Donaldson, S. K. *Floer homology groups in Yang–Mills theory* (Cambridge Tracts in Mathematics, no. 147, Cambridge University Press, 2002) 244 pp., 0 521 80803 0 (hardback), £50.

This is a relatively short but very informative, modern and clearly written introduction to the theory of Floer's homology groups, a version of Morse theory for certain infinite-dimensional spaces employing many ideas from quantum field theory. The Floer theory has recently found many applications in topology, geometry and theoretical physics.

In the early 1980s, Donaldson, the author of the book under review, obtained new invariants for compact smooth four-dimensional manifolds using moduli spaces of self-dual Yang–Mills gauge fields [**1**]. In 1985, Floer [**3**] used similar techniques to produce new topological invariants for certain three-dimensional manifolds. The relation between the two theories turned out to be much more than just a technical similarity: Floer's theory fits nicely into Donaldson's one as the necessary gadget to extend Donaldson's theory to 4-manifolds with boundary. In the late 1980s, Witten [**4**] was able to relate both theories to physics through his topological field theory.

A brief outline of Floer's homology is now given. For an oriented manifold N, denote by $A^k(N)$ the vector space of smooth $\mathfrak{su}(2)$-valued differential k-forms on N, $\mathfrak{su}(2)$ being the Lie algebra of the group $SU(2)$. The space $A^1(N)$ can be identified with the space of connections in the trivial $SU(2)$-bundle over N. The curvature $F_a \in A^2(N)$ of a particular connection $a \in A^1(N)$ is given by

$$F_a = da + a \wedge a.$$

A connection a is called flat if $F_a = 0$.

The group of gauge transformations

$$\mathcal{G}(N) := \{\text{smooth maps } N \to SU(2)\}$$

acts on $A^1(N)$ affinely,

$$a \to a' = gag^{-1} - dgg^{-1},$$

transforming the curvature linearly, $F_a \to F_{a'} = gF_a g^{-1}$.

Floer's original goal was to construct new invariants for compact 3-manifolds M with $H_1(M, \mathbb{Z}) = 0$ (the so-called homology 3-spheres) generalizing the one constructed by Casson. The idea was to equip M with a metric σ, then to replace (M, σ) by the tubular Riemannian 4-manifold $N = (M \times \mathbb{R},\ g = \sigma \times 1)$, and finally to consider the set $\mathcal{M}$ of all elements $[a]$ from the quotient $A^1(N)/\mathcal{G}(N)$ which

(i) are instantons, i.e. have curvature F_a satisfying the equation, $F_a = *_g F_a$, where $*_g$ is the usual Hodge duality isomorphism on 2-forms with respect to the product metric g; and

(ii) have the finite Yang–Mills action $\|F_a\|_2^2$.

This last condition of course forces each such element a to approach gauge-equivalence classes of flat connections, $[a_-]$ and $[a_+]$, as $t \to \pm\infty$, t being the usual coordinate on the factor $\mathbb{R}$. Thus $\mathcal{M}$ naturally decomposes into subspaces $\mathcal{M}(a_-, a_+)$ of instantons 'connecting' pairs of flat connections in $SU(2)$-bundles over M. Let us denote the space of such gauge-equivalence classes of flat connections on M by R, and set $R^* = R \setminus [0]$. Floer's main observations are that under certain conditions on a 3-manifold M, the following conditions hold.

(i) The spaces $\mathcal{M}(a_-, a_+)$ are smooth manifolds; moreover, there exists a 'relative Morse index', $\mu : R^* \to \mathbb{Z}_8$, such that $\dim \mathcal{M}(a_-, a_+) \bmod 8 = \mu(a_+) - \mu(a_-)$.

(ii) If $a_-, a_+ \in R^*$ are such that $\mu(a_+) - \mu(a_-) = 1$, then $\mathcal{M}(a_-, a_+)$ has finitely many one-dimensional components. If $\langle da_-, a_+\rangle$ denotes the sum over the signs of these components, and R_p denotes the set $\{a \in R^* \mid \mu(a) = p\}$, then the operators

$$d : R_p \to R_{p-1}, \qquad da := \sum_{\mu(a_+)=p-1} \langle da_-, a_+\rangle a_+$$

satisfy $d^2 = 0$.

(iii) The associated homology $\operatorname{Ker} d/\operatorname{Im} d$ does not depend on the choice of the metric σ on M and thus provides us with a set of invariants for the 3-manifold M.

The above outline shows that a rigorous construction of Floer's homology groups $\operatorname{Ker} d/\operatorname{Im} d$ requires fine tools from topology, geometry and analysis. It is a hard task to write a relatively short text which unites these three branches of mathematics into a single and coherent story. The author of the book under review manages to achieve it. I strongly recommend the book to both specialists and graduate students.

The contents of the book are as follows.

Introduction. The author gives an extended comment on the past, present and the future prospects of Floer's theory. Among other things, he draws a fascinating picture of Floer's groups as a Segal type topological field theory.

Chapter 2: Basic material. Yang–Mills theory on compact manifolds is discussed in detail, with a lot of useful technical results listed. The basic framework relating instantons on tubular 4-manifolds and the Chern–Simons functions associated to 3-manifolds is constructed. The chapter has three appendices on local models, pseudo-holomorphic maps, and relations to mechanics.

Chapter 3: Linear analysis. It is here (quoting the author) where 'we ... begin work in earnest'. The chapter provides the reader with the basic analytic background of Floer's theory. It gives a short and clear exposition of key results on linear differential operators over non-compact manifolds with tubular ends.

Chapter 4: Gauge theory and tubular tools. According to the author, 'This Chapter occupies a central position in the book as a whole'. Applying the tools developed in Chapter 3 and the gluing technique, the author develops the theory of instantons on manifolds with tubular ends.

Chapter 5: The Floer homology groups. A rigorous and detailed construction of Floer's homology groups of a homology 3-sphere M is given using the earlier developed theory of instantons on the four-dimensional tube $M \times \mathbb{R}$.

Chapter 6: Floer's homology and 4-manifold invariants. The main theme is the deep interrelations between Floer's and Donaldson's theories.

Chapter 7: Reducible connections and cup products. There are two basic stories: the first one deals with technical problems arising from reducibility, and the other one tells about the product structure on Floer's homology.

Chapter 8: Further directions. A critical analysis of problems yet to solve.

References

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3. A. Floer, An instanton invariant of 3-manifolds, *Commun. Math. Phys.* **118** (1989), 215–240.
4. E. Witten, Topological quantum field theory, *Commun. Math. Phys.* **117** (1988), 353–386.

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