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Asymptotic Formulae for the Lattice Point Enumerator

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Abstract. Let M be a convex body such that the boundary has positive curvature. Then by a well developed theory dating back to Landau and Hlawka for large λ the number of lattice points in λM is given by $G(\lambda M) = V(\lambda M) + O(\lambda^{d-1-\varepsilon(d)})$ for some positive $\varepsilon(d)$. Here we give for general convex bodies the weaker estimate

$$|G(\lambda M) - V(\lambda M)| \leq \frac{1}{2} S_{Z^d}(M)\lambda^{d-1} + o(\lambda^{d-1})$$

where $S_{Z^d}(M)$ denotes the lattice surface area of M. The term $S_{Z^d}(M)$ is optimal for all convex bodies and $o(\lambda^{d-1})$ cannot be improved in general. We prove that the same estimate even holds if we allow small deformations of M.

Further we deal with families $\{P_\lambda\}$ of convex bodies where the only condition is that the inradius tends to infinity. Here we have

$$|G(P_\lambda) - V(P_\lambda)| \leq dV(P_\lambda, K; 1) + o\big(S(P_\lambda)\big)$$

where the convex body K satisfies some simple condition, $V(P_\lambda, K; 1)$ is some mixed volume and $S(P_\lambda)$ is the surface area of P_λ.

1 Introduction

As we work with concepts from convex geometry and the geometry of numbers, our notation is taken from the standard books [S], [GL]. More specifically we denote by E^d the d-dimensional Euclidean space with norm $\|\cdot\|$ and by $\mathcal{K}^d$ the family of all convex bodies with non-empty interior in E^d. We write Λ for a lattice in E^d, Λ^* for its dual lattice, *i.e.*,

$$\Lambda^* = \{v \mid \langle v, u\rangle \in Z \text{ for } u \in \Lambda\}.$$

We note that the primitive vectors of Λ^* are normals to the lattice hyperplanes of Λ. We denote the determinant of Λ by $\det \Lambda$ and the lattice point enumerator of a set $M \subset E^d$ by G_Λ, *i.e.*, $G_\Lambda(M) = \#(\Lambda \cap M)$. In the special case $\Lambda = Z^d$ we frequently write $G(M)$ rather than $G_{Z^d}(M)$. For a set $M \subset E^d$ we write ∂M for its boundary, $\operatorname{cl} M$ for its closure, $\operatorname{int} M$ for its interior, $\operatorname{relint} M$ for its relative interior (interior with respect to its affine hull), and $\dim M$ for its affine dimension.

We are interested in the so called "circle problem"; namely, to determine $G_\Lambda(\lambda M)$ for $M \in \mathcal{K}^d$ and large real λ. For the unit ball B^d this is a well known problem in the theory of

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numbers which goes back to Gauss. For the more general case that M has positive curvature, $G(\lambda M)$ is estimated by the following formula which goes back to Landau and Hlawka (see [GL]), and was recently improved by Krätzel and Nowak ([KN]):

$$G_\Lambda(\lambda M) = \frac{V(M)}{\det \Lambda} \cdot \lambda^d + O\bigl(\lambda^{d-2+3/(2d)}(\log \lambda)^{2/d}\bigr). \tag{1}$$

Clearly (1) does not hold anymore if M contains a facet parallel to some lattice $(d-1)$-plane as then the error term can be no better than $O(\lambda^{d-1})$. Some more insight in the nature of the error term is given by Ehrhart's formula for the number of lattice points in lattice polytopes (see again [GL]). To state Ehrhart's result we need some more notation.

For a non-zero vector u and $M \in \mathcal{K}^d$ we write $u^\perp$ for the linear $(d-1)$-space orthogonal to u and $F_M(u)$ for the face of M with outer normal vector u. In addition, $\mathcal{H}^k$ denotes the k-dimensional Hausdorff-measure normalized so that it coincides with the k-dimensional Lebesgue-measure along hyperplanes. In particular, the surface-area $\mathcal{H}^{d-1}(\partial M)$ of M is denoted by $S(M)$.

For $M \in \mathcal{K}^d$ the "lattice surface area" $S_\Lambda(M)$ with respect to Λ is defined by

$$S_\Lambda(M) = \sum_{v \in \Lambda^* \text{primitive}} \frac{\mathcal{H}^{d-1}\bigl(F_M(v)\bigr)}{\det(v^\perp \cap \Lambda)}.$$

Now Ehrhart's formulae (see [GL]) for $G_\Lambda(\lambda P)$ for a lattice polytope P and natural λ make the role of $S_\Lambda(P)$ more transparent:

$$G_\Lambda(\lambda P) = \sum_{i=0}^{d} G_i(P)\lambda^i, \tag{2}$$

$$G_\Lambda\bigl(\operatorname{int}(\lambda P)\bigr) = \sum_{i=0}^{d} (-1)^{d-i} G_i(P)\lambda^i \tag{3}$$

where $G_d(P) = V(P)/\det \Lambda$, $G_{d-1}(P) = \frac{1}{2}S_\Lambda(P)$ and $G_0(P) = 1$, while the remaining G_i have a less obvious meaning (see [DR]).

Ehrhart's formula can easily be turned into an estimate of G_λ for all $\lambda > 0$ for a slightly more general class than lattice polytopes. For the sake of a better name we say that a polytope P is a lattice-facet polytope if for every facet some normal of the facet is in Λ^*, or in other words the hyperplanes spanned by the facets of P are parallel to lattice-hyperplanes of Λ.

Theorem A *Let Λ be a lattice in E^d and P be a lattice-facet polytope. Then*

$$\left| \frac{V(\lambda P)}{\det \Lambda} - G_\Lambda(\lambda P) \right| \le \frac{1}{2} S_\Lambda(P)\lambda^{d-1} + O(\lambda^{d-2}).$$

Ehrhart's formulae (2), (3) show that the estimate in Theorem A including the error term is optimal.

In fact the main result of our paper is a generalization of this result to general convex bodies which additionally allows some deformation of the shape of M.

Theorem B *Let Λ be a lattice in E^d and $M \in \mathcal{K}^d$. If a family $\{Q_\lambda\}$ of convex bodies tends to M as $\lambda \to \infty$ then*

$$\left| \frac{V(\lambda Q_\lambda)}{\det \Lambda} - G_\Lambda(\lambda Q_\lambda) \right| \leq \frac{1}{2} S_\Lambda(M)\lambda^{d-1} + o(\lambda^{d-1}).$$

At this point it seems worth while to mention that there is an application of Theorem B (and Theorem D below) to calculate the densities of large finite lattice packings (see [ABB], [BB]). If $d = 2$ then Theorem B is a trivial consequence of Pick's formula (this celebrated formula can be found in *e.g.* [GL]).

We note that for $M = Q_\lambda$, Theorem B becomes

Corollary C *For $M \in \mathcal{K}^d$,*

$$\left| \frac{V(\lambda M)}{\det \Lambda} - G_\Lambda(\lambda M) \right| \leq \frac{1}{2} S_\Lambda(M)\lambda^{d-1} + o(\lambda^{d-1}).$$

We remark that the same estimate holds if we consider arbitrary translates of λM.

If M is strictly convex then $S_\Lambda(M) = 0$, and hence

$$\det \Lambda \cdot G_\Lambda(\lambda M) = V(\lambda M) + o\big(S(\lambda M)\big).$$

In view of the formula of Landau and Hlawka and Theorem A, the error term in Theorem B and particularly in Corollary C appears to be very weak, but in fact it is best possible as a series of examples in Section 6 will show.

For the next theorem we consider a more general family $P_\lambda \in \mathcal{K}^d$, $\lambda \in \mathrm{N}$, such that for the inradius r we have $r(P_\lambda) \to \infty$. We prove a bound for the lattice point enumerator with the help of a suitable mixed volume.

Again we need some more notation. Let $M, K \in \mathcal{K}^d$ and let $H_K(\cdot)$ denote the support function of K. Then $V(\lambda M + K)$ is a polynomial in λ; namely,

$$V(\lambda M + K) = \sum_{i=0}^{d} \binom{d}{i} V(M, K; i)\lambda^{d-i}$$

(see [S]). We are interested in the term $V(M, K; 1)$. It is well known that

$$V(M, K; 1) = \frac{1}{d} \cdot \int_{\partial M} H_K(n_x)\, d\mathcal{H}^{d-1}(x), \tag{4}$$

where n_x is an exterior unit normal at $x \in \partial M$. We note that n_x is unique almost $\mathcal{H}^{d-1}$-everywhere on ∂M. For a polytope M, (4) becomes simply

$$V(M, K; 1) = \frac{1}{d} \cdot \sum_{u \in S^{d-1}} H_K(u) \cdot \mathcal{H}^{d-1}\big(F_M(u)\big), \tag{5}$$

where $\mathcal{H}^{d-1}\big(F_M(u)\big) \neq 0$ only if u is the exterior normal to a facet of M. The normalization reflects the definition of mixed volumes (see [S]).

After these preparations, we state our next

Theorem D *Let Λ be a lattice in E^d and assume that for some $K \in \mathcal{K}^d$, $H_K(v) \geq 1/2$ for any primitive $v \in \Lambda^*$. If $\{P_\lambda\}$ is a family of convex bodies with $r_d(P_\lambda) \to \infty$ as $\lambda \to \infty$ then*

$$|V(P_\lambda) - \det \Lambda \cdot G_\Lambda(P_\lambda)| \leq dV(P_\lambda, K; 1) + o\big(S(P_\lambda)\big).$$

The condition on $H_K(v)$ makes sure that K is sufficiently large with respect to Λ: For $v \in \Lambda^*$ primitive and $u \in \Lambda$ with $\langle u, v\rangle = 1$ we have that $H_K(v) \geq 1$ is equivalent to saying that K intersects $u + \operatorname{lin}(v^\perp \cap \Lambda)$, which is the closest non-linear affine lattice hyperplane to the origin normal to v.

While our work deals with large bodies we should remark that for the special case of $\mathbb{Z}^d$ there are estimates for $G(M) - V(M)$ for all bodies. A survey on these results can be found in [BW]. Especially a somewhat related lower bound was given in [BHW]; namely, $G_{\mathbb{Z}^d}(M) \geq V(M) - \frac{1}{2}S(M)$.

We proceed as follows: Sections 2, 3 and 4 provide the auxiliary statements which we need for the proofs of Theorem B and Theorem D. In Section 5 we start with a proof of Theorem A. While the statement of this theorem is folklore, we are not aware of a written proof. Furthermore the ideas of the proof are the same as in the rather more complicated Theorem B. Thus we use the proof of Theorem A as an outline of the proof of Theorem B and it might be useful to start to read the paper at that point. In Section 6, we discuss the exactness of the estimates in Theorem B. Finally Section 7 is devoted to the proof of Theorem D.

2 Approximation of Convex Bodies

In Sections 2 and 3, we discuss some elementary metrical properties of convex surfaces. The standard reference book for this and the next section is [S]. For the basic properties of Hausdorff measure, consult any monograph on geometric measure theory, for example the classical book [F].

The Euclidean distance function is denoted by $\delta(\cdot,\cdot)$ and $\Delta^H(\cdot,\cdot)$ stands for the Hausdorff distance of compact sets. We denote by $\angle(u, v)$ the angle of the vectors u and v. For $\sigma \subset E^d$ and $\omega \geq 0$, $N(\sigma, \omega)$ is the set of points with distance less than ω from σ.

Let M be some convex body containing o in its interior. Then for $x \neq o$, the radial projection $\pi_{\partial M}(x)$ of x into ∂M is well defined.

For the rest of the section we consider an $M \in \mathcal{K}^d$ such that for some positive r and R, $rB^d \subset \operatorname{int} M$ and $M \subset \operatorname{int} RB^d$.

Lemmas 2.1, 2.2 and 2.3 are easy consequences of the fact that for a convergent sequence of convex bodies, supporting hyperplanes can converge only to some supporting hyperplane of the limit.

Lemma 2.1 *Let $\Pi \subset \partial M$ have the property that for $x, y \in \operatorname{cl}\Pi$ and all n_x and n_y, $\angle(n_x, n_y) < \alpha$ holds. Then there exists a positive ω with the following property: Let $Q \in \mathcal{K}^d$ with $\Delta^H(Q, M) < \omega$ and $u, v \in \pi_{\partial Q}(\Pi)$. Then $\angle(n_u, n_v) < 2\alpha$ holds.*

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For any set $\sigma \subset S^{d-1}$ and convex body P we denote by $\psi_P(\sigma)$ the subset of ∂P whose points have an outer normal contained in σ.

Lemma 2.2 *Let $u \in S^{d-1}$. For positive θ there exist positive α and ω such that if $\Delta^H(M,Q) < \omega$ then $\psi_Q\big(S^{d-1} \cap N(u,\alpha)\big)$ is a subset of the radial projection of $\partial M \cap N\big(\psi_M(u),\theta\big)$ onto ∂Q.*

For $F \in \mathcal{K}^d$ we denote by $r(F)$ the relative inradius of F, *i.e.*, the radius of the largest ball with the same dimension as F that is contained in F. Then for $0 \le \theta \le r(F)$ we write $F_{-\theta}$ for the subset of F whose points are at least distance θ from each point of the relative boundary ∂F of F ("inner parallel body").

Lemma 2.3 *Let $F = F_M(u)$ such that $\dim F = d-1$ for some $u \in S^{d-1}$ and $0 < \theta < r(F)$. Then for any $\alpha > 0$ there exists an $\omega > 0$ with the following property: If $Q \in \mathcal{K}^d$ such that $\Delta^H(Q,M) < \omega$ and $x \in \pi_{\partial Q}(F_{-\theta})$ then $\angle(n_x,u) < \alpha$ for any normal vector n_x at x to Q.*

In order to compute $G_{\mathbb{Z}^d}(M) - V(M)$ we introduce some more notation. For $z = (z_1,\dots,z_d) \in \mathbb{Z}^d$ we denote by $W(z)$ the unit cube $W(z) = \{(x_1,\dots,x_d) \mid -\frac{1}{2} \le z_i - x_i < \frac{1}{2}, i = 1,\dots,d\}$. For a closed convex set Q let $z_\alpha = (z_1,\dots,z_{i-1},z_i+\alpha,z_{i+1},\dots,z_d) \in \partial Q$ with $z_i \in \mathbb{Z}$, $i = 1,\dots,d$ and $-\frac{1}{2} \le \alpha < \frac{1}{2}$ such that $\langle n_{z_\alpha}, e_i\rangle > 0$ (e_i is the i-th coordinate unit vector). Then the i-tower Z of Q at z_α is the union of all cubes $W(\bar z)$ with $\bar z = (z_1,\dots,z_{i-1},\bar z_i,z_{i+1},\dots,z_d)$ such that there is an $x \in W(\bar z) \cap \partial Q$ with $\langle n_x, e_i\rangle > 0$. For $\langle n_{z_\alpha}, e_i\rangle < 0$ the i-tower Z is defined correspondingly. If for all lattice points $z \in Z$ the points $x \in W(z) \cap \partial Q$ are in a common facet of Q then we obviously have

$$\sum_{z\in Z, z\in Q} V\big(W(z)\setminus Q\big) - \sum_{z\in Z, z\notin Q} V\big(W(z)\cap Q\big) = \begin{cases} \frac{1}{2} - \alpha & \text{for } \alpha \ge 0 \\ -\frac{1}{2} - \alpha & \text{for } \alpha < 0. \end{cases}$$

Thus for an i-tower Z at z_α we define the deviation $\operatorname{dev} Z$ of Z by

$$\operatorname{dev} Z = \left| \sum_{z\in Z, z\in Q} V\big(W(z)\setminus Q\big) - \sum_{z\in Z, z\notin Q} V\big(W(z)\cap Q\big) - \frac{1}{2} + \alpha \right|$$

for $\alpha \ge 0$ and

$$\operatorname{dev} Z = \left| \sum_{z\in Z, z\in Q} V\big(W(z)\setminus Q\big) - \sum_{z\in Z, z\notin Q} V\big(W(z)\cap Q\big) + \frac{1}{2} + \alpha \right|$$

for $\alpha < 0$.

Lemma 2.4 *Let $z_\alpha = (z_1,z_2,\dots,z_i+\alpha,\dots,z_d) \in \partial Q$ with $z_i \in \mathbb{Z}$ for $i = 1,\dots,d$ and $-\frac{1}{2} \le \alpha < \frac{1}{2}$, and $n \in S^{d-1}$ with $\langle e_i, n\rangle \ge \beta$ for $\beta > 0$. Let H denote the plane through z_α with normal n. Then for every $\varepsilon > 0$ there exists a $\gamma > 0$ depending only on β and ε with the following property: Let Z be the i-tower at z_α. If $\angle(n,n_x) \le \gamma$ holds for every $x \in Z \cap \partial Q$ then*

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(a) $\operatorname{dev} Z < \varepsilon$.
(b) $|\mathcal{H}^{d-1}(H \cap Z) - \mathcal{H}^{d-1}(\partial Q \cap Z)| < \varepsilon$.

The next lemma gives bounds for the angles between sections of certain planes:

Lemma 2.5 *Let H be a hyperplane with normal n and E be a two-dimensional plane spanned by the vectors u_1, u_2 such that $\angle(n, u_1) \leq \gamma < \pi/2$. Then for every $\xi > 0$ exists an $\eta > 0$ depending only on γ, ξ such that for any hyperplane H_1 with normal n_1 and $\angle(n_1, n) \leq \eta$ we have $\angle\big((E \cap H), (E \cap H_1)\big) \leq \xi$.*

In the last section we have to deal with convex bodies, whose extension in some directions is much larger than their extension in other directions. This situation is conveniently described by means of different inradii and best approximating planes: For $K \in \mathcal{K}^d$ we denote the k-th inradius, that is the radius of the largest k-dimensional ball contained in K by r_k. For every k-plane L exists an $\omega(L)$ for which $K \subset L + \omega(L)B^d$. Now the best approximating k-plane $L^k(K)$ is the plane L for which $\omega(L)$ becomes minimal. There is a well-known connection between radii and best approximating planes (see [P]):

$$K \subset L^k(K) + (k+2)r_{k+1}(K)B^d. \tag{6}$$

As we frequently need to consider orthogonal projections of sets onto planes we write $\pi_L(M)$ for the orthogonal projection of the set M onto the plane L. Further we write $L^\perp$ for the complementary orthogonal linear plane of L.

For some estimates we use a different notion of k-inradius, which was discussed in [BH]: The k-th inradius $r_k^\pi(K)$ with respect to projection is the radius of the largest k-ball, which is contained in a projection of K onto a k-dimensional plane. Of course the two notions of k-inradius are not independent:

Lemma 2.6 *Let $K \in \mathcal{K}^d$. Then*

$$r_k(K) \leq r_k^\pi(K) \leq k r_k(K).$$

Proof To prove the right inequality let L be a k-plane, for which $\pi_L(K)$ contains a k-ball B with radius $r_k^\pi(K)$. Let S be a regular k-dimensional simplex with vertices on the relative boundary of B. S is the projection of a simplex S' contained in K. As the ratio of the circumradius and the inradius of a regular k-simplex is k (see [BF]), S' contains a k-dimensional inball with radius $r_k^\pi(K)/k$. The other inequality is trivial. ■

The previous lemma and a result in [BH] immediately give a convenient tool to estimate volume and surface area of convex bodies:

Lemma 2.7 *Let $K \in \mathcal{K}^d$. Then there exist positive constants c_1, c_2, c_3, c_4 depending only on d such that*

$$c_1 r_1(K) \cdot \cdots \cdot r_d(K) \leq V(K) \leq c_2 r_1(K) \cdot \cdots \cdot r_d(K),$$
$$c_3 r_1(K) \cdot \cdots \cdot r_{d-1}(K) \leq S(K) \leq c_4 r_1(K) \cdot \cdots \cdot r_{d-1}(K).$$

Finally we need a sufficiently interior point of a convex set K. This is provided by the center $c(K)$ of a (relative) inball.

Now we can state the facts needed in the proof of Theorem D:

Lemma 2.8 *Let K_λ be a family of convex bodies, such that $r_d(K_\lambda)$ and $r_k(K_\lambda)/r_{k+1}(K_\lambda)$ tend to infinity, but $r_{k+1}(K_\lambda)/r_d(K_\lambda)$ is bounded. Let ω_λ be a sequence of positive numbers with $\omega_\lambda \to 0$. Then for $L_\lambda = L^k(K_\lambda)$ and $M_\lambda = (1-\omega_\lambda)\pi_{L_\lambda}(K_\lambda) + \omega_\lambda c\big(\pi_{L_\lambda}(K_\lambda)\big)$, we have*

(a) for every $\varepsilon > 0$ there is a λ_0 such that for all $\lambda > \lambda_0$ and $j = 1, \dots, k$

$$r_j\big(\pi_{L_\lambda}(K_\lambda)\big) \geq (1-\varepsilon) r_j(K_\lambda),$$

(b)

$$\lim_{\lambda\to\infty} \frac{\mathcal{H}^{d-1}\big((M_\lambda + L_\lambda^\perp) \cap \partial K_\lambda\big)}{S(K_\lambda)} = 1.$$

Proof Let B_λ be a j-ball of radius $r_j(K_\lambda)$ contained in K_λ, H_λ be the affine plane spanned by B_λ and $u_1^\lambda, \dots, u_j^\lambda$ be an orthonormal basis of the linear plane parallel to H^λ. We may write $u_i^\lambda = v_i^\lambda + w_i^\lambda$ for $i = 1, \dots, j$ where v_i^λ is in the linear plane parallel to L_λ and w_i^λ is in $L_\lambda^\perp$. Now $K_\lambda \subset L_\lambda + (k+2) r_{k+1}(K_\lambda) B^d$ and $r_{k+1}(K_\lambda)/r_j(K_\lambda) \to 0$ immediately show $w_i^\lambda \to 0$ for all i. (a) is a straightforward consequence.

We may assume that $c\big(\pi_{L_\lambda}(K_\lambda)\big)$ is the origin. Then we have $(1-\omega_\lambda)K_\lambda \subset (M_\lambda + L_\lambda^\perp) \cap K_\lambda$. From this we conclude

$$\lim_{\lambda\to\infty} \frac{S\big((M_\lambda + L_\lambda^\perp) \cap K_\lambda\big)}{S(K_\lambda)} = 1.$$

So it is sufficient to prove

$$\lim_{\lambda\to\infty} \frac{\mathcal{H}^{d-1}\big(\bigcup_{x\in\partial M_\lambda} (x + L_\lambda^\perp) \cap K_\lambda\big)}{S(K_\lambda)} = 0.$$

Among the sections $(x + L_\lambda^\perp) \cap K_\lambda$ let A be the one with largest $(d-k)$-measure. We have $\mathcal{H}^{d-k}(A) \leq c_1 r_{k+1}^{d-k}$ for some constant c_1 by (6). The result follows by (a) from the estimates given in Lemma 2.7. ∎

Finally we need that the volume of certain neighbourhoods of a piece of the boundary of a convex set cannot be too large.

Lemma 2.9 *Let K_λ, L_λ, ω_λ and M_λ be as in Lemma 2.8, and denote by σ_λ the closure of $\partial K_\lambda \cap \big((L_\lambda \backslash M_\lambda) + L_\lambda^\perp\big)$. Then for any $t > 0$,*

$$\lim_{\lambda\to\infty} \frac{V\big(N(\sigma_\lambda, t)\big)}{S(K_\lambda)} = 0.$$

Proof Approximating by polytopes, we may assume that K_λ is actually a polytope. Set

$$\tilde{M}_\lambda = (1 - 2\omega_\lambda)\pi_{L_\lambda}(K_\lambda) + 2\omega_\lambda c\big(\pi_{L_\lambda}(K_\lambda)\big)$$

and

$$\tilde{\sigma}_\lambda = \mathrm{cl}\Big(\partial K_\lambda \cap \big((L_\lambda \backslash \tilde{M}_\lambda) + L_\lambda^\perp\big)\Big).$$

Denote by N_k the set of points in $N(\sigma_\lambda, t)$ such that a closest point of ∂K_λ is in the relative interior of some k-face. Observe that for any point in N_{d-1} the closest point is in $\tilde{\sigma}_\lambda$ for λ sufficiently large, and hence

$$V(N_{d-1}) \leq 2t \cdot \mathcal{H}^{d-1}(\tilde{\sigma}_\lambda).$$

We deduce by Lemma 2.8 (b) that

$$\lim_{\lambda\to\infty} \frac{V(N_{d-1})}{S(K_\lambda)} = 0.$$

Now we assume $k < d - 1$. Then no point of N_k is contained in the interior of K_λ, and hence the definition of the mixed volumes yields

$$V(N_k) \leq \binom{d}{k} V(K_\lambda, B^d; d - k) \cdot t^{d-k}.$$

Since by the monotonicity of the mixed volumes, the inequalities

$$V(K_\lambda, B^d; d - k) \leq \frac{1}{r_d(K_\lambda)^{d-k-1}} \cdot V(K_\lambda, B^d; 1) = \frac{1}{d \cdot r(K_\lambda)^{d-k-1}} \cdot S(K_\lambda)$$

hold, we conclude the lemma. ■

3 Lipschitz Maps

We still keep M, r and R as in the previous section.

We need some very basic properties of a Lipschitz map. If f has Lipschitz constant γ, *i.e.*, $\|f(x) - f(y)\| \leq \gamma \cdot \|x - y\|$ for all x, y, then

$$\mathcal{H}^{d-1}\big(f(\sigma)\big) \leq \gamma^{d-1} \cdot \mathcal{H}^{d-1}(\sigma). \tag{7}$$

Lemma 3.1 *Let Π be a $(d-1)$-dimensional convex, compact set and $t > 0$ be smaller than the relative inradius of Π. If $f\colon \Pi \to E^d$ has Lipschitz constant γ then*

$$V\Big(N\big(f(\Pi), t\big)\Big) < 2^{2(d-1)} \frac{V(B^d)}{\mathcal{H}^{d-1}(B^{d-1})} (1 + \gamma)^d \mathcal{H}^{d-1}(\Pi) \cdot t.$$

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Proof We may assume that the origin is the center of the largest $(d-1)$-ball contained in Π and B^{d-1} is the unit ball in $\operatorname{lin}\Pi$. Let $X \subset \Pi$ have maximal cardinality with the condition that any two elements of X are at least distance t apart. Thus for $x, y \in X$ we have $\operatorname{int}\big((x+\frac{t}{2}B^{d-1})\cap(y+\frac{t}{2}B^{d-1})\big) = \emptyset$, $(X+tB^{d-1}) \subset 2\Pi$ and $X+tB^{d-1}$ covers Π. Thus X has at most $\mathcal{H}^{d-1}(2\Pi)/\mathcal{H}^{d-1}(\frac{1}{2}tB^{d-1})$ elements.

We deduce that

$$N\big(f(\Pi), t\big) \subset f(X) + (1+\gamma)tB^d,$$

which in turn yields the lemma. ■

Now let $Q \in \mathcal{K}^d$ such that $rB^d \subset Q \subset RB^d$. Then there exists some positive c_1 depending on r and R such that if $y, z \in \partial Q$ then

$$\frac{1}{c_1} \cdot \delta(y,z) < \angle(y,z) < c_1 \cdot \delta(y,z).$$

Similarly, there exists some positive c_2 depending on r and R such that if H is some hyperplane supporting M at $x \in \partial M$ and $y, z \in N(x, \frac{1}{2}r) \cap H$ then

$$\frac{1}{c_2} \cdot \delta(y,z) < \angle(y,z) < c_2 \cdot \delta(y,z).$$

We conclude

Lemma 3.2 *There exists a c depending on r, R and d and an $\omega > 0$ depending on M with the following property:*

Let H be a hyperplane supporting at $x \in \partial M$ and Q be a convex body with $\Delta^H(Q, M) < \omega$. Then for $y, z \in N(x, \frac{1}{2}r) \cap H$,

$$\frac{1}{c} \cdot \delta\big(\pi_{\partial Q}(y), \pi_{\partial Q}(z)\big) < \delta(y,z) < c \cdot \delta\big(\pi_{\partial Q}(y), \pi_{\partial Q}(z)\big).$$

We say that T is a tangent polytope of M if every facet F of T touches M and $F \subset N(F \cap M, \frac{1}{2}r)$. We deduce by Lemma 3.2 that for $\Delta^H(Q, M) < \omega$ (where ω comes from Lemma 3.2) $\pi_{\partial Q}$ is Lipschitz on ∂T and $\pi_{\partial T}$ is Lipschitz on ∂Q.

For positive k and a $(d-1)$-polytope F we construct k-patches on F in the following way: We choose a tiling of $\operatorname{aff} F$ by $(d-1)$-cubes with edge length $1/k$, and call a $(d-1)$-dimensional intersection of some tile and F a k-patch. We obtain a dissection of the boundary of a polytope T by taking all the k-patches of its facets. In general for $P \in \mathcal{K}^d$ with $o \in \operatorname{int} P$ and a polytope T with $o \in \operatorname{int} T$ the k-patches on ∂P are the radial projections of the k-patches on ∂T.

Lemma 3.3 *Let $\eta > 0$. There exists some compact $\varrho_\eta \subset \partial M$ with $\mathcal{H}^{d-1}(\varrho_\eta) = 0$ and a c depending only on r, R such that if $\delta(x,y) < \theta$ and $\angle(n_x, n_y) \geq \eta$ hold for outer normals n_x at $x \in \partial M$ and n_y at $y \in \partial M$ then $x, y \in N(\varrho_\eta, c \cdot \theta)$.*

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Proof Choose k so that any k patch on $2RS^{d-1}$ has diameter at most $R\sin\frac{1}{2}\eta$ and denote by ϱ_0 the union of the (relative) boundaries of these patches on $2RS^{d-1}$. Writing $p_M(x)$ for the closest point of x to M, we have that $\varrho = p_M(\varrho_0)$ has zero $\mathcal{H}^{d-1}$ measure since p_M is Lipschitz (see [S]) and $\mathcal{H}^{d-1}(\varrho_0) = 0$.

Assume that suitable x and y are given. There exists a continuous curve ξ on ∂M connecting x and y with length less then $c\cdot\theta$ where c depends on r, R. We claim that $\xi_0 = p_M^{-1}(\xi)\cap 2RS^{d-1}$ is connected: Else there exist disjoint compact sets ξ_1, ξ_2 with $\xi_0 = \xi_1\cup\xi_2$. Now there exists an $x\in p_M(\xi_1)\cap p_M(\xi_2)$, as $p_M(\xi_1)$ and $p_M(\xi_2)$ are compact and ξ is connected. Thus $p_M^{-1}(x)\cap 2RS^{d-1}$ is disconnected. But this is a contradiction as clearly $p_M^{-1}(z)\cap 2RS^{d-1}$ is connected for all $z\in\xi$.

Let x_0 (y_0) be the inverse image of x (y) generated by n_x (n_y), and let y_0' be the intersection of $2RS^{d-1}$ and the ray starting from x parallel to n_y. Thus $\delta(x_0, y_0') > 2R\sin\frac{1}{2}\eta$, and for small θ, we have $\delta(x_0, y_0) > R\sin\frac{1}{2}\eta$. Thus x_0, y_0 are contained in different patches of $2R\,S^{d-1}$. We conclude, that $\xi_0\cap\varrho_0\neq\varnothing$ as otherwise there would be a dissection of ξ_0 in two non-empty open sets. Consequently $\xi\cap\varrho$ is non-empty too. ■

In view of the previous lemma it is of interest to look at the neighbourhood of compact sets with zero measure. Here we have

Lemma 3.4 *Let T be a polytope, $\varrho\subset\partial T$ compact and $\mathcal{H}^{d-1}(\varrho) = 0$. For every $k\in\mathbb{N}$ let K denote the set of k-patches Π on ∂T such that for every $\Pi\in K$ we have $\Pi\cap N(\varrho, 1/k)\neq\varnothing$. Then for every $\varepsilon > 0$ there exists a $k\in\mathbb{N}$ such that $\sum_{\Pi\in K}\mathcal{H}^{d-1}(\Pi) < \varepsilon$.*

Proof By considering the facets separately, it is sufficient to prove for compact $\varrho\subset E^{d-1}$ with $\mathcal{H}^{d-1}(\varrho) = 0$ that there is a $\tau > 0$ such that $\mathcal{H}^{d-1}\big(N(\varrho,\tau)\big) < \varepsilon$.

As $(d-1)$-dimensional Hausdorff-measure and Lesbesgue-measure coincide we have by definition of the measure an open set G containing ϱ with $\mathcal{H}^{d-1}(G) < \varepsilon$. For each $x\in\varrho$, let $\tau(x) > 0$ be the maximal radius such that $N\big(x,\tau(x)\big)\subset G$. Clearly $\tau(x)$ is continuous, and hence τ can be chosen as the positive minimum of $\tau(x)$. ■

4 Some Properties of Bounded Lattice Vectors

We establish some simple properties for lattice vectors. The first lemma shows that a family of lattice vectors with bounded length is not too sparse, but it is also not too "dense" according to the second lemma. We close with an observation concerning the approximation of arbitrary planes by lattice planes in a way that the approximating plane contains short lattice vectors.

Lemma 4.1 *For every $\chi > 0$ there exists an n_0 with the following property: Let $0\le\psi\le 1$. Then there is an $(n,m)\in\mathbb{Z}^2$ with $n\le n_0$ such that $\angle\big((1,\psi),(n,m)\big)\le\chi$ and $|n\cdot\psi - m| < 1/n$.*

Proof We observe that for given χ there exists a $\delta > 0$ such that $\angle\big((1,\psi),(1,\eta)\big) < \chi$ for all η with $|\psi-\eta| < \delta$. Let $n_0 = \lceil 1/\delta\rceil$. By a fundamental theorem from Diophantine Approximation there exists an $n\le n_0$ and $m\in\mathbb{Z}$ such that $|\psi - \frac{m}{n}| < \frac{1}{n\cdot n_0}$ (see [C], [GL]).

Apparently (n,m) has the required properties. ■

Let $e_1, \ldots, e_d$ denote the canonical basis of $\mathbb{Z}^d$, and set $P_{ij} = \operatorname{lin}\{e_i, e_j\}$.

Lemma 4.2 *For (large) φ, let $u_1, \ldots, u_m \in S^{d-1}$ be the normals of the sub $(d-1)$-lattices of $\mathbb{Z}^d$ such that $\langle u_i, e_1\rangle > 1/(2\sqrt{d})$ and each P_{1i}, $i = 2, \ldots, d$, contains a non-zero lattice vector of length at most φ of the sublattice. Then for any (small) positive α there exists a positive β with the following property:*

For any $v \in S^{d-1}$, with $\angle(v, u_j) > \alpha$ for $j = 1, \ldots, m$, and $\langle v, e_1\rangle > 1/(2\sqrt{d})$, there exists an $i \geq 2$ such that if $v_i \in S^{d-1} \cap P_{1i}$ is perpendicular to v and $\angle(w, v_i) < \beta$ for a primitive $w \in \mathbb{Z}^d \cap P_{1i}$ then the length of w is greater than φ.

Proof We denote the set of unit vectors v satisfying $\langle v, e_1\rangle \geq 1/(2\sqrt{d})$ by Ω. For any $i = 2, \ldots, d$ the map $v \mapsto v_i'$ on Ω is continuous where the unit vector $v_i' \in S^{d-1} \cap P_{1i}$ is parallel to the orthogonal projection of v onto P_{1i}. We may choose $v_i \in S^{d-1} \cap P_{1i}$ orthogonal to v_i' (and hence also to v) so that the map $v \mapsto v_i$ is still continuous. We set

$$f(v) = \max_{i=2,\ldots,d} \min\{\angle(v_i, w) \mid w \in P_{1i} \cap \mathbb{Z}^d \text{ and } \|w\| \leq \varphi\}.$$

Clearly we have $f(v) \neq 0$ for $v \notin \{u_1, \ldots, u_m\}$. Now for $\alpha > 0$, let

$$\Omega_\alpha = \{v \in \Omega \mid \angle(v, u_j) \geq \alpha \, \forall j = 1, \ldots, m\}.$$

Since Ω_α is compact and $f(v)$ is continuous, β can be chosen as the minimum of f on Ω_α (assuming that $\Omega_\alpha \neq \emptyset$). ∎

A Minkowski reduced basis $w_1, \ldots, w_d$ of a lattice Λ is defined as follows: w_1 is a shortest vector, and w_k is a shortest vector of Λ not contained in $\operatorname{lin}\{w_1, \ldots, w_{k-1}\}$. Then there exists a positive constant c depending only on d (see [GL, p. 150]) such that $c\|w_d\|B^d$ contains no d independent lattice points.

Lemma 4.3 *For every linear k-plane $\tilde{L} \subset \mathbb{R}^d$ and every $\varepsilon > 0$ there exist a lattice k-plane L of $\mathbb{Z}^d$ such that for a Minkowski reduced basis $w_1, \ldots, w_k$ of $L \cap \mathbb{Z}^d$,*

(a) the distance of any point of

$$T = \Big\{\sum_{i=1}^{k} \alpha_i w_i \mid 0 \leq \alpha_i < 1\Big\}$$

from $\tilde{L}$ is at most ε, and

(b) for any unit normal u to $\tilde{L}$ there exists an unit normal v to L such that $\angle(u, v) < \varepsilon$.

Proof We choose an orthonormal basis $(u_{11}, \ldots, u_{1d}), \ldots, (u_{k1}, \ldots, u_{kd})$ of $\tilde{L}$. Now we use simultaneous Diophantine approximation for the kd numbers u_{ij}. By a well known theorem (see [GL, p. 44] or [C]) there are infinitely many $q \in \mathbb{N}$ such that there exist $p_{ij} \in \mathbb{Z}$, $1 \leq i \leq k$, $1 \leq j \leq d$ with

$$\left| u_{ij} - \frac{p_{ij}}{q} \right| \leq \frac{1}{q^{1+\frac{1}{kd}}}.$$

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From this it follows that there are constants c_1 and c_2 depending only on d such that for infinitely many $q \in \mathbb{N}$ there are $v_1, \ldots, v_k \in \mathbb{Z}^d$ such that $\|v_i\| \leq c_1 \cdot q$ and

$$\angle(v_i, u_i) < \frac{c_2}{q^{1+\frac{1}{kd}}}.$$

Let L be the plane spanned by $v_1, \ldots, v_k$ which satisfies (b) for large q, even replacing ε with $c_3/q^{1+\frac{1}{kd}}$, where c_3 is a constant. If $w_1, \ldots, w_k$ is a Minkowski reduced bases of $L \cap \mathbb{Z}^d$ then $\operatorname{diam} T < c_4 q$, where c_4 is a constant, which in turn yields (a) for large q. ■

5 The Proof of Theorem B

We observe that all our statements are invariant under simultaneous linear transformations of the lattice and the convex bodies. Thus we may assume that $\Lambda = \mathbb{Z}^d$. In this section z always is a lattice point and $W(z) = z + [-\frac{1}{2}, \frac{1}{2})^d$.

For $Q \in \mathcal{K}^d$ we have the trivial identity

$$|G(Q) - V(Q)| = \Bigg| \sum_{\substack{z \in Q,\\ W(z) \cap \partial Q \neq \varnothing}} V\big(W(z) \backslash Q\big) - \sum_{\substack{z \notin Q,\\ W(z) \cap \partial Q \neq \varnothing}} V\big(W(z) \cap Q\big) \Bigg|. \tag{8}$$

The basic idea of Theorem B is that we can rather easily estimate $G - V$ for the union of certain towers. To this end we introduce the notion of an i-box on the boundary of Q for $i = 1, \ldots, d$. We say that for $z = (z_1, \ldots, z_d)$ that U is an i-box at z, if

$$U = \left\{ x = (x_1, \ldots, x_d) \mid x \in \partial Q, -\frac{1}{2} \leq z_p - x_p < \frac{1}{2}, p \neq i, \langle n_x, e_i \rangle > 0 \right\}.$$

Analogously there are i-boxes for $-e_i$. We say that a box U is simple, if $Z \cap U = Z \cap \partial Q$ for the i-tower Z with $Z \cap U \neq \varnothing$. For the union of certain simple boxes we can easily estimate $G - V$.

Lemma 5.1 *Let $\{U_0, \ldots, U_{q-1}\}$ be a set of simple 1-boxes at z_i, $i = 0, \ldots, q-1$ such that*

$$z_i + \left(\frac{i}{q} + \alpha_i\right) e_1 \in \partial Q$$

with $-1/q < \alpha_i < 1/q$ for $i = 0, \ldots, q-1$. Finally let $\operatorname{dev} Z_i < \epsilon$ for every one tower Z_i at $z_i + (\frac{i}{q} + \alpha_i)e_1$. Then

$$\left| \sum_{i=0}^{q-1} \Big(\sum_{z \in Z_i, z \in Q} V\big(W(z) \setminus Q\big) - \sum_{z \in Z_i, z \notin Q} V\big(W(z) \cap Q\big) \Big) \pm \frac{1}{2} + \sum_{i=0}^{q-1} \alpha_i \right| \leq q\varepsilon.$$

Here we have "+", if $\alpha_0 < 0$ and "−" otherwise.

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Proof By the definition of the deviation in Section 2 and writing $\{a\}$ for the fractional part of a we have

$$\Bigl|\sum_{\substack{z\in Z_i\\ z\in Q}} V\bigl(W(z)\setminus Q\bigr) - \sum_{\substack{z\in Z_i\\ z\notin Q}} V\bigl(W(z)\cap Q\bigr) - \frac{1}{2} + \frac{i}{q} + \alpha_i\Bigr| \le \varepsilon$$

for $\frac{i}{q} + \alpha_i \in [0, 1/2]$ and

$$\Bigl|\sum_{\substack{z\in Z_i\\ z\in Q}} V\bigl(W(z)\setminus Q\bigr) - \sum_{\substack{z\in Z_i\\ z\notin Q}} V\bigl(W(z)\cap Q\bigr) + \frac{1}{2} + \frac{i}{q} + \alpha_i\Bigr| \le \varepsilon$$

for $\frac{i}{q} + \alpha_i \in (1/2, 1)$. The lemma is an immediate consequence. ■

We shall see that Theorem B is a consequence of the fact, that for large λ most of $\partial\lambda Q$ can be covered appropriate unions of sufficiently flat simple boxes. As a first application we give a proof of Theorem A:

Proof of Theorem A Let F be a facet of the lattice-facet polytope Q with primitive exterior normal $u \in \mathbb{Z}^d$. Set $L = \mathbb{Z}^d \cap \operatorname{lin}(F-F)$ and $q = \det L$. We may assume that $u = (u_1, \ldots, u_d)$ with $u_1 > 0$.

Let P be a fundamental cell of L. There are exactly q points $z_0, \ldots, z_{q-1} \in \mathbb{Z}^d$ such that $w_i = z_i + \frac{i}{q}e_1 \in P$. We have $\operatorname{aff} F = \operatorname{lin}(F-F) + te_1$ for some $t \in \mathbb{R}$. There is a renumbering $\tilde{w}_0, \ldots, \tilde{w}_{q-1}$ of $w_1, \ldots, w_{q-1}$ such that

$$\tilde{w}_i + \lambda t e_1 = \tilde{z}_i + \left(\frac{i}{q} + \alpha\right) e_1$$

for suitable $\tilde{z}_i \in \mathbb{Z}^d$ and

$$-1/(2q) \le \alpha < 1/(2q). \tag{9}$$

Now let $\nu > 0$ be fixed and

$$L_0(\lambda) = \{l \in L \mid w_i + \lambda t e_i + l \in \lambda F_{-\nu/2}, i = 0, \ldots, q-1\}.$$

Then every $l \in L_0(\lambda)$ defines a set of simple 1-boxes $U_{0,l}, \ldots, U_{q-1,l}$ at $\tilde{z}_0 + l, \ldots, \tilde{z}_{q-1} + l$ such that

$$\lambda F_{-\nu} \subset \bigcup_{l\in L_0} (U_{1,l} \cup \cdots \cup U_{q-1,l}).$$

Thus Lemma 5.1 yields with $\epsilon = 0$

$$\begin{aligned}\Big| \sum_{l \in L_0(\lambda)} \sum_{i=0}^{q-1} \Big(\sum_{\substack{z \in \lambda Q,\\ W(z)\cap U_{il} \neq \varnothing}} V\big(W(z)\lambda Q\big) - \sum_{\substack{z \notin \lambda Q,\\ W(z)\cap U_{il} \neq \varnothing}} V\big(W(z) \cap \lambda Q\big)\Big)\Big| \\ = \sum_{l \in L_0(\lambda)} \left(\frac{1}{2} - q|\alpha|\right) \\ = \left(\frac{1}{2} - q|\alpha|\right) \frac{\mathcal{H}^{d-1}\lambda F}{\det L} + O(\nu \cdot \lambda^{d-2}).\end{aligned}$$

Since at most $O(\nu \cdot \lambda^{d-2})$ cubes $W(z)$ and at most $O(\nu \cdot \lambda^{d-2})$ of the surface area were not taken into account, the theorem follows. ■

The proof of Theorem B is quite analoguous to that of Theorem A. We approximate the boundary of M by k-patches (*cf.* the definition in Section 3). We distinguish between flat patches which can be considered as a substitute for facets and bended patches. Among the flat patches we distinguish further between facets parallel to lattice hyperplanes, "good" patches, for which the normals are not to close to the normals of lattice hyperplanes with small determinant and some remaining "bad" patches. For the lattice facets and the good patches we can apply Lemma 5.1 and for the bended and bad patches we show that there are not too many of them. We prove Theorem B in the apparently equivalent form:

Theorem 5.2 *Let $M \in \mathcal{K}^d$. For any $\varepsilon > 0$ there exist positive λ_0 and ω such that for all $Q \in \mathcal{K}^d$ with $\Delta^H(M, Q) < \omega$ and all $\lambda > \lambda_0$*

$$|G(\lambda Q) - V(\lambda Q)| \leq \tfrac{1}{2} S_{\mathbb{Z}^d}(M) \cdot \lambda^{d-1} + \varepsilon \cdot \lambda^{d-1}.$$

As the proof of Theorem 5.2 is somewhat longish, we shall split it into several lemmas. First we observe that we may assume that $S(M) = 1$, and as the surface area is continuous, also that $S(Q) = 1$. Further there are positive r, R such that for some $t \in E^d$ $rB^d + t \subset \operatorname{int} M$ and $M \subset \operatorname{int} RB^d + t$. Now let some $\varepsilon > 0$ be given.

We start with the construction of a suitable tangent polytope and associated patches (for the definitions see Section 3). First we identify the lattice hyperplanes such that the facets in the hyperplanes could make significant contributions to $G - V$. Let H be a lattice hyperplane, such that $e_i \notin H$. Now let us assume that for $j = 1, \ldots, d$, $j \neq i$, there is a lattice vector of length at most ϕ in $H \cap P_{ij}$ (the definition of P_{ij} is in Section 4). Then the lattice $\mathbb{Z}^d \cap H$ has determinant less than ϕ^{d-1}. Thus for $\phi = 5/\varepsilon$ there exists an $m_0 \in \mathbb{N}$ such that $u_1, \ldots, u_{m_0} \in S^{d-1}$ are the normals of the sub $(d-1)$-lattices such that for some i we have $\langle e_i, u_k \rangle \neq 0$ and each P_{ij}, $j \neq i$, contains a non-zero lattice vector of length at most ϕ of the sublattice. We enumerate the u_k such that for $k = 1, \ldots, m_1$ we have $\mathcal{H}^{d-1}\big(F_M(u_k)\big) > 0$ and for $k = m_1 + 1, \ldots, m$ we have $\mathcal{H}^{d-1}\big(F_M(u_k)\big) = 0$.

Now let T be a fixed tangent polytope of M that has $u_1, \ldots, u_{m_0}$ in its set of normal vectors. We note $F_M(u_k) \subset \partial T$ for $k = 1, \ldots, m_0$. By Lemma 3.2, there exist c, ω_0 depending only on M such that for $\Delta^H(Q, M) < \omega_0$ the maps $\pi_{\partial Q} \colon \partial T \to \partial Q$ and $\pi_{\partial T} \colon \partial Q \to \partial T$

have Lipschitz constant c. By possibly taking c larger we may further assume by (7) that for measurable $\sigma \subset \partial T$ the formula

$$\mathcal{H}^{d-1}\big(\pi_{\partial Q}(\sigma)\big) \leq c \cdot \mathcal{H}^{d-1}(\sigma) \tag{10}$$

holds. From now on we assume without further mentioning that $\Delta^H(Q, M) < \omega_0$.

In the next step we construct a k such that the k-patches have the right properties. We write ϱ_i for the boundary of $\pi_{\partial T}\big(F_M(u_i)\big)$ with respect to ∂T and $\sigma_i(k)$ for the union of k patches on ∂T which intersect $\partial T \cap N(\varrho_i, 1/k)$. By Lemma 3.1 there exists a k_1 and a λ_0 such that for all $\lambda > \lambda_0$

$$\sum_{i=1}^{m} V\left(N\left(\lambda \cdot \pi_{\partial Q}\big(\sigma_i(k_1)\big), \sqrt{d}\right)\right) < \frac{\varepsilon}{8}\lambda^{d-1}. \tag{11}$$

Further we observe that by Lemma 2.2 there exist positive α and $\omega_1 < \omega_0$ such that for all $\omega < \omega_1$

$$\psi_Q\big(N_{S^{d-1}}(u_i, \alpha)\big) \subset \pi_{\partial Q}\big(\sigma_i(k_1) \cup F_M(u_i)\big). \tag{12}$$

Consequently we assume from now that $\Delta^H(Q, M) < \omega_1$ and k is a suitable multiple of k_1.

In the next step we assure that most patches become sufficiently flat. Let β_1 be the angle given by Lemma 4.2 for the α above, n_0 the smallest integer which satisfies $n_0 \geq 5/\varepsilon$ and $1/n_0^2 \leq \tan\beta_1$, and $\beta_2 = \angle\big((1,1), (1, 1 - 1/n_0^2)\big)$. Now let $\eta_1 = \eta/2$ for the angle η given by Lemma 2.5 for $\gamma = \arccos\frac{1}{\sqrt{2d}}$ and $\xi = \min\{\beta_1, \beta_2\}$, η_2 be half of the angle γ provided by Lemma 2.4 for $\beta = 1/(2\sqrt{d})$, $\eta = \min\{\eta_1, \eta_2\}$ and for $\varepsilon/10$ in place of ε.

For this η we construct the set ϱ_η from Lemma 3.3. We write $\varrho(k)$ for the union of k patches which intersect $N\big(\pi_{\partial T}(\varrho_\eta), 1/k\big)$. By Lemma 3.4 and Lemma 3.1 we can find a multiple k_2 of k_1 such that

$$V\left(N\left(\lambda \cdot \pi_{\partial Q}\big(\varrho(k_2)\big), \sqrt{d}\right)\right) < \frac{\varepsilon}{8}\lambda^{d-1}. \tag{13}$$

Now we can further subdivide the patches, such that most of them become very flat on M: By Lemma 3.3 there is a multiple k_3 of k_2 such that all k_3-patches Π on M satisfying $\Pi \not\subset \pi_{\partial M}\big(\varrho(k_2)\big)$ have the property that for $x, y \in \Pi$ the normals n_x, n_y satisfy $\angle(n_x, n_y) < \eta$. We subsume our discussion in the following

Lemma 5.3 *There is a $k \in \mathrm{N}$ such that there are k-patches on M which can be partitioned into three classes:*

(a) Patches, which are contained in the facets $F_M(u_i)$, $i = 1, \ldots, m_1$ of M.
(b) m_2 patches Π_q, such that for the numbers α, η given above all $x, y \in \Pi_q$ satisfy $\angle(n_x, n_y) < \eta$ and $\angle(n_x, u_i) > \alpha$ for $i = 1, \ldots, m_0$.
(c) Patches, which are not enumerated in (a) and (b). Their union ϱ satisfies $V\left(N\big(\lambda\pi_Q(\varrho), \sqrt{d}\big)\right) < \frac{\varepsilon}{4}\lambda^{d-1}$.

We do the summation in (8) separately for the patches listed in (a), (b) and (c) above.

Let us start with the facets $F_M(u_r)$, $r = 1, \dots, m_1$. For a fixed facet $F = F_M(u_r)$ and $u = u_r$, define L, q, $z_0, \dots, z_{q-1}$ and ν as in the the proof of Theorem A. By Lemma 2.3 we may further assume that if Z is a 1-tower at $w \in \pi_{\partial\lambda Q}\big((\lambda F)_{-\nu/2}\big)$ then $\operatorname{dev} Z < \frac{1}{16}\varepsilon$. Then we can proceed exactly as in the proof of Theorem A. We only have to observe, that the parameters t, α now depend on l and i, we have to replace (9) by

$$-1/(2q) - \epsilon/16 \leq \alpha < 1/(2q) + \epsilon/16,$$

and have to apply Lemma 5.1 with $\epsilon/16$ instead of 0. Altogether we obtain (a) below and Lemma 3.1 yields (b):

Lemma 5.4 *Let F be a facet of M with normal u_r, $r \in \{1, \dots, m_1\}$. Then there exist $\nu, \omega, \lambda_0 > 0$ such that for $\Delta^H(Q, M) \leq \omega$ and $\lambda > \lambda_0$*

(a)

$$\Bigg| \sum_{\substack{z \in \lambda Q, \\ W(z) \cap \pi_{\partial\lambda Q}((\lambda F)_{-\nu}) \neq \varnothing}} V\big(W(z) \backslash \lambda Q\big) - \sum_{\substack{z \notin \lambda Q, \\ W(z) \cap \pi_{\partial\lambda Q}((\lambda F)_{-\nu}) \neq \varnothing}} V\big(W(z) \cap \lambda Q\big) \Bigg| \leq \frac{1}{2} \cdot \frac{\mathcal{H}^{d-1}(\lambda F)}{\det(u_i^{\perp} \cap \mathbb{Z}^d)} + \frac{\varepsilon}{4} \mathcal{H}^{d-1}(\lambda F),$$

(b) $V\Big(N\big(\lambda \pi_{\partial Q}(F \setminus F_{-\nu}), \sqrt{d}\big)\Big) < \dfrac{\varepsilon}{8m_1} \lambda^{d-1}$.

Next, consider the patches in Lemma 5.3 (b). For $z = (z_1, \dots, z_d) \in \mathbb{Z}^d$, we call an $U \subset \lambda\partial Q$ an (i, j)-strip of length q at z, if

$$U = \Big\{ x = (x_1, \dots, x_d) \,\Big|\, x \in \partial Q, |z_p - x_p| \leq \frac{1}{2}, \quad p \neq i, j, \langle n_x, e_i \rangle > 0, -\frac{1}{2} \leq x_j - z_j \leq q - \frac{1}{2} \Big\}.$$

Analogously there are (i, j)-strips for $-e_i$. We say that U is simple if it is the union of q simple i-boxes.

Lemma 5.5 *Let Π be a patch from Lemma 5.3 (b).*

(a) There exist $\nu > 0$ and $\lambda_0 > 0$ such that for all $\lambda > \lambda_0$

$$V\Big(N\big(\lambda \pi_{\partial Q}(\Pi \setminus \Pi_{-\nu}), \sqrt{d}\big)\Big) < \frac{\varepsilon}{8m_2} \lambda^{d-1}.$$

(b) Let $x \in \Pi$ and $\langle e_i, n_x \rangle \geq 1/(2\sqrt{d})$ for some $i \in \{1, \dots, d\}$ and $q \in \mathbb{N}$. Then there are positive ω_2, λ_0 with the following property: For all $j \in \{1, \dots, d\}$, $j \neq i$, all $\lambda > \lambda_0$, and all $Q \in \mathcal{K}^d$ with $\Delta^H(M, Q) < \omega_2$ there are simple (i, j)-strips U_l of length q, $l = 1, \dots, m(\lambda)$, such that $U = \bigcup U_l$ covers $\lambda\pi_{\partial Q}(\Pi_{-\nu})$ and each U_l is contained in $\pi_{\partial Q}(\Pi_{-\nu/2})$.

Proof The first statement is an immediate consequence of Lemma 3.1. Next observe that $W(z) \cap \lambda\partial Q = W(z) \cap \lambda\pi_{\partial Q}(\Pi_{-\nu/2})$ holds for every $z \in \mathbb{Z}^d$ with $W(z) \cap \lambda\pi_{\partial Q}(\Pi_{-\nu}) \neq \varnothing$. ∎

For each Π we choose fixed $i,\ j,\ q$ in Lemma 5.5 (b). For this choice we write $\Sigma(\Pi) = \{z \in \mathbb{Z}^d \mid W(z) \cap U \neq \varnothing\}$. We observe by our previous lemma that the sets $\Sigma(\Pi)$ are mutually disjoint.

In the sequel we must take into account the difference between Q and M and that we have only approximate normals of patches. Thus for given P_{ij}, $v \in P_{ij}$, $x \in Q$, and $\theta \geq 0$ we say that v θ-approximates Q at x if for $0 \leq \mu \leq 1$ there exists a $\tau(\mu)$ with $|\tau(\mu)| \leq \theta$ and $x + \mu v + \tau(\mu)e_i \in \partial Q$.

Let Π be a fixed patch from Lemma 5.3 (b) and ν the number from Lemma 5.5. Let $x \in \pi_{\partial Q}(\Pi_{-\nu})$ and n_x a normal at x. We may assume that $|\langle e_1, n_x\rangle| = \max\{|\langle e_i, n_x\rangle|\} \geq 1/\sqrt{d}$ and P_{12} is the plane given by Lemma 4.2 for n_x. Let $u = (u_1, 1, 0, \ldots, 0)$ be a support vector at x to $Q \cap P_{12}$. Let n_0 be the number given in Lemma 4.1 for $\chi = \eta$, where η is the angle used in the construction of the patches. We may assume that the conditions of Lemma 5.5 (b) are satisfied for $q = n_0$. Thus we have a lattice vector (m, n) with $n \leq n_0$ such that $\angle\big(u, (m, n)\big) \leq \eta$. By the choice of η we have on the other hand by Lemma 4.2 that $\|(m, n)\| \geq \phi$.

We now ascertain that nearly all strips on $\pi_{\partial Q}(\Pi)$ behave nicely: We may assume that $\Delta^H(Q, M)$ is sufficiently small and λ sufficiently large, that by Lemma 2.1 $\angle(n_x, n_y) \leq 2\eta$ for all $y \in \pi_{\partial Q}(\Pi_{-\nu/2})$ and by Lemma 5.5 $\lambda\pi_{\partial Q}(\Pi_{-\nu})$ is covered by simple $(1,2)$-strips U_j of length n such that each U_j is contained in $\pi_{\partial Q}(F_{-\nu/2})$.

Let now U_j be such a simple $(1,2)$-strip at $z = (z_1 + \alpha, z_2, \ldots, z_d)$ of length n. Let $\bar{z} = \big(z_1 + \bar{\alpha}, z_2 + (n-1)/2, z_3, \ldots, z_d\big)$ such that $\bar{z} \in U_j$. Now for $-n/2 \leq \mu \leq n/2$ let $z(\mu)$ be defined by the condition, that $z(\mu) = \big(z_1 + \alpha(\mu), z_2 + \mu, z_3, \ldots, z_d\big) \in U_j$. Then we have by our construction that $|z(\mu) - \bar{z} - \mu u| \leq \mu/n$ and $|\mu\big(u - (\frac{m}{n}, 1)\big)| \leq \mu/n$. Thus (m, n) $1/n$-approximates Q at $(z_1 + \alpha, z_2 - 1/2, z_3, \ldots, z_d)$.

Therefore Lemma 5.1 and Lemma 2.4 yield

$$\left|\sum_{i=0}^{n-1}\Big(\sum_{z \in Z_i, z \in \lambda Q} V\big(W(z) \setminus \lambda Q\big) - \sum_{z \in Z_i, z \notin \lambda Q} V\big(W(z) \cap \lambda Q\big)\Big)\right| \leq \frac{1}{2} + n\frac{\varepsilon}{5}.$$

Since $n \geq \phi = 5/\varepsilon$, we deduce by adding up:

Lemma 5.6 *Let Π be a patch given by Lemma 5.3 (b). Then there exist $\nu, \omega, \lambda_0 > 0$ such that for $\Delta^H(Q, M) \leq \omega$ and $\lambda > \lambda_0$*

$$\left|\sum_{z \in \lambda Q, z \in \Sigma(\Pi)} V\big(W(z) \setminus \lambda Q\big) - \sum_{z \notin \lambda Q, z \in \Sigma(\Pi)} V\big(W(z) \cap \lambda Q\big)\right| \leq \frac{\varepsilon}{4} \cdot \mathcal{H}^{d-1}(\lambda\Pi).$$

Finally, we need some book keeping in order to prove Theorem B. We deduce by Lemma 5.4 (a) and Lemma 5.6 that the facets from Lemma 5.3 (a) and the patches from Lemma 5.3 (b) cause all together an error of at most $\frac{1}{2}\varepsilon\lambda^{d-1}$ (remember that $S(M) = 1$). Now the error caused by the cubes which intersect the rest of λQ is at most $\frac{1}{2}\varepsilon\lambda^{d-1}$ by Lemma 5.3 (c), Lemma 5.4 (b) and Lemma 5.5 (a). Summing up these estimates completes the proof of Theorem B.

6 The Optimality of the Estimates of Theorem B and Corollary C

We present series of examples to show that the estimates of Corollary C and hence of Theorem B are optimal in general. Again it is sufficient to consider the case $\Lambda = \mathbb{Z}^d$.

First we look at the coefficient of λ^{d-1}. Here we have optimality for all M.

Example 6.1 *The term $S_\Lambda(M)\lambda^{d-1}$ is optimal in Corollary C for any $M \in \mathcal{K}^d$.*

As the statement is trivial if $S_{\mathbb{Z}^d}(M) = 0$, we assume $S_{\mathbb{Z}^d}(M) > 0$. Let $\varepsilon > 0$, and we prove the existence of arbitrary large λ so that

$$G(\lambda M) > V(M)\lambda^d + \frac{1}{2}S_{\mathbb{Z}^d}(M)\lambda^{d-1} - \varepsilon\lambda^{d-1}. \tag{14}$$

Note that the optimality of the lower bound can be similarly proved, choosing $\lambda = q - \frac{1}{q^{1/m}}$ in (c) below.

As in the proof of Theorem B, set $W(z) = z + [-\frac{1}{2}, \frac{1}{2})$ and assume $S(M) = 1$ and $o \in \operatorname{int} M$. The proof of Theorem B shows that we may choose finitely many lattice facets with outer unit normals $u_1, \dots, u_m$ such that

(a) $\displaystyle \frac{1}{2}\sum \frac{\mathcal{H}^{d-1}\big(F_M(u_i)\big)}{\det(u_i^\perp \cap \mathbb{Z}^d)} > \frac{1}{2}S_{\mathbb{Z}^d}(M) - \frac{\varepsilon}{4}$,

(b) for large λ and $\Omega_\lambda = \{z \mid W(z) \cap \lambda\big(\partial M \setminus \bigcup F_M(u_i)\big) \neq \varnothing\}$,

$$\Big|\sum_{z\in\Omega\cap\lambda M} V\big(W(z)\setminus M\big) - \sum_{z\in\Omega\setminus\lambda M} V\big(W(z)\cap M\big)\Big| < \frac{\varepsilon}{4}\lambda^{d-1}.$$

Now observe that $\tau_i^{-1}F_M(u_i)$ is contained in some lattice hyperplane for $\tau_i = H_M(u_i)\det(u_i^\perp \cap \mathbb{Z}^d)$. Applying simultaneous Diophantine approximation to $\tau_1, \dots, \tau_m$ (compare the proof of Lemma 4.3) results in an arbitrarily large integer q and corresponding $p_1, \dots, p_m \in \mathbb{Z}$ satisfying $|p_i - q\tau_i| < \frac{1}{q^{1/m}}$, and hence

$$\left|\frac{p_i}{\det(u_i^\perp \cap \mathbb{Z}^d)} - qH_M(u_i)\right| < \frac{1}{q^{1/m}}.$$

In particular, Lemma 5.1 yields for large q and $i = 1, \dots, m$ that

(c) for $\lambda = q + \frac{1}{q^{1/m}}$ and $\Omega_\lambda^i = \{z \mid W(z) \cap \lambda F_M(u_i) \neq \varnothing\}$,

$$\sum_{z\in\Omega_\lambda^i\cap M} V\big(W(z)\setminus M\big) - \sum_{z\in\Omega_\lambda^i\setminus M} V\big(W(z)\cap M\big) > \frac{1}{2}\,\frac{\mathcal{H}^{d-1}\big(F_M(u_i)\big)}{\det(u_i^\perp \cap \mathbb{Z}^d)}\lambda^{d-1} - \frac{\varepsilon}{4m}\lambda^{d-1}.$$

Since the number of fundamental cells intersecting the relative boundary of any of the $F_M(u_i)$ is $O(\lambda^{d-2})$ (see the proof of Theorem A), combining (a), (b) and (c) yields (14) by the formula (8).

Next we look at the term $o(\lambda^{d-1})$. Here we have

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Example 6.2 *The error term $o(\lambda^{d-1})$ in Corollary C is optimal.*

For sake of simplicity, we provide an example only for $d = 2$ and $S_{\mathbb{Z}^2}(M) = 0$.

For our example we need the following well-known statement from the theory of numbers:

Lemma 6.3 *Let $\{\delta_q\}$ be a sequence of positive numbers. Then there exists a τ with $0 < \tau < 1$ such that for infinitely many pairs (p, q) of relatively prime natural numbers we have*

$$0 < p \cdot \tau - q < \delta_q.$$

Proof The lemma is proved in a constructive way, completely analogously to the construction of "Liouville-type" transcendental numbers (see [Sc]). ∎

Our examples are given explicitly in the next lemma:

Lemma 6.4 *Let $\varepsilon \colon \mathbb{R}_+ \to \mathbb{R}_+$ be a function satisfying $\lim_{\lambda\to\infty} \varepsilon(\lambda) = 0$. Then there exists an o-symmetric parallelogram M with $S_{\mathbb{Z}^d}(M) = 0$ such that for any natural number N there exists a $\lambda > N$ satisfying*

$$|G(\lambda M) - V(\lambda M)| > \varepsilon(\lambda) \cdot S(\lambda M). \tag{15}$$

Proof By replacing $\varepsilon(\lambda)$ by

$$\sup\{\varepsilon(t) \mid t \geq \lambda\},$$

we may assume that $\varepsilon(\lambda)$ is decreasing.

For $M \in \mathcal{K}^2$ we define for $\lambda > 0$ and $0 < l < 1$ the function $f_M(\lambda, l)$ by

$$f_M(\lambda, l) = \frac{G\big((\lambda + l)M\big) - G(\lambda M) - V\big((\lambda + l)M\big) + V(\lambda M)}{S(\lambda M)}.$$

Now we assume that M is given so that for any large $\lambda > 0$,

$$|G(\lambda M) - V(\lambda M)| \leq \varepsilon(\lambda) \cdot S(\lambda M). \tag{16}$$

For such an M (16) and $\varepsilon(\lambda + l) \leq \varepsilon(\lambda)$ yield that for $\lambda > 1$,

$$f_M(\lambda, l) < 4 \cdot \varepsilon(\lambda). \tag{17}$$

We prove the lemma by constructing a parallelogram M which does not satisfy (17) for certain pairs (λ, l) where λ can be arbitrarily large. For every positive integer q we choose a positive integer $m = m(q)$, so that

$$4 \cdot \varepsilon(m \cdot q) < \frac{1}{q^2},$$

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and a $\delta_q > 0$ satisfying

$$\delta_q < \frac{1}{m \cdot q^2}. \tag{18}$$

Let τ be the number provided by Lemma 6.3. We observe that τ is irrational.

Now we set $u = e_1 + \tau \cdot e_2$, $v = \tau \cdot e_1 - e_2$. Then the parallelogram M is given by

$$M = \operatorname{conv}\{\pm u \pm v\}.$$

We observe that the length of an edge of M is between 2 and $2\sqrt{2}$. We denote by L the line through o and u. Now let

$$0 < p \cdot \tau - q < \delta_q$$

for relatively prime natural numbers p, q. We set $m = m(q)$ and $w = qe_1 + pe_2$. For any integer t with $|t| \leq 2m$, the distance of tw from L is at most $2m\delta_q$. Thus for large q, we may choose λ with

$$m \cdot q < \lambda < m \cdot q + 1,$$

so that each edge of λM contains one lattice point, and there exist $2m$ lattice points along each side of λM which are not in that square but the distance of these points from λM is at most $2m\delta_q$. Let l be minimal so that all of these $4 \times 2m$ points are contained in $(\lambda + l)M$.

It is easy to see that

$$G\big((\lambda + l)M\big) - G(\lambda M) > c_1 \cdot m,$$

$$V\left((\lambda + l)M\right) - V(\lambda M) < c_2 \cdot m^2 \cdot q \cdot \delta_q,$$

and finally,

$$S(\lambda M) < c_3 \cdot m \cdot q$$

for fixed positive c_i, $i = 1, 2, 3$. Now $\varepsilon(\lambda) \leq \varepsilon(mq)$, and the definition of $m(q)$ and δ_q yield that if q is chosen large enough then

$$f_M(\lambda, l) > \frac{c_1}{2c_3} \cdot \frac{1}{q} > 4 \cdot \varepsilon(\lambda).$$

Therefore the lemma follows by (17). ■

For a higher dimensional example, one can use a parallelotope such that all but two coordinates of each facet normal are zero, and the two non-zero coordinates are 1 and the τ above. In order to ensure $S_{Z^d}(M) > 0$, one just cuts the parallelotope by a lattice hyperplane.

Finally we remark that even under the additional assumption of differentiability, we can not hope to improve our estimates very much as Theorem 1 in [MN] shows that in the planar case $o(\lambda)$ cannot be replaced by $o(\lambda^{1-\varepsilon})$ even if the boundary of M is assumed to be analytical.

7 Number of Lattice Points in Large Bodies

We deduce by (5) that

$$d \cdot V(M, K; 1) \geq \sum_{u \in \Lambda^* \text{prim.}} \frac{H_K(u)}{\|u\|} \cdot \mathcal{H}^{d-1}\big(F_M(u)\big).$$

It follows that if $H_K(u) \geq 1/2$ for any primitive $u \in \Lambda^*$ then

$$\frac{\det \Lambda}{2} \cdot S_\Lambda(M) \leq d \cdot V(M, K; 1).$$

Since $V(\,\cdot, K; 1)$ is continuous, Theorem B yields

Lemma 7.1 *Let Λ be a lattice in E^d and assume that for some $K \in \mathcal{K}^d$, $H_K(u) \geq 1/2$ for any primitive $u \in \Lambda^*$. If $M \in \mathcal{K}^d$ and $\{P_\lambda\}$ is a family of convex bodies such that P_λ/λ tends to M then*

$$|\det \Lambda \cdot G_\Lambda(P_\lambda) - V(P_\lambda)| \leq dV(P_\lambda, K; 1) + o\big(S(P_\lambda)\big).$$

Theorem D generalizes this statement to the case where the only condition is that the inradius of the P_λ tends to infinity. As we frequently need the volume of lower dimensional convex bodies we write $|K|$ rather than $\mathcal{H}^{\dim K}(K)$.

Proof of Theorem D As for Theorem B we observe that the inequality in Theorem D is invariant with respect to simultaneous nondegenerate linear transformations of Λ, K and the P_λ. Thus we may assume that $\Lambda = \mathbb{Z}^d$. Further approximating P_λ by polytopes with the same number of lattice points shows that it is sufficient to consider the case where all P_λ are polytopes. We do this by induction on d where the case $d = 1$ is trivial. So we assume that the theorem holds for all dimensions less than d, $d \geq 2$. In addition, we assume that contradicting our statement, there exist an $\varepsilon > 0$ and a sequence $\{P_\lambda\}$ of polytopes with $r_d(P_\lambda) \to \infty$ for $\lambda \to \infty$ and

$$|G(P_\lambda) - V(P_\lambda)| \geq dV(P_\lambda, K; 1) + \varepsilon \cdot S(P_\lambda). \tag{19}$$

If $R(P_\lambda)/r_d(P_\lambda)$ is bounded then by possibly taking a suitable subsequence, we may assume that $P_\lambda/r_d(P_\lambda)$ tends to some convex body M. Then $S(P_\lambda) \sim S\big(r_d(P_\lambda)M\big)$, and hence we easily obtain a family based on $\{P_\lambda\}$ contradicting Lemma 7.1. So we may assume that for some $1 \leq k \leq d-1$ and $c > 0$, we have $r_{k+1}(P_\lambda) < c \cdot r_d(P_\lambda)$ but $r_k(P_\lambda)/r_{k+1}(P_\lambda)$ tends to infinity.

Let L_λ a best approximating affine k-plane for P_λ (*cf.* Section 2). By taking a suitable subsequence, we may assume that the linear k-planes $L_\lambda - L_\lambda$ tend to a linear k-plane $\tilde{L}$.

In the following z always denotes a point of $\mathbb{Z}^d$. The main idea of the inductive step is as follows: We choose a lattice k-plane L close to $\tilde{L}$, and a semi-open fundamental cell T for $L \cap \mathbb{Z}^d$. We define a tiling $W(z)$, $z \in \mathbb{Z}^d$, where each $W(z)$ is congruent to $T + T_0$ and T_0 is a semi-open fundamental cell for $\pi_{L^\perp}(\mathbb{Z}^d)$. We have in analogy to formula (8)

$$G(P_\lambda) - V(P_\lambda) = \sum_{\substack{z \in \mathbb{Z}^d \cap P_\lambda \\ W(z) \cap P_\lambda \neq \varnothing}} V\big(W(z) \backslash P_\lambda\big) - \sum_{\substack{z \in \mathbb{Z}^d \backslash P_\lambda \\ W(z) \cap P_\lambda \neq \varnothing}} V\big(W(z) \cap P\big). \tag{20}$$

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Thus it is sufficient to consider tiles around the boundary of P_λ. We split up P_λ into pieces $(z+T+L^\perp)\cap P_\lambda$. For large λ most of the pieces are almost orthogonal prisms with a basis of the form $(z+L^\perp)\cap P_\lambda$. Splitting up the summation with respect to the pieces and projecting on $L^\perp$ will give a counterexample in $L^\perp$ which contradicts the inductive hypothesis.

For every k-plane L we have for $D = \pi_{L^\perp}(K)$ that $H_K(u) = H_D(u)$ for all u satisfying $\|\pi_L(u)\| = 0$. Thus by Lemma 4.3 for every $\delta > 0$ there is a lattice k-plane L with the following properties.

(a) There exists a Minkowski reduced basis $w_1, \ldots, w_k$ of $L \cap \mathbb{Z}^d$ such that the distance of any point of

$$T = \Big\{ \sum_{i=1}^{k} \alpha_i w_i \;\Big|\; 0 \le \alpha_i < 1 \Big\}$$

from $\tilde{L}$ is at most δ.

(b) There exists an $\varepsilon_0 > 0$ such that for λ sufficiently large and $\|\pi_{L_\lambda}(u)\| < \varepsilon_0$

$$|H_K(u) - H_D(u)| < \delta \tag{21}$$

holds.

Next let $w_{k+1}, \ldots, w_d$ be a basis of $\pi_{L^\perp}(\mathbb{Z}^d)$, and define

$$W = \Big\{ \sum_{i=1}^{d} \alpha_i w_i \;\Big|\; 0 \le \alpha_i < 1 \Big\}.$$

For a $z \in \mathbb{Z}^d$, consider the $y \in \mathbb{Z}^d \cap L$ satisfying $\pi_L(z) \in y + T$, and set

$$W(z) = \pi_{L^\perp}(z) + y + W.$$

Then $\{W(z)\}$, $z \in \mathbb{Z}^d$ is a tiling of E^d.

In the next step we split up the summation in (20). First we identify the "bad" part and show that it is not too large. To do this we define

$$\omega_\lambda = \max \left\{ \frac{1}{\sqrt{r_d(P)}}, \sqrt{\frac{r_{k+1}(P_\lambda)}{r_k(P_\lambda)}} \right\}$$

and $M(\omega) = \pi_{L_\lambda}\big((1-\omega)P_\lambda\big) + \omega c\big(\pi_{L_\lambda}(P_\lambda)\big)$ (*cf.* Section 2). The union of all $W(z)$, which intersect ∂P_λ and are not contained in $M(\omega_\lambda) + L^\perp$, is denoted by N_λ. We have for suitable $t > 0$ that

$$N_\lambda \subset N\left(\partial P \cap \Big(\big(\pi_{L_\lambda}(P_\lambda) \setminus M(\omega_\lambda)\big) + L^\perp\Big), t\right).$$

We deduce by the definition of ω_λ and Lemma 2.8 (a) that

$$\lim_{\lambda\to\infty} \frac{r_{k+1}(P_\lambda)}{\omega_\lambda \cdot r_k\big(\pi_{L_\lambda}(P_\lambda)\big)} = 0, \tag{22}$$

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which in turn yields by Lemma 2.9 that

$$\lim_{\lambda\to\infty} \frac{V(N_\lambda)}{S(P_\lambda)} = 0. \tag{23}$$

We set $\Omega_\lambda = \mathbb{Z}^d \cap L \cap \big(M(\omega_\lambda) + L^\perp\big)$. Now (20) and (23) yield that

$$\begin{aligned} G(P_\lambda) - V(P_\lambda) = &\sum_{z\in\Omega_\lambda} G\big((z + T + L^\perp) \cap P_\lambda\big) \\ &- \sum_{z\in\Omega_\lambda} V\left((z + T + L^\perp) \cap P_\lambda\right) + o\big(S(P_\lambda)\big). \end{aligned} \tag{24}$$

For large λ, we have

$$\pi_{L_\lambda}(P_\lambda) \setminus M\left(\frac{1}{2}\omega_\lambda\right) \subset \pi_{L_\lambda}(N_\lambda) \tag{25}$$

by (22). Let $x \in \partial P_\lambda \subset L_\lambda + (k+1)r_{k+1}(P_\lambda)B^d$ such that $\pi_{L_\lambda}(x) = y \in M(\frac{1}{2}\omega_\lambda)$. The k-ball in L_λ centered at y with radius $\frac{1}{2}\omega_\lambda r_k(P_\lambda)$ is contained in $\pi_{L_\lambda}(P_\lambda)$, and hence if u_x is an unit outer normal at x then

$$\|\pi_{L_\lambda}(u_x)\| < c \cdot \frac{r_{k+1}(P_\lambda)}{\omega_\lambda \cdot r_k(P_\lambda)}. \tag{26}$$

On the other hand, Lemma 2.9 implies that if n_x is some unit outer normal at $x \in \partial P_\lambda$ then

$$dV(P_\lambda, K; 1) = \int_{\bigcup_{z\in\Omega_\lambda}(z+T+L_0)\cap\partial P_\lambda} H_K(n_x)\,dx + o\big(S(P_\lambda)\big). \tag{27}$$

Now we make use of the estimates, how well L approximates L_λ for large λ. For $z \in \Omega_\lambda$, let $A_\lambda(z) \subset L^\perp$ be the maximal and $C_\lambda(z) \subset L^\perp$ be the minimal convex, compact set such that

$$z + T + A_\lambda(z) \subset (z + T + L^\perp) \cap P_\lambda \subset z + T + C_\lambda(z).$$

Denote by $v(\cdot)$ the volumes or mixed volumes in $L^\perp$, and by $s(\cdot)$ the surface area in $L^\perp$.

We observe that by the construction of T for sufficiently small δ and sufficiently large λ

$$|T|v\big(A_\lambda(z)\big) \le V\left((z + T + L^\perp) \cap P_\lambda\right) \le |T|v\big(C_\lambda(z)\big) \le |T|\left(v\big(A_\lambda(z)\big) + \frac{\varepsilon}{4}\right).$$

Writing $P_\lambda(y) = (y + L^\perp) \cap P_\lambda$ for $y \in L$ we have further for $B_\lambda(z) = A_\lambda(z)$ or $B_\lambda(z) = C_\lambda(z)$,

$$\begin{aligned} &\left|\int_{(z+T+L^\perp)\cap\partial P_\lambda} H_K(u_x)\,dx - |T|(d-k)v\big(B_\lambda(z), D; 1\big)\right| \\ &\quad\le \left|\int_T \int_{(z+y+L^\perp)\cap\partial P_\lambda} H_D(u_x)\,dx\,dy - |T|(d-k)v\big(B_\lambda(z), D; 1\big)\right| + \frac{\varepsilon}{8}|T|s\big(B_\lambda(z)\big) \\ &\quad= \left|(d-k)\int_T v\big(P_\lambda(z+y), D; 1\big)\,dy - |T|v\big(B_\lambda(z), D; 1\big)\right| + \frac{\varepsilon}{8}|T|s\big(B_\lambda(z)\big) \\ &\quad\le \frac{\varepsilon}{4}|T|s\big(B_\lambda(z)\big). \end{aligned}$$

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Therefore (19) yields for large λ by (21), (24) and (27) that either

$$\sum_{z\in\Omega_\lambda} G\big(T + A_\lambda(z)\big) \le |T| \cdot \sum_{x\in\Omega_\lambda} v\big(A_\lambda(z)\big) - |T| \cdot \sum_{x\in\Omega_\lambda} \Big[(d-k)\cdot v\big(A_\lambda(z), D; 1\big) + \frac{\varepsilon}{4}\cdot s\big(A_\lambda(z)\big)\Big] \tag{28}$$

or

$$\sum_{z\in\Omega_\lambda} G\big(T + C_\lambda(z)\big) \ge |T| \cdot \sum_{x\in\Omega_\lambda} v\big(C_\lambda(z)\big) + |T| \cdot \sum_{x\in\Omega_\lambda} \Big[(d-k)\cdot v\big(C_\lambda(z), D; 1\big) + \frac{\varepsilon}{4}\cdot s\big(C_\lambda(z)\big)\Big]. \tag{29}$$

We denote by Λ' the orthogonal projection of $\mathbb{Z}^d$ onto $L^\perp$ and note that $\det \mathbb{Z}^d = \det \Lambda' \cdot |T|$ and for any $\sigma \subset L^\perp$, we have $G(\sigma + T) = G_{\Lambda'}(\sigma)$. We deduce by (28) and (29) that for any large λ there exists an $z \in \Omega_\lambda$ such that for $A_\lambda = A_\lambda(z)$ and $C_\lambda = C_\lambda(z)$, either

$$\det \Lambda' \cdot G_{\Lambda'}(A_\lambda) \le v(A_\lambda) - (d-k)\cdot v(A_\lambda, D; 1) - \frac{\varepsilon}{4}\cdot s(A_\lambda)$$

or

$$\det \Lambda' \cdot G_{\Lambda'}(C_\lambda) \ge v(C_\lambda) + (d-k)\cdot v(C_\lambda, D; 1) + \frac{\varepsilon}{4}\cdot s(C_\lambda).$$

Finally, $\omega_\lambda \cdot r_d(P_\lambda) \to \infty$ yields that $r_{d-k}(A_\lambda) \to \infty$. On the other hand, for any primitive u from the dual of Λ' in $L^\perp$, the relation $H_D(u) \ge 1/2$ readily holds. This contradiction with the induction hypothesis implies the theorem. ∎

Remark 1 Assume that $K \in \mathcal{K}^d$ is minimal with the property that $H_K(v) \ge 1/2$ for any primitive $v \in \Lambda^*$. Then there exists some lattice d-polytope P such that for the primitive outer facet normals $v_1, \dots, v_k \in \Lambda^*$ the formula $H_K(v_i) = 1/2$ holds. Set $P_\lambda = \lambda P$ for $\lambda \in \mathbb{N}$. We deduce by Ehrhart's formula that

$$\det \Lambda \cdot G_\Lambda(P_\lambda) = V(P_\lambda) + dV(P_\lambda, K; 1) + O(\lambda^{d-2}),$$

and hence Theorem D can not be improved in general.

Remark 2 Assume that $\Lambda = \mathbb{Z}^d$ and $K = [-\frac{1}{2}, \frac{1}{2}]^d$. Choosing $P_\lambda = [0, \mu_\lambda]^{d-1} \times [0, \lambda]$ where μ_λ tends arbitrarily slowly to infinity shows that the error term is optimal also in Theorem D.

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