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Two structure theorems for homeomorphism groups

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Let $H(C)$ be the group of homeomorphisms of the Cantor set, C, onto itself. Let $p : C \to M$ be a (continuous) map of C onto a compact metric space M, and let $G(p, M)$ be $\{h \in H(C) \mid \forall x \in C, p(x) = ph(x)\}$. $G(p, M)$ is a group. The map $p : C \to M$ is *standard*, if for each $(x, y) \in C \times C$ such that $p(x) = p(y)$, there is a sequence $\{x_n\}_{n=1}^{\infty} \subset C$ and a sequence $\{h_n\}_{n=1}^{\infty} \subset G(p, M)$ such that $x_n \to x$ and $h_n(x_n) \to y$. Standard maps and their associated groups characterize compact metric spaces in the sense that: Two such spaces, M and N, are homeomorphic if and only if, given p standard from C onto M, there is a standard q from C onto N for which $G(p, M) = h^{-1}G(q, N)h$, for some $h \in H(C)$. That is, two compact metric spaces are homeomorphic if and only if they determine, via standard maps, the same classes of conjugate subgroups of $H(C)$.

The present note exhibits two natural structure theorems relating algebraic and topological properties: First, if $M = H \cup K$ $(H, K \neq \emptyset)$, compact metric, and $p : C \to M$ are given, then $G(p, M)$ is isomorphic to a subdirect product of $G(p, M)/S(p, H\backslash K)$ and $G(p, M)/S(p, K\backslash H)$ where, generally, $S(p, N)$ is the normal subgroup of homeomorphisms supported on $p^{-1}(N)$. Second, given M and N compact metric and $p : M \to N$ continuous and onto, let $M \neq M - \mathrm{Cl}D_{\alpha}^{*} \neq \emptyset$, where $\{D_{\alpha}\}_{\alpha \in A}$ is

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the collection of non-degenerate preimages of points in N .
Then there is a standard $p : C \to M$ such that $fp : C \to N$ is standard and there is a homomorphism

$$H : G(p, M) \to G(fp, N)/S(fp, \mathrm{Cl}D^*_\alpha) .$$

Introduction

The results described in the Abstract appeared in [2]. Later, in [3], it was shown that if M_1 and M_2 are compact metric spaces, then there are standard maps $p : C \to M_1 \times M_2$ and $p_i : C \to M_i$, $i = 1, 2$, such that $G(p, M_1 \times M_2) = G(p_1, M_1) \cap G(p_2, M_2)$.

In some vague sense "dual" to this is the observation: if $M = H \cup K$ where $H \cap K = \emptyset$ and each of H and K is closed, then, for $p : C \to M$, $G(p, M)$ is the (interior) direct product of the two normal subgroups supported on (the identity on the complement of) the preimages of H and of K , respectively.

The first of the following theorems concerns the less fortuitous circumstance in which H and K intersect.

The second theorem, below, grows out of the observation in [2], that if $f : M \to N$ is continuous and onto, for M and N compact metric, then there is a standard map $p : C \to M$ such that $fp : C \to N$ is also standard and $G(p, M) \subset G(fp, N)$. Theorem 2 is an indication, for many cases, of "how much bigger" $G(fp, N)$ need be.

First a simple preparatory lemma:

LEMMA. *Given compact metric* M *,* $N \subset M$ *and a map* $p : C \to M$ *, then* $S(p, N) := \{h \in G(p, M) \mid \forall x \in C \backslash p^{-1}(N), h(x) = x\}$ *is a normal subgroup of* $G(p, M)$.

Proof. Since $S(p, N)$ is obviously a subgroup, we show only normality: if $x \notin p^{-1}(N)$, $h(x) \notin p^{-1}(N)$ for any $h \in G(p, M)$. Hence for $f \in S(p, N)$ and $h \in G(p, M)$, $h^{-1}fh(x) = x$, for $x \notin p^{-1}(N)$; and $h^{-1}fh \in S(p, N)$.

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Note that, while $H(C)$ is simple (see [1]), the groups $G(p, M)$ have many different normal subgroups.

THEOREM 1. *Let compact metric* $M = H \cup K$ *and let* $p : C \to M$ *be given. Then* $G(p, M)$ *is isomorphic to a subdirect product of* $G(p, M)/S(p, H\backslash K)$ *and* $G(p, M)/S(p, K\backslash H)$.

Proof. Clearly, $S(p, H\backslash K) \cap S(p, K\backslash H)$ is the identity subgroup. Hence, by a well known theorem about groups, $G(p, M)$ is isomorphic to a subdirect product of $G(p, M)/S(p, H\backslash K)$ and $G(p, M)/S(p, K\backslash H)$. The isomorphism may be chosen so that the elements of $[G(p, M)/S(p, H\backslash K)] \times [G(p, M)/S(p, K\backslash H)]$ in the isomorph of $G(p, M)$ are precisely those of the form $\left(gS(p, H\backslash K), gS(p, K\backslash H)\right)$, $g \in G(p, M)$.

Generally, "subdirectness" indicates the necessary coupling between restrictions of $G(p, M)$ - homeomorphisms to preimages of non-separated sets.

THEOREM 2. *Let* M *and* N *be compact metric spaces and let* $f : M \to N$ *be continuous and onto. Let* $M \neq M - \mathrm{Cl}D^*_\alpha \neq \emptyset$ *, where* $\{D_\alpha\}_{\alpha \in A}$ *is the collection of non-degenerate preimages of points in* N *and* D^*_α *is their union. Then there is a standard* $p : C \to M$ *such that* $fp : C \to N$ *is standard and there is an onto homomorphism* $H : G(p, M) \to G(fp, N)/S\left(fp, \mathrm{Cl}D^*_\alpha\right)$. *If* $\mathrm{Cl}D^*_\alpha$ *is nowhere dense in* M *,* p *and* H *can be chosen so that* H *is an isomorphism.*

Proof. Let $\left\{T^1_i\right\}_{i=1}^{n(1)}$, $\left\{T^2_i\right\}_{i=1}^{n(2)}$, ... , be a sequence of finite closed covers of M with the properties:

1) mesh of $\left\{T^k_i\right\}_{i=1}^{n(k)}$ and of $\left\{f\left(T^k_i\right)\right\}_{i=1}^{n(k)} < 1/k$;

2) $T^k_i \cap T^k_j \neq \emptyset$ $\left(\text{and hence } f\left(T^k_i\right) \cap f\left(T^k_j\right) \neq \emptyset\right)$ is the union of two or more elements of $\left\{T^{k+1}_i\right\}_{i=1}^{n(k+1)}$ $\left(\left\{f\left(T^{k+1}_i\right)\right\}_{i=1}^{n(k+1)}\right)$; and

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3) for each k, $\mathrm{Cl}(M-\mathrm{Cl}D_\alpha^*)$ is the union of elements of $\{T_i^k\}_{i=1}^{n(k)}$, each of which is the closure of an open set in M.

Let the T_i^k's be listed so that the closures of the open sets which cover $\mathrm{Cl}(M-\mathrm{Cl}D_\alpha^*)$ occur first: $T_i^k \subset \mathrm{Cl}(M-\mathrm{Cl}D_\alpha^*)$, $1 \le i \le N(k) < n(k)$.

Now divide the interval $[0, 1]$ into $2n(1)$ equal subintervals, labeling every second one of these, end points included, as $E_1^1, E_2^1, \ldots, E_{n(1)}^1$. Given the interval $E_{i(1),i(2),\ldots,i(k)}^k$, where $i(1), \ldots, i(k)$ is such that $T_{i(1)}^1 \supset T_{i(2)}^2 \supset \ldots \supset T_{i(k)}^k$, divide it into $2m(i(k))$ equal subintervals where $m(i(k))$ is the number of elements of $\{T_i^{k+1}\}_{i=1}^{n(k+1)}$ contained in $T_{i(k)}^k$. Denote every second one of these by

$$E_{i(1),i(2),\ldots,i(k),j(1)}^{k+1}, E_{i(1),i(2),\ldots,i(k),j(2)}^{k+1}, \ldots, E_{i(1),i(2),\ldots,i(k),j(m(i(k)))}^{k+1},$$

where the $j(r)$'s are the subscripts of the elements of the $(k+1)$-st cover which are contained in $T_{i(k)}^k$.

Set $C = \bigcap_{k=1}^{\infty} \left(\bigcup E_{i(1),\ldots,i(k)}^k\right)$ for all sequences $i(1), \ldots, i(k)$ for which $T_{i(1)}^1 \supset T_{i(2)}^2 \supset \ldots \supset T_{i(k)}^k$. C is a Cantor set. Let $F_n(x) = T_{i(n)}^n$ for $x \in C \cap E_{i(1),\ldots,i(n)}^n$, where $i(1), \ldots, i(n)$ is such that $T_{i(1)}^1 \supset \ldots \supset T_{i(n)}^n$. Let $p(x) = \bigcap_{n=1}^{\infty} F_n(x)$, a (continuous) map of C onto M. Observe, as in [2], that $fp : C \to N$ is also standard by condition 2) above.

(The routine constructions of C and p have been repeated to

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permit an identification of the parts of C with which we deal next.)

Letting

$$C_1 = \bigcup_{k=1}^{\infty} \bigcup_{1 \le i(k) \le N(k)} \left(C \cap E^k_{i(1),\ldots,i(k)}\right) = \bigcup_{1 \le i(1) \le N(1)} \left(C \cap E^1_{i(1)}\right) ,$$

and $C_2 = C \backslash C_1$, observe that $h(C_1) = C_1$ by the third condition above, on the covers of M for $h \in G(p, M)$. Otherwise, points in the fp-preimage of $M - \mathrm{Cl}D^*_\alpha$ would be carried onto points of its complement. Since C_1 is both open and closed, each homeomorphism in $G(p, M)$ may be expressed, uniquely except for order, as the product of homeomorphisms supported on (the identity on the complement of) each of C_1 and C_2 . Thus, $G(p, M)$ is the interior direct product of its normal subgroups G_1 and G_2 , the homeomorphisms supported on C_1 and C_2 , respectively.

Likewise, $G(fp, N)$ is the interior direct product of its normal subgroups of homeomorphisms supported on each of C_1 and C_2 , respectively. The first of these is G_1 again. The second is $S\left(fp, \mathrm{Cl}D^*_\alpha\right)$, since none of the points in C_1 which are in the preimage of a point of $\mathrm{Cl}D^*_\alpha$ can be permuted without moving points in the preimages of points outside $\mathrm{Cl}D^*_\alpha$.

The obvious homomorphism H is suggested by the diagram

$$G(p, M) \to \left[G(p, M)/G_2 \approx G_1\right] \to \left[G_1 \approx G(fp, N)/S\left(fp, \mathrm{Cl}D^*_\alpha\right)\right] .$$

If $\mathrm{Cl}D^*_\alpha$ is nowhere dense in M , one can, for each k , let $N(k) = n(k)$, so that each of G_2 and $S\left(fp, \mathrm{Cl}D^*_\alpha\right)$ is the identity subgroup, and H becomes an isomorphism. If $\mathrm{Cl}D^*_\alpha$ is not nowhere dense, obvious examples show the above construction must fail to yield an isomorphism.

References

[1] R.D. Anderson, "The three conjugates theorem", (to appear).

https://doi.org/10.1017/S000497270004627X Published online by Cambridge University Press

[2] Arnold R. Vobach, "On subgroups of the homeomorphism group of the Cantor set", *Fund. Math.* 60 (1967), 47-52.

[3] A.R. Vobach, "A theorem on homeomorphism groups and products of spaces", *Bull. Austral. Math. Soc.* 1 (1969), 137-141.

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