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DYNAMICS OF PSEUDO ALMOST PERIODIC SOLUTION FOR IMPULSIVE NEOCLASSICAL GROWTH MODEL

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Abstract

This paper analyses the pseudo almost periodicity of the impulsive neoclassical growth model. We investigate the existence, uniqueness and exponential stability of the pseudo almost periodic solution. Moreover, an example is given to illustrate the significance of the main findings.

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1. Introduction

The examination of economic growth models is one of the most interesting and important topics in mathematical economics. Based on discrete time scales and a mound-shaped production function, Day [7, 8] introduced and investigated a neoclassical growth model. Since then, many authors have made important contributions to this model. However, only a few of those are devoted to the case of continuous time scales. Matsumoto and Szidarovszky [13] introduced the neoclassical growth model

$$x'(t) = sF(x(t)) - \alpha x(t), \tag{1.1}$$

where x is the capital per labour, $s \in (0, 1)$ is the average propensity to save and $\alpha = n + s\mu$ with μ being the depreciation ratio of capital and n being the growth rate of labour. Since nonlinearities of production functions and production delay are inevitable, Matsumoto and Szidarovszky [14] considered equation (1.1) with the mound-shaped production function $F(x) = \varepsilon x^{\gamma} e^{-\delta x}$, so that

$$x'(t) = \beta x^{\gamma}(t - \tau_0)e^{-\delta x(t-\tau_0)} - \alpha x(t), \tag{1.2}$$

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where α, γ, δ and $\beta = s\varepsilon$ are positive parameters, τ_0 is the delay in the production process and δ reflects the strength of the 'negative effect' caused by increasing concentration of capital. The parameter $\gamma > 0$ can be thought of as a proxy for measuring returns to the scale of the production function. In fact, when x is small, output increases more than unity, exactly unity and less than unity if $\gamma > 1$, $\gamma = 1$ and $\gamma < 1$, respectively.

For equation (1.2), the local asymptotical stability of the positive equilibrium was studied by Matsumoto and Szidarovszky [14]. The permanence of the solutions and global exponential stability of the positive equilibrium were investigated by Chen and Wang [5]. In particular, if $\gamma = 1$, then (1.2) is the well-known Nicholson's blowflies model. An extensive study concerning the dynamic behaviour of Nicholson's blowflies model exists in the literature (see [2, 4, 6, 10, 15, 17] for more details).

The variation of the environment plays an important role in the real world, and impulsive phenomena appear widely in economics. Incorporating these phenomena, the following impulsive generalized neoclassical growth model with delay is considered here:

$$\begin{cases} x'(t) = -\alpha(t)x(t) + \displaystyle\sum_{i=1}^{m} \beta_i(t)x^{\theta}(t-\tau_i)e^{-\eta_i(t)x(t-\tau_i)}, & t \in \mathbb{R}^+, t \neq t_k, \\ \Delta x(t_k) = \gamma_k x(t_k) + I_k(x(t_k)) + \delta_k, \quad k \in \mathbb{N}, \end{cases} \tag{1.3}$$

where $\theta \geq 1$, $\tau_i > 0$, $\alpha(t)$, $\beta_i(t)$, $\eta_i(t)$ $(i = 1, 2, \ldots, m)$, γ_k, δ_k and $I_k(x)$, $k \in \mathbb{N}$, are (pseudo) almost periodic functions or sequences. The main aim of this paper is to obtain the sufficient conditions for the existence, uniqueness and exponential stability of a pseudo almost periodic solution for equation (1.3).

2. Preliminaries and basic results

Throughout this paper, let T be the set consisting of all real sequences $\{t_k\}_{k\in\mathbb{Z}}$ such that $\kappa = \inf_{k\in\mathbb{Z}}(t_{k+1} - t_k) > 0$; then $\lim_{k\to+\infty} t_k = +\infty$ and $\lim_{k\to-\infty} t_k = -\infty$. Let $PC(\mathbb{R}, \mathbb{R})$ denote the space formed by all piecewise continuous functions such that $f(\cdot)$ is continuous at t for $t \notin \{t_k\}_{k\in\mathbb{Z}}$, $f(t_k^+)$ and $f(t_k^-)$ exist and $f(t_k^-) = f(t_k)$ for $k \in \mathbb{Z}$. Define $l^\infty(\mathbb{Z}, \mathbb{R}) = \{x : \mathbb{Z} \to \mathbb{R} \mid \|x\| = \sup_{n\in\mathbb{Z}} |x(n)| < \infty\}$.

Definition 2.1 [11]. A function $f \in C(\mathbb{R}, \mathbb{R})$ is said to be almost periodic in the sense of Bohr if, for each $\varepsilon > 0$, there exists an $l(\varepsilon) > 0$ such that every interval J of length $l(\varepsilon)$ contains a number τ with the property that $|f(t+\tau) - f(t)| < \varepsilon$ for $t \in \mathbb{R}$. Let $AP(\mathbb{R}, \mathbb{R})$ denote the set of all such functions.

Definition 2.2 [16]. A sequence $\{x_n\}$ is called almost periodic if, for any $\varepsilon > 0$, there exists a natural number $l = l(\varepsilon)$ such that for $k \in \mathbb{Z}$, there is at least one number p in $[k, k+l]$ for which inequality $|x_{n+p} - x_n| < \varepsilon$ holds for all $n \in \mathbb{Z}$. Let $AP(\mathbb{Z}, \mathbb{R})$ denote the set of all such sequences.

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Define

$$PAP_0(\mathbb{Z},\mathbb{R}) = \Big\{x_n \in l^\infty(\mathbb{Z},\mathbb{R}) \,\Big|\, \lim_{n\to+\infty} \frac{1}{2n}\sum_{k=-n}^{n} |x_k| = 0\Big\},$$

$$PAP_0(\mathbb{R},\mathbb{R}) = \Big\{f \in C(\mathbb{R},\mathbb{R}) \,\Big|\, \lim_{r\to+\infty} \frac{1}{2r}\int_{-r}^{r} |f(t)|\,dt = 0\Big\}.$$

Definition 2.3 [12]. A function $f \in C(\mathbb{R},\mathbb{R})$ is said to be pseudo almost periodic if it can be decomposed as $f = g + \varphi$, where $g \in AP(\mathbb{R},\mathbb{R})$, $\varphi \in PAP_0(\mathbb{R},\mathbb{R})$. Let $PAP(\mathbb{R},\mathbb{R})$ denote the set of all such functions.

Definition 2.4 [1]. A sequence $\{x_n\}_{n\in\mathbb{Z}} \in l^\infty(\mathbb{Z},\mathbb{R})$ is called pseudo almost periodic if $x_n = x_n^1 + x_n^2$, where $x_n^1 \in AP(\mathbb{Z},\mathbb{R})$, $x_n^2 \in PAP_0(\mathbb{Z},\mathbb{R})$. Let $PAP(\mathbb{Z},\mathbb{R})$ denote the set of all such sequences.

Definition 2.5 [16]. A function $f \in PC(\mathbb{R},\mathbb{R})$ is said to be piecewise almost periodic (denoted by $AP_T(\mathbb{R},\mathbb{R})$) if the following conditions are satisfied.

(1) $\{t_k^j = t_{k+j} - t_k\}, k, j \in \mathbb{Z}$ are equipotentially almost periodic, that is, for any $\varepsilon > 0$, there exists a relatively dense set in $\mathbb{R}$ of ε-almost periods which is common for all the sequences $\{t_k^j\}$.
(2) For any $\varepsilon > 0$, there exists a positive number $\delta = \delta(\varepsilon)$ such that, if the points t' and t'' belong to the same interval of continuity of f and $|t' - t''| < \delta$, then $|f(t') - f(t'')| < \varepsilon$.
(3) For any $\varepsilon > 0$, there exists a relatively dense set Ω_ε in $\mathbb{R}$ such that, if $\tau \in \Omega_\varepsilon$, then $|f(t+\tau) - f(t)| < \varepsilon$ for all $t \in \mathbb{R}$ which satisfy the condition $|t - t_k| > \varepsilon, k \in \mathbb{Z}$.

Define

$$PC_T^0(\mathbb{R},\mathbb{R}) = \Big\{f \in PC(\mathbb{R},\mathbb{R}) \mid \lim_{t\to+\infty} |f(t)|\,dt = 0\Big\},$$

$$PAP_T^0(\mathbb{R},\mathbb{R}) = \Big\{f \in PC(\mathbb{R},\mathbb{R}) \,\Big|\, \lim_{r\to+\infty} \frac{1}{2r}\int_{-r}^{r} |f(t)|\,dt = 0\Big\}.$$

Definition 2.6 [12]. A function $f \in PC(\mathbb{R},\mathbb{R})$ is said to be piecewise pseudo almost periodic if it can be decomposed as $f = g + \varphi$, where $g \in AP_T(\mathbb{R},\mathbb{R})$ and $\varphi \in PAP_T^0(\mathbb{R},\mathbb{R})$. Let $PAP_T(\mathbb{R},\mathbb{R})$ denote the set of all such functions.

Note that $PAP_T(\mathbb{R},\mathbb{R})$ is a Banach space when endowed with the supremum norm $\|\cdot\|$.

Remark 2.7. $PAP_T^0(\mathbb{R},\mathbb{R})$ is a translation-invariant set and $PC_T^0(\mathbb{R},\mathbb{R}) \subset PAP_T^0(\mathbb{R},\mathbb{R})$.

The following is similar to a result of Diagana [9, Lemma 2.5].

Lemma 2.8. *Let $\{f_n\}_{n\in\mathbb{N}} \subset PAP_T^0(\mathbb{R},\mathbb{R})$ be a sequence of functions. If $\{f_n\}$ converges uniformly to f, then $f \in PAP_T^0(\mathbb{R},\mathbb{R})$.*

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3. Main results

Consider the following impulsive neoclassical growth model with delay:

$$\begin{cases} x'(t) = -\alpha(t)x(t) + \sum_{i=1}^{m} \beta_i(t)x^{\theta}(t-\tau_i)e^{-\eta_i(t)x(t-\tau_i)}, & t \in \mathbb{R}^+,\ t \neq t_k, \\ \Delta x(t_k) = \gamma_k x(t_k) + I_k(x(t_k)) + \delta_k, & k \in \mathbb{N}, \end{cases} \tag{3.1}$$

where $\theta \geq 1$, $\alpha(t)$, $\beta_i(t)$, $\eta_i(t) \in C(\mathbb{R}^+, \mathbb{R}^+)$, $\tau_i \geq 0$ $(i = 1, 2, \ldots, m)$, γ_k, $\delta_k \in l^{\infty}(\mathbb{Z}, \mathbb{R}^+)$ and $I_k(x) \in C(\Omega, \mathbb{R}^+)$ for $x \in \Omega$, $k \in \mathbb{N}$. Together with (3.1), we consider the initial condition

$$x_{\sigma} = \xi, \tag{3.2}$$

where $\xi \in PC([\sigma - \tau, \sigma], \mathbb{R}^+)$ and $\tau = \max_{1 \leq i \leq m} \tau_i$. Since we are interested in solutions of economics significance, we restrict our attention to positive ones. Related to (3.1), we consider the linear system

$$\begin{cases} x'(t) = -\alpha(t)x(t), & t \in \mathbb{R}^+,\ t \neq t_k, \\ \Delta x(t_k) = \gamma_k x(t_k), & k \in \mathbb{N}. \end{cases} \tag{3.3}$$

From the work of Samoilenko and Perestyuk [16], it is known that the linear system (3.3) with an initial condition $x(t_0) = x_0$ has a unique solution $x(t; t_0, x_0) = W(t, t_0)x_0, t_0, x_0 \in \mathbb{R}^+$, where W is the Cauchy matrix of (3.3) defined as

$$W(t, s) = \begin{cases} e^{-\int_s^t \alpha(r)dr}, & t_{k-1} < s \leq t \leq t_k, \\ \prod_{i=m}^{k+1}(1 + \gamma_i)e^{-\int_s^t \alpha(r)dr}, & t_{m-1} < s \leq t_m \leq t_k < t \leq t_{k+1}. \end{cases}$$

For convenience, if $f(t)$ is a bounded continuous function, let

$$f^+ = \sup_{t \in \mathbb{R}^+} f(t), \qquad f^- = \inf_{t \in \mathbb{R}^+} f(t).$$

In this paper, we make the following assumptions.

(A_1) The set of sequences $\{t_k^j\}$ are equipotentially almost periodic.
(A_2) $\alpha \in C(\mathbb{R}^+, \mathbb{R}^+)$ is almost periodic in the sense of Bohr and there exists a constant $\mu > 0$ such that $\alpha(t) \geq \mu$.
(A_3) The sequence $\{\gamma_k\}$ is almost periodic and $-1 \leq \gamma_k \leq 0$, $k \in \mathbb{N}$.
(B_1) The functions $\beta_i(t), \eta_i(t) \in PAP(\mathbb{R}^+, \mathbb{R}^+)$, $i = 1, 2, \ldots, m$.
(B_2) The sequence $\{\delta_k\}$ is pseudo almost periodic and $\sup_{k \in \mathbb{N}} |\delta_k| < \varpi$, $k \in \mathbb{N}$.
(B_3) The sequence of functions $\{I_k(x)\}$ are pseudo almost periodic uniform with respect to $x \in \Omega$ and there exist constants $\lambda, L > 0$ such that $|I_k(x)| < \lambda$, $|I_k(x) - I_k(y)| \leq L|x - y|$, $k \in \mathbb{N}$, $x, y \in \Omega$.
(H_1) There exists a constant $\vartheta > 0$ such that $\sum_{i=1}^m \beta_i^+ \vartheta^{\theta} + \mu(\lambda + \varpi)/(1 - e^{-\mu\kappa}) < \mu\vartheta$.
(H_2) The condition $\sum_{i=1}^m \beta_i^+ \vartheta^{\theta-1}(\theta + \vartheta\eta_i^+) + \mu L/(1 - e^{-\mu\kappa}) < \mu$ holds.

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Lemma 3.1 [3]. *Let* (A_1)–(A_3) *be satisfied; then, for* $W(t,s)$ *in* (3.3)*, there exists a positive constant* μ *such that* $|W(t,s)| \le e^{-\mu(t-s)},\ t \ge s,\ t,s \in \mathbb{R}^+$.

Lemma 3.2 [3]. *Let* (A_1)–(A_3) *be satisfied; then, for* $\varepsilon > 0,\ t \in \mathbb{R}^+,\ s \in \mathbb{R}^+,\ t \ge s,$ $|t-t_k| > \varepsilon,\ |s-t_k| > \varepsilon,\ k \in \mathbb{N}$*, there exist a relatively dense set* Λ *of* ε*-almost periods of the function* $\alpha(t)$ *and a constant* $M > 0$ *such that* $|W(t+\omega, s+\omega) - W(t,s)| \le \varepsilon M e^{\mu(t-s)/2}$ *for* $\omega \in \Lambda$.

First, we give the result for the almost periodic case, that is, the case for which the following conditions are satisfied.

- (C_1) $\beta_i(t), \eta_i(t) \in C(\mathbb{R}^+, \mathbb{R}^+)$ are almost periodic in the sense of Bohr, $i = 1, 2, \ldots, m$.
- (C_2) The sequence $\{\delta_k\}$ is almost periodic and $\sup_{k\in\mathbb{N}} |\delta_k| < \varpi$, $k \in \mathbb{N}$.
- (C_3) The sequence of functions $\{I_k(x)\}$, $k \in \mathbb{N}$, are almost periodically uniform with respect to $x \in \Omega$ and there exist constants $\lambda, L > 0$ such that $|I_k(x)| < \lambda$, $|I_k(x) - I_k(y)| \le L|x-y|$, $k \in \mathbb{N}, x, y \in \Omega$.

For $\vartheta > 0$, define $D = \{\varphi \in AP_T(\mathbb{R}^+, \mathbb{R}^+) : \|\varphi\| \le \vartheta\}$.

Lemma 3.3. *Assume that* (A_1)–(A_3), (C_1)–(C_3) *and* (H_1) *hold and* $\varphi \in D$*. Then*

$$(\mathcal{F}\varphi)(t) = \int_{-\infty}^{t} W(t,s) g_\varphi(s)\,ds + \sum_{t_k<t} W(t,t_k)(I_k(\varphi(t_k)) + \delta_k) \in D, \tag{3.4}$$

where $g_\varphi(s) = \sum_{i=1}^{m} \beta_i(s)\varphi^\theta(s-\tau_i)e^{-\eta_i(s)\varphi(s-\tau_i)}$.

Proof. Note that for $t_j \le t < t_{j+1}$, $j \in \mathbb{Z}$,

$$\sum_{t_k<t} e^{-\mu(t-t_k)} \le \sum_{-\infty<k\le j} e^{-\mu(j-k)\kappa} = \sum_{0\le m=j-k<+\infty} e^{-\mu m\kappa} = \frac{1}{1-e^{-\mu\kappa}};$$

then, by Lemma 3.1, for $\varphi \in D$,

$$\begin{aligned}
\|\mathcal{F}\varphi\| &\le \sup_{t\in\mathbb{R}}\Big\{\int_{-\infty}^{t} |W(t,s)|\Big(\sum_{i=1}^{m} |\beta_i(s)\varphi^\theta(s-\tau_i)e^{-\eta_i(s)\varphi(s-\tau_i)}|\Big)ds \\
&\qquad + \sum_{t_k<t} |W(t,t_k)|(|I_k(\varphi(t_k))| + |\delta_k|)\Big\} \\
&\le \sup_{t\in\mathbb{R}}\Big\{\int_{-\infty}^{t} |W(t,s)|\Big(\sum_{i=1}^{m} \beta_i^+\vartheta^\theta\Big)ds + \sum_{t_k<t} e^{-\mu(t-t_k)}(|I_k(\varphi(t_k))| + |\delta_k|)\Big\} \\
&\le \frac{1}{\mu}\Big(\sum_{i=1}^{m} \beta_i^+\vartheta^\theta\Big) + \frac{\lambda+\varpi}{1-e^{-\mu\kappa}} \le \vartheta.
\end{aligned}$$

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For $t_k < t \le t_{k+1}$, Samoilenko and Perestyuk [16] have shown that there exist ω, q such that

$$\begin{aligned}\|\mathcal{F}\varphi(t+\omega) - \mathcal{F}\varphi(t)\| &\le \sup_{t\in\mathbb{R}}\Big\{\int_{-\infty}^{t} |W(t+\omega, s+\omega) - W(t,s)||g_\varphi(s+\omega)|\,ds\Big\} \\ &\quad + \sup_{t\in\mathbb{R}}\Big\{\int_{-\infty}^{t} |W(t,s)||g_\varphi(s+\omega) - g_\varphi(s)|\,ds\Big\} \\ &\quad + \sup_{t\in\mathbb{R}}\Big\{\sum_{t_k<t} |W(t+\omega, t_{k+q}) - W(t,t_k)|(|\sigma_{k+q}| + |\delta_{k+q}|)\Big\} \\ &\quad + \sup_{t\in\mathbb{R}}\Big\{\sum_{t_k<t} |W(t,t_k)||\sigma_{k+q} - \sigma_k| + \sum_{t_k<t} |W(t,t_k)||\delta_{k+q} - \delta_k|\Big\} \\ &\le C\varepsilon,\end{aligned}$$

where

$$C = \frac{2M}{\mu}\sum_{i=1}^{m}\beta_i^{+}\vartheta^{\theta} + \frac{1}{\mu}\Big(\sum_{i=1}^{m}\beta_i^{+} + m\vartheta^{\theta}\Big) + \frac{M(\lambda+\varpi)}{1-e^{-\mu\kappa/2}} + \frac{2}{1-e^{-\mu\kappa}},$$

which implies that $\mathcal{F}\varphi \in AP_T(\mathbb{R}^+, \mathbb{R}^+)$; thus, $\mathcal{F}\varphi \in D$. □

Theorem 3.4. *If* (A_1)–(A_3), (B_1)–(B_3) *and* (H_1)–(H_2) *hold, then equation (3.1) has a unique positive solution* $x \in PAP_T(\mathbb{R}^+, \mathbb{R}^+)$.

Proof. Define $\mathcal{D} = \{\varphi \in PAP_T(\mathbb{R}^+, \mathbb{R}^+) \mid \|\varphi\| \le \vartheta\}$ and $\mathcal{F}$ in $\mathcal{D}$ as in equation (3.4). Since $\theta_k = I_k(\varphi(t_k)) \in PAP(\mathbb{Z}, \mathbb{R}^+)$, $\delta_k \in PAP(\mathbb{Z}, \mathbb{R}^+)$, let $\theta_k = \mu_k + \nu_k$, $\delta_k = a_k + b_k$, where $\mu_k, a_k \in AP(\mathbb{Z}, \mathbb{R}^+)$, $\nu_k, b_k \in PAP_0(\mathbb{Z}, \mathbb{R}^+)$. Let $g_\varphi = g_1 + g_2$, $g_1 \in AP_T(\mathbb{R}^+, \mathbb{R}^+)$, $g_2 \in PAP_T^0(\mathbb{R}^+, \mathbb{R}^+)$. Hence, $\mathcal{F}\varphi = \mathcal{F}_1\varphi + \mathcal{F}_2\varphi$, where

$$\mathcal{F}_1\varphi = \int_{-\infty}^{t} W(t,s)g_1(s)\,ds + \sum_{t_k<t} W(t,t_k)(a_k + \mu_k),$$

$$\mathcal{F}_2\varphi = \int_{-\infty}^{t} W(t,s)g_2(s)\,ds + \sum_{t_k<t} W(t,t_k)(b_k + \nu_k) := \mathcal{H}_1 + \mathcal{H}_2.$$

Similarly as in the proof of Lemma 3.3, $\mathcal{F}_1\varphi \in AP_T(\mathbb{R}^+, \mathbb{R}^+)$. Next, we show that $\mathcal{F}_2\varphi \in PAP_T^0(\mathbb{R}^+, \mathbb{R}^+)$. In fact, for $r > 0$,

$$\begin{aligned}\frac{1}{2r}\int_{-r}^{r}\Big\|\int_{-\infty}^{t} W(t,s)g_2(s)\,ds\Big\|\,dt &\le \frac{1}{2r}\int_{-r}^{r}\int_{-\infty}^{t} e^{-\mu(t-s)}\|g_2(s)\|\,ds\,dt \\ &= \int_{0}^{\infty} e^{-\mu s}\Phi_r(s)\,ds,\end{aligned}$$

where $\Phi_r(s) = (1/2r)\int_{-r}^{r}\|g_2(t-s)\|\,dt$. Since $g_2 \in PAP_T^0(\mathbb{R}^+, \mathbb{R}^+)$, it follows that $g_2(\cdot - s) \in PAP_T^0(\mathbb{R}^+, \mathbb{R}^+)$ for each $s \in \mathbb{R}$ by Remark 2.7; hence, $\lim_{r\to+\infty}\Phi_r(s) = 0$

for all $s \in \mathbb{R}$. By using the Lebesgue dominated convergence theorem, $\mathcal{H}_1 \in PAP_T^0(\mathbb{R}^+, \mathbb{R}^+)$. Now, we show that $\mathcal{H}_2 \in PAP_T^0(\mathbb{R}^+, \mathbb{R}^+)$. For a given $k \in \mathbb{Z}$, define the function $\xi(t)$ by $\xi(t) = W(t, t_k)(b_k + \nu_k)$, $t_k < t \leq t_{k+1}$; then

$$\lim_{t\to+\infty} |\xi(t)| = \lim_{t\to+\infty} |W(t, t_k)(b_k + \nu_k)| \leq \lim_{t\to+\infty} e^{-\mu(t-t_k)}(|b_k| + |\nu_k|) = 0,$$

so that $\xi \in PC_T^0(\mathbb{R}^+, \mathbb{R}^+) \subset PAP_T^0(\mathbb{R}^+, \mathbb{R}^+)$. Define $\xi_m : \mathbb{R} \to \mathbb{R}^+$ by $\xi_m(t) = W(t, t_{k-m})(b_{k-m} + \nu_{k-m})$, $t_k < t \leq t_{k+1}$, $m \in \mathbb{N}^+$, so that $\xi_m \in PAP_T^0(\mathbb{R}^+, \mathbb{R}^+)$. Moreover, $|\xi_m(t)| \leq \sup_{k\in\mathbb{Z}}(|b_k| + |\nu_k|)e^{-\mu(t-t_{k-m})} \leq \sup_{k\in\mathbb{Z}}(|b_k| + |\nu_k|)e^{-\mu(t-t_k)}e^{-\mu\kappa m}$. Therefore, $\sum_{m=1}^{\infty} \xi_m$ is uniformly convergent on $\mathbb{R}$, so that $\mathcal{H}_2 \in PAP_T^0(\mathbb{R}^+, \mathbb{R}^+)$ by Lemma 2.8. Hence, $\mathcal{F}$ is a self-mapping from $\mathcal{D}$ to $\mathcal{D}$. Let $\varphi, \psi \in \mathcal{D}$; then

$$\begin{aligned}
\|\mathcal{F}\varphi - \mathcal{F}\psi\| \leq \int_{-\infty}^{t} W(t, s)\Big[&\sum_{i=1}^{m} \beta_i(s)e^{-\eta_i(s)\varphi(s-\tau_i)}|\varphi^\theta(s-\tau_i) - \psi^\theta(s-\tau_i)| \\
&+ \sum_{i=1}^{m} \beta_i(s)\psi^\theta(s-\tau_i)|e^{-\eta_i(s)\varphi(s-\tau_i)} - e^{-\eta_i(s)\psi(s-\tau_i)}|\Big]\,ds \\
&+ \sum_{t_k<t} W(t, t_k)|I_k(\varphi(t_k)) - I_k(\psi(t_k))| \\
\leq{}& \int_{-\infty}^{t} W(t, s)\Big[\sum_{i=1}^{m} \beta_i^+\theta\vartheta^{\theta-1} + \sum_{i=1}^{m} \beta_i^+\eta_i^+\vartheta^\theta\Big]\,ds \cdot \|\varphi - \psi\| \\
&+ L\sum_{t_k<t} W(t, t_k)|\varphi(t_k) - \psi(t_k)| \\
\leq{}& \Big[\frac{1}{\mu}\sum_{i=1}^{m} \beta_i^+\vartheta^{\theta-1}(\theta + \vartheta\eta_i^+) + \frac{L}{1 - e^{-\mu\kappa}}\Big]\|\varphi - \psi\|.
\end{aligned}$$

Hence, equation (3.1) has a unique positive solution $x \in PAP_T(\mathbb{R}^+, \mathbb{R}^+)$. □

Next, if $\tau_i = 0$, the following result holds for the PAP_T solution.

Theorem 3.5. *Assume that (A_1)–(A_3), (B_1)–(B_3) and (H_1)–(H_2) hold. If $[\ln(1 + L)]/\kappa + \sum_{i=1}^{m} \beta_i^+\vartheta^{\theta-1}(\theta + \vartheta\eta_i^+) < \mu$, then the unique PAP_T solution, $x(t)$, of (3.1) is exponentially stable.*

Proof. Let $y(t)$ be an arbitrary solution of (3.1) and (3.2), and $x(t)$ be a unique positive PAP_T solution of (3.1) with the initial condition $x_\sigma = \zeta$. Then

$$\begin{aligned}
|y(t) - x(t)| \leq e^{-\mu(t-\sigma)}|\xi - \zeta| + \int_{\sigma}^{t} e^{-\mu(t-s)}\Big[\sum_{i=1}^{m} \beta_i^+\vartheta^{\theta-1}(\theta + \vartheta\eta_i^+)\Big]|y(s) - x(s)|\,ds \\
+ \sum_{\sigma<t_k<t} e^{-\mu(t-t_k)}L|y(t_k) - x(t_k)|.
\end{aligned}$$

Let $u(t) = |y(t) - x(t)|e^{\mu t}$; then

$$u(t) \leq u(\sigma) + \int_{\sigma}^{t} \Big\{\sum_{i=1}^{m} \beta_i^+\vartheta^{\theta-1}(\theta + \vartheta\eta_i^+)\Big\}u(s)\,ds + \sum_{\sigma<t_k<t} Lu(t_k).$$

By the generalized Gronwall–Bellman inequality [16],

$$\begin{aligned} u(t) &\le u(\sigma) \prod_{\sigma<t_k<t} (1+L) \exp\Big((t-\sigma)\Big\{\sum_{i=1}^m \beta_i^+ \vartheta^{\theta-1}(\theta+\vartheta\eta_i^+)\Big\}\Big) \\ &\le u(\sigma) \exp\Big(\Big\{\frac{\ln(1+L)}{\kappa} + \sum_{i=1}^m \beta_i^+ \vartheta^{\theta-1}(\theta+\vartheta\eta_i^+)\Big\}(t-\sigma)\Big), \end{aligned}$$

that is, $|y(t)-x(t)| \le |\xi-\zeta| \exp((t-\sigma)\{\ln(1+L)/\kappa + \sum_{i=1}^m \beta_i^+\vartheta^{\theta-1}(\theta+\vartheta\eta_i^+) - \mu\})$. Then it follows that the unique PAP_T solution of (3.1) is exponentially stable. □

4. Example

Consider the following impulsive neoclassical growth model with delay:

$$\begin{cases} x' = -\alpha(t)x(t) + \beta(t)x^2(t-\tau)e^{-\eta(t)x(t-\tau)}, & t \in \mathbb{R},\ t \ne t_k, \\ \Delta x(t_k) = \gamma_k x(t_k) + \delta_k, & k \in \mathbb{N}, \end{cases} \tag{4.1}$$

where $\tau \ge 0$ and

$$t_k = k + \frac{1}{4}|\sin k - \sin \sqrt{2}k|, \quad \gamma_k = -\frac{1}{5}(|\sin k| + |\sin \pi k|),$$

$$\delta_k = \frac{1}{3}(|\sin k| + |\sin \sqrt{2}k|) + \frac{1}{3(1+k^2)}, \quad \alpha(t) = |\sin t| + |\sin \sqrt{2}t| + 5,$$

$$\beta(t) = \frac{1}{10}|\cos t| + \frac{1}{10}|\cos \sqrt{2}t| + \frac{3}{10(1+t^4)},$$

$$\eta(t) = \frac{1}{3}\cos^2 t + \frac{1}{3}\cos^2 \sqrt{2}t + \frac{1}{3(1+t^4)}.$$

Then $\delta_k \in PAP(\mathbb{Z}, \mathbb{R}^+)$ and $\gamma_k \in AP(\mathbb{Z}, \mathbb{R}^+)$, $-1 < \gamma_k \le 0$, so that (A_3) and (B_2) hold with $\varpi = 1$. Note that $\{t_k^j\}$, $k \in \mathbb{Z}$, $j \in \mathbb{Z}$, are equipotentially almost periodic and $\kappa = \inf_{k\in\mathbb{Z}}(t_{k+1} - t_k) > 2/5 > 0$ (see [12, 16] for more details). Hence, (A_1) holds. It is not difficult to see that (A_2), (B_1) and (B_3) hold with $\mu = 5$, $L = 0$. Since $\beta^+ = 0.5$, $\eta^+ = 1$, (H_1) and (H_2) hold with $\vartheta = 2$. By Theorems 3.4 and 3.5, equation (4.1) has a unique solution $x \in PAP_T(\mathbb{R}^+, \mathbb{R}^+)$ which is exponentially stable.

5. Conclusion

In this paper, dynamics of the pseudo almost periodic solution for the impulsive neoclassical growth model are investigated. The Banach contraction mapping principle and the Gronwall–Bellman inequality are the main tools used in carrying out the proofs.

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