there is a dense network of walls within our cosmological horizon and that the network does not lose a significant amount of energy to radiation, we can work out the expansion rate of the universe and the scaling of the density of walls.

The energy-momentum of the scalar field that forms the domain walls is given in Eq. (8.1). If we denote an average over a large volume by $\langle\cdot\rangle$ we have

$$\langle T_{\mu\nu}\rangle = \langle \partial_\mu\phi\partial_\nu\phi\rangle - \left\langle g_{\mu\nu}\left\{\frac{1}{2}(\partial_\alpha\phi)^2 - V(\phi)\right\}\right\rangle \tag{8.38}$$

[1] The null energy condition is $N^\mu N^\nu T_{\mu\nu} > 0$ where N^μ is any null vector and $T_{\mu\nu}$ is the energy-momentum tensor. For fluids with energy density ρ and isotropic pressure p, the null energy condition is $\rho + p > 0$.

We assume that $g_{\mu\nu}$ is a background metric and only dependent on time. Also the field distribution is assumed to be isotropic so that

$$\langle(\partial_x\phi)^2\rangle = \langle(\partial_y\phi)^2\rangle = \langle(\partial_z\phi)^2\rangle \tag{8.39}$$

and

$$\langle\partial_i\phi\partial_j\phi\rangle = 0, \quad i \neq j \tag{8.40}$$

Define

$$\langle\phi'^2\rangle = \frac{1}{3}\langle(\partial_x\phi)^2 + (\partial_y\phi)^2 + (\partial_z\phi)^2\rangle \tag{8.41}$$

If we further assume that the field is dominantly in the form of domain walls that satisfy Eq. (1.31) to a good approximation, we get

$$\langle\phi'^2\rangle = \frac{2}{3}\langle V\rangle \tag{8.42}$$

which leads to

$$\langle T_{xx}\rangle = \frac{5}{6}\langle\dot{\phi}^2\rangle - \frac{2}{3}\langle T_{tt}\rangle \tag{8.43}$$

For slowly varying fields this leads to the effective equation of state $p = -2\rho/3$ where p is the (isotropic) pressure and ρ the energy density [186]. If we assume that the time dependence of ϕ is only due to a boost of the domain walls, we can use $\dot{\phi} = v\gamma\partial_X\phi = v\gamma\sqrt{2V(\phi)}$ and $\partial_x\phi = \gamma\partial_X\phi$, where $X = \gamma(x - vt)$ and γ is the Lorentz factor (see Eq. (1.10)). This leads to

$$\langle T_{xx}\rangle = \left(\langle v^2\rangle - \frac{2}{3}\right)\langle T_{tt}\rangle \tag{8.44}$$

Following Appendix F and treating the wall network as a fluid with equation of state $p = -2\rho/3$, we can write down the solutions for the scale factor and the scaling of the energy density in walls. If the initial conditions are such that the wall density is ρ_0 when the scale factor is a_0, the solution is

$$\rho_{\text{walls}}(a) = \rho_0\frac{a_0}{a}, \qquad a(t) = a_0\left(\frac{t}{t_0}\right)^2 \tag{8.45}$$

Note that this derivation ignores processes by which the wall network could lose energy into scalar and gravitational radiation. In addition, the walls interact with surrounding matter and experience friction. These effects make the problem of understanding the evolution of the wall network much more challenging. We now describe some numerical [125, 36, 97, 59] and analytical [77, 78] efforts to understand the evolution of the network.

8.11 Evolution: numerical results

There are two numerical schemes for evolving a network of domain walls. The first is to use the zero thickness approximation for walls. In this approximation, it is hard to treat the collision of walls and the loss of energy from the network into radiation. The second approach is to solve the field theory equations of motion. In this approach, all the degrees of freedom of the system are retained. In fact, a lot of degrees of freedom that are evolved are inessential to the domain wall network and this additional baggage slows down the simulations. In an expanding universe the problem is even more severe because the overall length scales grow larger with time while the domain wall thickness remains the same. Thus the simulation needs to handle very disparate length scales.

In [125, 36, 97, 59], the authors get around these problems by solving the field theory equations of motion but by letting the domain walls expand with the universe.

More specifically, consider the Z_2 model in an expanding space-time with metric

$$g_{\mu\nu} = a^2(\tau)\eta_{\mu\nu} \tag{8.46}$$

where $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$ and τ is the conformal time. The equation of motion is

$$\partial_\tau^2\phi + 2\frac{\dot{a}}{a}\partial_\tau\phi - \nabla^2\phi + \lambda(\phi^2 - a^2\eta^2)\phi = 0 \tag{8.47}$$

In the approach pioneered in [125] the $a^2\eta^2$ in the last term is replaced by a constant, effectively decreasing the vacuum expectation value, η, with Hubble expansion. Since the width of the domain wall is proportional to $1/\eta$, this amounts to letting the thickness of the walls grow in proportion to the scale factor.

The result of this numerical study shows that the areal density, $\mathcal{A}$, i.e. area of walls in a given region divided by the volume of the region, scales inversely as the first power of conformal time

$$\mathcal{A} = \mathcal{A}_0\left(\frac{\tau_0}{\tau}\right)^p, \quad p \approx 1 \tag{8.48}$$

where the subscript 0 refers to some initial time. This result holds in Minkowski space-time ($a \propto \tau^0$), radiation-dominated ($a \propto \tau^{1/2}$), and matter-dominated ($a \propto \tau^{1/3}$) cosmologies.

The domain wall network has also been studied by a combination of numerical and analytical techniques that use scaling arguments [13, 14].

8.12 Evolution: analytical work

An analytic technique to study the evolution of non-relativistic interfaces in the condensed matter context was developed in [116] (also see [24, 65, 111]). The technique has been extended to relativistic systems in [77, 78] and we now summarize the main features of this analysis.

The starting point is to define a fictitious scalar field $u(x^\mu)$ such that it vanishes on the domain wall network

$$u(X^\mu(\sigma^a)) = 0, \quad a = 0, 1, 2 \tag{8.49}$$

where the domain wall network is located at $X^\mu(\sigma^a)$ and σ^a denote world-volume coordinates. While $u(x^\mu)$ could have been taken to be the scalar field in the original field theory (say for the Z_2 model), this is not suitable since u is later assumed to be a random field with a Gaussian distribution. The next step is to derive an equation of motion for u.

We define the domain wall world-volume metric as in Eq. (7.16)

$$h_{ab} = g_{\mu\nu}(X)\partial_a X^\mu \partial_b X^\nu \tag{8.50}$$

where $g_{\mu\nu}$ is the ambient space-time metric and the indices a, b refer to world-volume coordinates. Two derivatives of Eq. (8.49) lead to

$$\frac{1}{\sqrt{|h|}}\partial_a(\sqrt{|h|}h^{ab}\partial_b X^\mu)\partial_\mu u + h^{ab}\partial_a X^\mu \partial_b X^\nu \partial_\mu \partial_\nu u = 0 \tag{8.51}$$

As long as the thin-wall limit is valid and, in particular, walls do not intersect, X^μ satisfies the Nambu-Goto equation. When walls do intersect, the Nambu-Goto formalism breaks down. The formalism can continue to be valid provided we impose additional boundary conditions by hand at the intersection point. Depending on the boundary conditions that one imposes at the intersection point, the Nambu-Goto equation can describe intercommuting walls or walls that pass through each other. In the present formalism, the boundary conditions automatically arise from the evolution of the u field. The dynamics of the u field are such that they always describe walls that intercommute [77]. Using the Nambu-Goto equations of motion, Eq. (7.21), then leads to the equation of motion for the fictitious field u

$$[(\partial u)^2 g^{\mu\nu} - \partial^\mu u \partial^\nu u]\big(\partial_\mu \partial_\nu u - \Gamma^\rho_{\mu\nu}\partial_\rho u\big) = 0 \tag{8.52}$$

where $\Gamma^\rho_{\mu\nu}$ is the Christoffel symbol defined in Eq. (7.22).

To solve Eq. (8.52) we must find a way to handle the non-linear terms. The key point now is that the domain wall network contains a random distribution of walls and hence u is a statistical field. One approach to treat the non-linear terms is to use the mean field approximation. In this approach non-linear terms are replaced by averages of non-linear terms multiplied by a single power of u. For example

$$u^3 \to \langle u^2 \rangle u \tag{8.53}$$

Further, the distribution of u is assumed to be Gaussian.

After defining the correlators that enter the mean field theory version of Eq. (8.52), the field u satisfies the equation of motion

$$\partial_\tau^2 u + \frac{\mu(\tau)}{\tau}\partial_\tau u - v^2\nabla^2 u = 0 \tag{8.54}$$

where, as in the previous section, τ is the conformal time, The functions μ and v^2 are defined in terms of the assumed two-point correlation functions of u (for details see [77, 78]). Once the solution for u is obtained from the linear differential equation, Eq. (8.54), the average areal density, $\mathcal{A}$, and other quantities may be calculated. The results agree with the scaling in Eq. (8.48).

8.13 Cosmological constraints

The cosmological constraint on domain walls is remarkably robust, being almost independent of the field theory, details of the phase transition, and cosmology [186]. At any time after the domain wall forming phase transition, the vacua in different cosmological horizons are uncorrelated. This means that there is at least one domain wall per horizon. The minimum area of a horizon size domain wall is $\sim H^{-2}$ where H^{-1} is the horizon size. Therefore the domain wall energy density averaged over a horizon volume is $\rho_{\rm walls} \sim \sigma H$. Comparing this to the critical density of the universe,[2] we get

$$\Omega_{\rm walls} \equiv \frac{\rho_{\rm walls}}{\rho_{\rm c}} \sim \frac{G\sigma}{H} \sim G\sigma t \tag{8.55}$$

where t is the cosmic time. (We have taken $H \sim 1/t$ which is true in a Friedman-Robertson-Walker cosmology in which $a(t) \propto t^\alpha$ with $0 < \alpha < 1$.) Hence, as time proceeds, there comes an epoch when the domain walls are the dominant form of energy in the universe. This happens at time t_* given by

$$t_* \sim \frac{1}{G\sigma} \tag{8.56}$$

Now $\sigma \sim \eta^3$ (e.g. Eq. (1.20)) up to factors of coupling constants which we assume are order unity. We also know particle physics fairly well up to an energy scale of about 100 GeV (approximately the electroweak scale) and have not seen any scalar fields yet. So the minimum value of σ is about $(100\ {\rm GeV})^3$. Walls of this tension would have started dominating the universe at (see Appendix A for numerical values)

$$t_*\Big|_{\rm min} \sim \frac{m_{\rm P}^2}{\eta^3} \sim 10^8\ {\rm s} \tag{8.57}$$

[2] The critical density of the universe is defined as $\rho_{\rm c} = 3H^2/8\pi G$, where $H(t) = \dot{a}/a$ is the Hubble expansion rate defined in terms of the scale factor $a(t)$ and its time derivative, $\dot{a}$.

https://doi.org/10.1017/9781009290456.009 Published online by Cambridge University Press

or approximately 10 years after the Big Bang. Once the walls dominate, the universal expansion becomes $a \propto t^2$ (Eq. (F.5)). This is unacceptable for several reasons. For example, since the domain wall dominated universe accelerates ($\ddot{a} > 0$), density perturbations that are larger than the horizon keep getting stretched and stay larger than the horizon. This means that super-horizon density perturbations can never re-enter the horizon, which is an essential condition for them to start growing to form the galaxies, clusters, and large-scale structures that we currently observe. Even the growth of sub-horizon density perturbations is suppressed owing to cosmic acceleration.

A second constraint on a network of cosmic domain walls acting as a fluid with equation of state $p = -2\rho/3$ comes from the measured expansion rate of the universe using supernovae data [127, 118]. These surveys find that the equation of state parameter, $w \equiv p/\rho$, for our universe is less than about -0.8 [11, 128]. However, a universe dominated by a network of static ("frustrated") domain walls [25] would have $w \approx -0.67$.

Another possibility that has been considered is that perhaps there are some features that are missing in the standard model of particle physics, and that there indeed are very light domain walls in the universe [76]. Such light walls, if light enough, would be benign and could potentially play a role in cosmology. If we require that the domain walls not dominate the universe until the present time ($\sim 10^{17}$ s), Eq. (8.57), gives $\eta < 100$ MeV. Other cosmological constraints, such as arising from the isotropy of the cosmic microwave background can be used to put similar or somewhat stronger bounds on η [144, 158].

8.14 Constraints on and implications for particle physics

Let us summarize the picture that has emerged in this chapter.

- If a field theory has discrete symmetries that are spontaneously broken in the ground state, it must contain domain wall solutions.
- If high-energy particle physics is described by such a field theory and the discrete symmetry gets spontaneously broken in the early universe, cosmic domain walls are produced.
- If the standard model is complete at energies below 100 GeV, then there can be no domain walls in the universe and no spontaneously broken discrete symmetries in particle physics.

A closer examination of this sequence of arguments reveals a few loopholes that allow for spontaneously broken discrete symmetries in particle physics. First, there is the possibility that the discrete symmetry was broken right from the moment of the Big Bang. Then the whole universe could have been in one of the many discrete vacua at its very creation and no domain walls would be formed even though the

https://doi.org/10.1017/9781009290456.009 Published online by Cambridge University Press

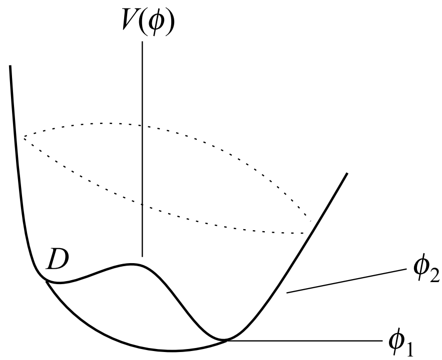

Figure 8.2 Sketch of the potential in the model of Eq. (8.58). Domain walls arise owing to a 2π change in the angular field variable and the location of the field inside the wall is marked by D. Such domain walls can terminate on strings, and the field within the string is located at $\phi = 0$.

underlying particle-physics theory could have broken discrete symmetries. This kind of scenario has been studied for magnetic monopoles in [50]. A related possibility is that, if the universe went through a period of superluminal expansion ("inflation"),[3] then correlations extend to scales that are vastly larger than our current horizon and our region of the universe is very likely to be free of any domain walls [97]. In another variant, domain walls are formed but subsequently inflated away.

All the above loopholes only apply to very high-energy domain walls where quantum gravity and/or inflation effects are relevant. If a particle-physics model has spontaneously broken discrete symmetries at lower energy scales (but still larger than ~ 100 MeV) no loopholes are known and the model is ruled out based on the "cosmological domain wall catastrophe." However, there is still the possibility that metastable or biased domain walls (see Section 6.8) can exist for some time in the universe. We now describe these two possibilities.

8.15 Metastable domain walls

In certain field theories, it is possible for domain walls to get punctured. To see how this can happen, consider the potential for a *complex* scalar field ϕ

$$V(\phi) = \frac{\lambda}{4}(|\phi|^2 - \eta^2)^2 - \frac{\alpha\eta}{32}(\phi + \phi^*)^3 \tag{8.58}$$

where we assume $0 < \alpha << \lambda$. The shape of this potential is shown in Fig. 8.2. The first term is minimized when $|\phi| = \eta$ and, restricting ϕ to the submanifold $|\phi| = \eta$,

[3] Inflation occurs when the universe is dominated by a field that has an equation of state with $-\rho < p < -\rho/3$. Then the expansion rate of the universe is superluminal and volumes that are larger than the horizon can get correlated.

https://doi.org/10.1017/9781009290456.009 Published online by Cambridge University Press

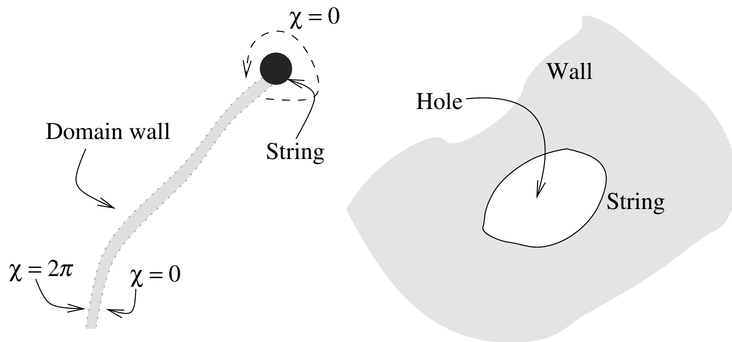

Figure 8.3 Cross-section of a wall that terminates on a string is shown on the left, and a wall with a puncture bordered by a string is shown on the right.

the second term is minimized when $\phi + \phi^* = +2\eta$. Another way of writing the potential is by setting $\phi = \psi \exp(\mathrm{i}\chi)$ and then ψ, χ are real fields. Then

$$V(\phi) = \frac{\lambda}{4}(\psi^2 - \eta^2)^2 - \frac{\alpha\eta}{4}\psi^3 \cos^3 \chi \tag{8.59}$$

The extrema of V are at $\psi = 0$ and at

$$\psi = \eta \left[\frac{3\alpha + \sqrt{9\alpha^2 + 64\lambda^2}}{8\lambda} \right], \quad \chi = n\pi \tag{8.60}$$

where n is an integer. The true vacua occur when n is an even integer. For example, domain wall solutions exist with the boundary condition $\chi(x = -\infty) = 0$, $\chi(x = +\infty) = 2\pi$.

Now consider a domain wall in the model in $3 + 1$ dimensions. Such a domain wall can terminate as shown in Fig. 8.3 since the path from $\chi = 0$ to $\chi = 2\pi$ can be contracted by lifting it over the top ($\psi = 0$) of the potential. While we have not described cosmic strings here, in models such as Eq. (8.58), we can have finite sections of open walls that are bordered by strings. Walls can also get punctured by holes that are bounded by strings. For further discussion of walls bounded by strings, we refer the reader to [88, 168, 171].

The evolution of a network of walls that can have punctures is very different from that of stable walls because a puncture can grow and eat up the wall. This provides a very efficient way for the wall network to lose energy and so the network never dominates the universe [168].

Another scheme that allows for the universe to have a finite period of time with domain walls is if a discrete symmetry is broken and then restored (see Section 6.1). Walls would be formed at the first phase transition and then they would dissolve at the second phase transition when the symmetry is restored. However, this scheme would imply an unbroken discrete symmetry in the low-energy particle physics. We do not know of such a discrete symmetry although the possibility cannot be excluded.

Finally, domain walls could have existed for some time in the early universe if there is an approximate discrete symmetry in the high-energy particle-physics model [60]. We have already seen an example of an approximate discrete symmetry in the $SU(5)$ Grand Unification model discussed in Section 2.1. If the cubic coupling in the potential in Eq. (2.5) is small, it can be ignored and the resulting model has $SU(5) \times Z_2$ symmetry, with all the domain wall solutions discussed in Section 2.2. A simpler example is that of the $\lambda\phi^4$ together with a small cubic term. The potential is

$$V(\phi) = -\frac{m^2}{2}\phi^2 + \gamma m\phi^3 + \frac{\lambda}{4}\phi^4 \tag{8.61}$$

Now the model still has two local minima but they are not exactly degenerate if γ is very small (see Fig. 6.11). At the phase transition, a network of domain walls is formed and the typical separation and curvature scale of the domain walls is given by the correlation length ξ_0. With time the curvature scale grows and is denoted by $R(t)$. So the force per unit area on the wall owing to tension is $\sim \sigma/R(t)$ where σ is the energy density of the wall. There is also a pressure difference pushing the wall toward the vacuum with the lowest energy. This pressure is given by the energy difference between the vacua and hence is proportional to γ

$$p \sim \gamma m\eta^3 \tag{8.62}$$

where η is the vacuum expectation value of the field. Therefore the tension is much larger than the pressure, and the dynamics of the wall network are unaffected by the pressure difference coming from the cubic term as long as

$$R(t) < \frac{\sigma}{\gamma m\eta^3} \sim \frac{1}{\gamma\eta} \tag{8.63}$$

Once the network has evolved to a point where this condition is not met, the pressure becomes important and drives the domain wall network such that the whole system reaches the true vacuum. From the area scaling law in Eq. (8.48), it follows that $R(t)$ grows linearly with conformal time.[4] Therefore, in a radiation dominated universe,

$$R = R_0\frac{\tau}{\tau_0} = R_0\left(\frac{t}{t_0}\right)^{1/2} \tag{8.64}$$

[4] The scaling law holds at late times after friction becomes unimportant. At earlier times, R grows as a different power of conformal time [87].

where R_0 can be taken to be the correlation length at the time of the phase transition t_0. Inserting this relation in Eq. (8.63) and using $R_0 \sim 1/\eta$, we get that the walls survive for a duration

$$t_{\text{walls}} \sim \frac{t_0}{\gamma^2} \tag{8.65}$$

If γ is small, the walls can survive for many Hubble expansions. In fact, if the walls survive for a long time, they might start dominating the density of the universe before they disappear.

Even if domain walls are present in the universe for a relatively short time, they can still have important implications for cosmology. As the wall network evolves, the ambient matter interacts with the walls. Magnetic monopoles can get trapped on domain walls, leading to faster annihilation. This is the sweeping scenario discussed in [52]. In addition, the eventual collapse of domain walls can lead to black hole formation. These issues have received some attention but have yet to be studied in detail.

8.16 Open questions

1. What happens when a black hole collides with a domain wall? Does it get stuck on the wall? Or does it pass through? For a discussion from the gravitational point of view, see [29, 148].
2. Develop an analytical formulation (perhaps along the lines of Section 3.5) to calculate the scalar radiation rate from collapsing domain walls.
3. Discuss the cosmology of superconducting domain walls.
4. What is the outcome of the $SU(5)$ Grand Unified phase transition when the cubic coupling is small? Are domain walls formed? How does the network evolve?

https://doi.org/10.1017/9781009290456.009 Published online by Cambridge University Press