ZL-amenability Constants of Finite Groups with Two Character Degrees

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Abstract. We calculate the exact amenability constant of the centre of $\ell^1(G)$ when G is a finite group and is either dihedral, extraspecial, or Frobenius with abelian complement and kernel. This is done using a formula that applies to all finite groups with two character degrees. In passing, we answer in the negative a question raised in work of the third author with Azimifard and Spronk.

1 Introduction

Let G be a compact group. While its group algebra $L^1(G)$ is always amenable as a Banach algebra, the centre of $L^1(G)$ need not be amenable.[1] The third author of this paper, together with A. Azimifard and N. Spronk [1], examined this phenomenon in various cases and showed that if G is an infinite product of non-abelian finite groups, given the product topology, then the centre of $L^1(G)$ fails to be amenable. (Theorem 1.10, *op cit.*) The proof works by examining the so-called *amenability constant* of the centre of $L^1(G)$ in the case where G is finite, and observing that this constant is nothing but the ℓ^1-norm of a certain central idempotent in the complex group ring of $G \times G^{op}$. It then follows from results of D. Rider that the infimum of the possible amenability constants as G varies over all finite, non-abelian groups is strictly greater than 1; however, the lower bound provided by Rider's theorem seems far from best possible.

Motivated by this and other questions in [1], we investigate the amenability constant of the centre of $L^1(G)$ in the case where G is a finite group with two character degrees (that is, nonabelian finite groups where all irreducible non-linear characters have the same degree). For this special class of groups, we obtain an exact formula for the amenability constant that is easily computed for various natural classes of such groups, including dihedral groups, extraspecial p-groups, and certain Frobenius groups. It will be seen that in some cases interesting patterns emerge, which remain to be fully explored.

Notation and terminology Given a group G (with its discrete topology) we denote its ℓ^1-group algebra by $\ell^1(G)$ and denote the centre of $\ell^1(G)$ by $Z\ell^1(G)$. In this paper

[1]It is not clear where credit belongs for this observation, but it is mentioned in [9] as a personal communication from B. E. Johnson.

we are only concerned with finite groups. We assume the reader has a basic knowledge of finite group theory. For all other terminology and definitions pertaining to character theory, the book [6] will suffice, as should many other choices. See also Chapters 3 and 4 of Gorenstein's book [3].

Given a finite group G, we shall define the amenability constant of $Z\ell^1(G)$ in Section 2.1. This constant will be denoted throughout by $\mathrm{AM}_Z(G)$, and to save cumbersome repetition in verbal descriptions, we will refer to $\mathrm{AM}_Z(G)$ as the *ZL-amenability constant of G*. Although this is, to our knowledge, not standard terminology, it emphasizes that $\mathrm{AM}_Z(G)$ can be computed from the character table of G without any reference to Banach algebras and the general theory of amenability. Indeed, one can take equation (2.2) below as the *definition* of $\mathrm{AM}_Z(G)$ when G is finite.

2 A Formula for the Amenability Constant

2.1 General Background

We start with some general remarks on amenability constants. This material is not original. It is well known to specialists and is either folklore or can be extracted from parts of [1]. Nevertheless we include these remarks to provide context for the formulas that will follow. We will not discuss amenability for general Banach algebras here and instead focus on the finite-dimensional setting.

If A is a finite-dimensional Banach algebra with identity, then A is *amenable* if and only if there exists some $M \in A \otimes A$ satisfying $a \cdot M = M \cdot a$ and $\pi(M) = 1_A$. Such an M is called a *diagonal element* for the algebra A. (Thus far we have not used the norm on A. Indeed, the characterization of amenability that we have given for finite-dimensional algebras is equivalent to *separability* in the sense of ring-theory; what we call a diagonal element is usually called a *separability idempotent* in that context.)

Example 2.1 Let H be a finite group. Then it is straightforward to check that

$$N := |H|^{-1} \sum_{h \in H} \delta_h \otimes \delta_{h^{-1}}$$

is a diagonal element for the algebra $\ell^1(H)$, and that $\|N\| = 1$ as an element of the projective tensor product $\ell^1(H) \widehat{\otimes} \ell^1(H) = \ell^1(H \times H)$.

If A is a finite-dimensional, semisimple, and commutative Banach algebra, then it is isomorphic as an algebra to $\mathbb{C}^n$ for some n. Since $\mathbb{C}^n$ has a unique diagonal element, so does A. More precisely, if the minimal idempotents of A are $e_1, \ldots, e_n$, then the unique diagonal element for A is just $M = \sum_{j=1}^{n} e_j \otimes e_j$. In this setting the amenability constant of A is the norm of M as an element of the projective tensor product $A \widehat{\otimes} A$.

Now let G be a finite group. By basic results from representation theory, the centre of $\mathbb{C}G$ is commutative and semisimple, and its minimal idempotents are the functions of the form $|G|^{-1}d_\chi\chi$, where χ is an irreducible character on G. (See also [6, Corollary 15.4] for an alternative approach to this result.) Therefore, letting $\mathrm{Irr}(G)$ denote

the set of irreducible characters on G, the unique diagonal element of $Z\ell^1(G)$ is

$$(2.1) \qquad M := \sum_{\chi \in \mathrm{Irr}(G)} \frac{d_\chi}{|G|} \chi \otimes \frac{d_\chi}{|G|} \chi.$$

We remark that this observation is also made in the proof of [1, Theorem 1.8]. [2]

Let $\mathrm{Conj}(G)$ denote the set of conjugacy classes in G, and let $\chi(C)$ denote the value taken by a character χ on any (hence every) element of a conjugacy class $C \in \mathrm{Conj}(G)$. Then, as shown in the proof of [1, Theorem 1.8], we find that

$$(2.2) \qquad \mathrm{AM}_Z(G) = \|M\| = \frac{1}{|G|^2} \sum_{C,D \in \mathrm{Conj}(G)} \Big| \sum_{\chi \in \mathrm{Irr}(G)} d_\chi^2 \chi(C)\overline{\chi(D)} \Big| \, |C| \, |D|.$$

Equation (2.2) is difficult to work with if one wants to calculate the *exact* ZL-amenability constant of G, as it seems to require knowledge of the full character table of G. When G has two character degrees, things simplify greatly.

2.2 The Case of Two Character Degrees

Following standard terminology, we say that a character on a finite group is *linear* if it is actually a homomorphism $G \to \mathbb{T}$ (*i.e.,* the trace of a 1-dimensional representation) and *non-linear* otherwise.

Definition 2.2 We say that a finite group G has *two character degrees* if (i) G is non-abelian, (ii) every non-linear irreducible character has the same degree.

Remark 2.3 Groups with two character degrees were studied in an unrelated context by Isaacs and Passman [5]. (In the terminology of that paper, the groups we consider *have a.c. m* for some integer $m > 1$.) It is known that such groups must be metabelian [4, Corollary 12.6], and they seem a natural class of examples to consider when looking for large non-abelian groups with small ZL-amenability constants. We shall see some evidence to support this approach in Examples 3.6 and 3.13.

Given a finite group G, we denote by G' its *derived subgroup* (also called its *commutator subgroup*). This can be constructed as the normal subgroup generated by all commutators in G, or characterized as the smallest normal subgroup N for which the quotient G/N is abelian.

Theorem 2.4 Let G be a finite group wth two character degrees, and let m be the degree of any (hence every) non-linear irreducible group character of G. Then

$$(2.3) \qquad \mathrm{AM}_Z(G) = 1 + 2(m^2 - 1)\Big(1 - \frac{1}{|G| \, |G'|} \sum_{C \in \mathrm{Conj}(G)} |C|^2 \Big).$$

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[2]The reader should be aware that in [1, Section 1.5], G is equipped with the normalized counting measure. Here we have chosen to equip G with counting measure, since this ensures that $Z\ell^1(G)$ has an identity element of norm 1.

The advantage of (2.3) over (2.2) is that, once we know that every non-linear character of G has dimension m, we only need two other pieces of information: the order of the derived subgroup (see Remark 2.5) and the size of each conjugacy class. In particular, we do not need the full character table of G.

Remark 2.5 Given an arbitrary finite group G, let L denote the set of linear characters on G. Then L is in bijection, in a natural way, with the set of characters on the abelian group $G_{ab} := G/G'$ (see for instance, [6, Theorem 17.11]); and consequently $|L| = |\widehat{G_{ab}}| = |G_{ab}| = |G| / |G'|$). We shall use this fact in the proof of Theorem 2.4. For some of our examples, it will be more convenient to count the number of linear characters than to work out the derived subgroup.

Before proving Theorem 2.4, we isolate one of the steps as a separate lemma.

Lemma 2.6 *Let G be an arbitrary finite group, and let $L = \{\chi \in \mathrm{Irr}(G) \colon d_\chi = 1\}$. Then*

$$(2.4) \qquad \frac{1}{|G|^2} \sum_{C,D \in \mathrm{Conj}(G)} |C|\,|D| \Big| \sum_{\chi \in L} \chi(C)\overline{\chi(D)} \Big| = 1.$$

Proof Let G_{ab} denote the quotient group G/G' and let $q : G \to G_{ab}$ be the quotient homomorphism. The left-hand side of (2.4) is the norm, in $Z\ell^1(G \times G)$, of the idempotent

$$\widetilde{M} := \frac{1}{|G|^2} \sum_{\chi \in L} \chi \otimes \chi.$$

Since each χ is constant on cosets of the derived subgroup, $\widetilde{M}$ factors through the quotient map $q \otimes q : Z\ell^1(G \times G) \to Z\ell^1(G_{ab} \times G_{ab}) = \ell^1(G_{ab} \times G_{ab})$. For each $x, y \in G$, put

$$M\big(q(x), q(y)\big) := |G'|^2 \widetilde{M}(x, y).$$

Then M is a well-defined element of $Z\ell^1(G_{ab} \times G_{ab})$, and a little thought shows that $\|M\| = \|\widetilde{M}\|$. On the other hand, since $\chi \in L$ if and only if $\chi = \phi \circ q$ for some $\phi \in \widehat{G_{ab}}$ (see Remark 2.5),

$$M = \frac{1}{|G_{ab}|^2} \sum_{\phi \in \widehat{G_{ab}}} \phi \otimes \overline{\phi}.$$

Comparing this with (2.1), we see that M is the unique diagonal element for the Banach algebra $Z\ell^1(G_{ab}) = \ell^1(G_{ab})$. On the other hand, by Example 2.1, $\ell^1(G_{ab})$ has a diagonal element of norm 1. It follows that M has norm 1, which completes the proof. ∎

Proof of Theorem 2.4 To ease notation, we write Conj instead of Conj(G) throughout this proof. Let

$$\mathrm{AM}_{\mathrm{diag}} = \frac{1}{|G|^2} \sum_{C \in \mathrm{Conj}} |C|^2 \sum_{\chi \in \mathrm{Irr}(G)} d_\chi^2 |\chi(C)|^2$$

and

$$\mathrm{AM}_{\mathrm{off}} = \frac{1}{|G|^2} \sum_{(C,D) \in \mathrm{Conj}^2 : \, C \neq D} |C|\,|D| \, \Big| \sum_{\chi \in \mathrm{Irr}(G)} d_\chi^2 \chi(C) \overline{\chi(D)} \Big|,$$

so that, by (2.2), $\mathrm{AM}_{\mathrm{Z}}(G) = \mathrm{AM}_{\mathrm{diag}} + \mathrm{AM}_{\mathrm{off}}$.

Let $L = \{\chi \in \mathrm{Irr}(G) \colon d_\chi = 1\}$. Using Schur column orthogonality and the fact that $|\chi(\,\cdot\,)|^2 = 1$ for every $\chi \in L$, we get

$$(2.5) \qquad |G|^2 \, \mathrm{AM}_{\mathrm{diag}} = \sum_{C \in \mathrm{Conj}} |C|^2 \Big(\sum_{\chi \in \mathrm{Irr}(G)} m^2 |\chi(C)|^2 - \sum_{\chi \in L} (m^2 - 1)|\chi(C)|^2 \Big)$$

$$= \sum_{C \in \mathrm{Conj}} |C|^2 \Big(m^2 \frac{|G|}{|C|} - (m^2 - 1)|L| \Big)$$

$$= m^2 |G|^2 - (m^2 - 1)\,|L| \sum_{C \in \mathrm{Conj}} |C|^2.$$

Similarly,

$$(2.6)$$
$$|G|^2 \, \mathrm{AM}_{\mathrm{off}} = \sum_{(C,D) \colon C \neq D} |C|\,|D| \, \Big| \sum_{\chi \in \mathrm{Irr}(G)} m^2 \chi(C) \overline{\chi(D)} - \sum_{\chi \in L} (m^2 - 1)\chi(C)\overline{\chi(D)} \Big|$$

$$= \sum_{(C,D) \colon C \neq D} |C|\,|D| \, \Big| \sum_{\chi \in L} (m^2 - 1)\chi(C)\overline{\chi(D)} \Big|$$

$$\text{(by Schur column orthogonality)}$$

$$= (m^2 - 1) \sum_{(C,D)} |C|\,|D| \, \Big| \sum_{\chi \in L} \chi(C)\overline{\chi(D)} \Big| - (m^2 - 1) \sum_{C} |C|^2 \, |L|$$

$$\text{(since } |\chi(\,\cdot\,)|^2 = 1 \text{ for all } \chi \in L\text{)}$$

$$= (m^2 - 1)|G|^2 - (m^2 - 1) \sum_{C} |C|^2 \, |L|,$$

with the last equation following from Lemma 2.6. Combining (2.5) and (2.6), using the equality $|L| = |G|\,/|G'|$, and rearranging terms, we obtain the desired formula. ∎

2.3 Motivation: Lower Bounds on ZL-amenability Constants

As mentioned in the introduction, it is observed in [1, §1.5] that

$$(2.7) \qquad \inf\{\mathrm{AM}_{\mathrm{Z}}(G) \colon G \text{ finite and non-abelian}\} > 1.$$

(See the proof of Theorem 1.10, *op. cit.*) The proof relies on the following hard result of D. Rider.

Theorem 2.7 (Rider; see [8, Lemma 5.2]) *Let G be a compact group, λ a Haar measure on it, and ψ a finite linear combination of irreducible group characters on G. Suppose that $\psi * \psi = \psi$ as elements of $L^1(G, \lambda)$ and that $\int_G |\psi(x)| \, d\lambda(x) > 1$. Then $\int_G |\psi(x)| \, d\lambda(x) \geq 301/300$.*

Remark 2.8 Rider's result is stated for the case where $\lambda(G) = 1$. However, if we let $\mu = \lambda(G)^{-1}\lambda$, then $\psi * \psi = \psi$ in $L^1(G, \lambda)$ if and only if $\lambda(G)\psi * \lambda(G)\psi = \lambda(G)\psi$ in $L^1(G, \mu)$. So by rescaling, our formulation reduces to the one given by Rider.

Question Can we get an improved bound on the infimum on the left-hand side of (2.7) beyond the lower bound 301/300 provided by Rider's theorem?

Remark 2.9 To put this question in context, we note that the smallest explicitly known value of $\mathrm{AM}_Z(G)$ for a non-abelian group G is 7/4 (see Remark 3.15 in the next section). Rider remarks that his estimates are not intended to be best possible, but it seems unlikely that his techniques can get near 11/10, let alone 7/4, without substantial new input. Of course, his results concern much more general central idempotents, whereas our concern is with the very particular idempotent described in (2.1).

It seems difficult to attack this problem directly using (2.2). One might hope that for groups with two character degrees one can use (2.3) to obtain a lower bound on the ZL-amenability constants that is strictly greater than 1. While we were unable to do this in full generality, we can do better for particular classes of groups; these calculations are the topic of the next section.

Another question raised in [1, §1.5] is the following:

Question Given a finite non-abelian group G, can we get a lower bound on $\mathrm{AM}_Z(G)$ in terms of $\max_{\pi \in \widehat{G}} d_\pi$?

If this were the case, one could obtain further results on (non-)amenability of the centre of $L^1(G)$ for certain profinite groups G. Unfortunately, as we shall see below (Remark 3.8), there exists a sequence of finite groups (G_i) such that $\sup_i \max_{\pi \in \widehat{G_i}} d_\pi = +\infty$, yet $\sup_i \mathrm{AM}_Z(G_i) = 5$. Therefore this question has a negative answer.

3 ZL-amenability Constants of Particular Groups

Using Theorem 2.4, we can find the ZL-amenability constants for several well-known families of finite groups.

3.1 Dihedral Groups

Let us fix some notation: D_n denotes the *dihedral group of order* $2n$, whose standard presentation is

$$D_n = \langle r, t \mid r^n = t^2 = 1, tr = r^{-1}t \rangle.$$

The character table of D_n is well known and can be found in standard sources, for instance, see [6, pp. 182–183]. We note, nevertheless, that we only need to know the number of linear characters and the cardinalities of the conjugacy classes, both of which can be determined by straightforward *ad hoc* arguments that we leave to the reader. As usual, we must treat the cases of odd and even n separately.

The case of even n. Suppose $n = 2\nu$ for some integer $\nu \geq 2$. Then D_n has four linear characters (so that its derived subgroup has order $n/2$), and all other characters have degree 2. Also, D_n has two conjugacy classes of size 1 (namely $\{1\}$ and $\{r^\nu\}$), two of size $n/2$ (namely $[t]$ and $[rt]$), and $\nu - 1$ of size 2 (the remaining rotations, paired up). Thus,

$$\sum_{C \in \mathrm{Conj}(D_{2\nu})} |C|^2 = 2 \cdot 1^2 + 2\left(\frac{n}{2}\right)^2 + \left(\frac{n}{2} - 1\right) \cdot 2^2 = \frac{1}{2}(n^2 + 4n - 4),$$

and so, by our general formula (2.3),

$$(3.1) \qquad \mathrm{AM}_\mathrm{Z}(D_{2\nu}) = 1 + 2(2^2 - 1)\left(1 - \frac{n^2 + 4n - 4}{2n^2}\right)$$
$$= 1 + 6\frac{n^2 - 4n + 4}{2n^2} = 1 + 3\left(1 - \frac{2}{n}\right)^2.$$

The case of odd n. Suppose $n = 2\nu + 1$, where ν is an integer ≥ 1. Then D_n has two linear characters (so that its derived subgroup has order n), and all other characters have degree 2. Also, its conjugacy classes are as follows: the trivial conjugacy class of the identity; the conjugacy class consisting of all involutions, which has size n; and $(n - 1)/2$ conjugacy classes of size 2 (each consisting of a rotation and its inverse). Thus,

$$\sum_{C \in \mathrm{Conj}(D_{2\nu+1})} |C|^2 = 1^2 + n^2 + \frac{n-1}{2} \cdot 2^2 = n^2 + 2n - 1,$$

and so, by our general formula (2.3),

$$(3.2) \qquad \mathrm{AM}_\mathrm{Z}(D_{2\nu+1}) = 1 + 2(2^2 - 1)\left(1 - \frac{n^2 + 2n - 1}{2n^2}\right)$$
$$= 1 + 6\frac{n^2 - 2n + 1}{2n^2} = 1 + 3\left(1 - \frac{1}{n}\right)^2.$$

In fact, when n is odd, D_n fits into a family of more general examples, for which one can simplify (2.3) even further. These groups are the topic of the next subsection.

3.2 Frobenius Groups with Abelian Complement and Kernel

Frobenius groups admit various characterizations or equivalent definitions. The following one is convenient for our purposes.

Definition 3.1 (*cf.* [7, Theorem 8.2]) A finite group G is a *Frobenius group* if it has a finite, proper, non-trivial subgroup H that is *malnormal, i.e.,* that satisfies $H \cap gHg^{-1} = \{e\}$ for all $g \in G \setminus H$. We say that H is a *Frobenius complement* in G.

Given a Frobenius complement $H < G$, let $K := (G \setminus \bigcup_{g \in G} gHg^{-1}) \cup \{e\}$. Clearly K is a conjugation-invariant susbet of G. By a deep result of Frobenius, K is actually a subgroup of G, called the *Frobenius kernel* of G, and G is the semidirect product $K \rtimes H$. (See Passman's book, in particular the proof of [7, Theorem 17.1], for further details.)

Remark 3.2 *A priori*, K depends on the particular choice of Frobenius complement H. However, it turns out that if G has a Frobenius complement H and K is the corresponding Frobenius kernel, then K is equal to the Fitting subgroup of G; moreover, all proper, non-trivial, malnormal subgroups of G are conjugate in G ([7, Corollary 17.5]). These highly non-obvious results are sometimes summarized in the slogan "a finite group can be Frobenius in at most one way".

For sake of brevity, we write "let $G = K \rtimes H$ be Frobenius" as an abbreviation for "let G be a finite Frobenius group, with Frobenius complement H and Frobenius kernel K."

Proposition 3.3 *Let $G = K \rtimes H$ be Frobenius. Suppose H is an abelian group of order h and K is an abelian group of order k. Then h divides $k - 1$. Moreover:*

(i) *G has trivial centre, $(k - 1)/h$ conjugacy classes of size h, and $h - 1$ conjugacy classes of size k;*
(ii) *G has exactly h linear characters, the remaining characters each having degree h.*

The proposition is an assembly of several standard facts about Frobenius groups. However, as it is difficult to locate a reference that states what we need concisely, we give a proof in Appendix A.

Theorem 3.4 *Let G be a Frobenius group whose complement and kernel are both abelian; let h and k be the orders of the complement and kernel, respectively. Then*

$$(3.3) \qquad \mathrm{AM}_Z(G) = 1 + 2 \cdot \frac{h^2 - 1}{h} \left(1 - \frac{h - 1}{k} \right) \left(1 - \frac{1}{k} \right).$$

Proof By Proposition 3.3,

$$\sum_{C \in \mathrm{Conj}(G)} |C|^2 = 1 + \frac{k - 1}{h} h^2 + (h - 1) k^2 = 1 + h(k - 1) + (h - 1)k^2,$$

and substituting the remaining information from Proposition 3.3 into the general

formula (2.3) yields

$$
\frac{\mathrm{AM}_Z(G) - 1}{2} = (h^2 - 1)\Big(1 - \frac{1 + h(k-1) + (h-1)k^2}{hk^2}\Big)
$$

$$
= \frac{h^2 - 1}{h} \cdot \frac{-1 - hk + h + k^2}{k^2}
$$

$$
= \frac{h^2 - 1}{h}\Big(1 - \frac{h}{k} + \frac{h-1}{k^2}\Big).
$$

Factoring and rearranging this gives the formula (3.3), as required. ∎

Example 3.5 (Dihedral groups of odd order, revisited) Let n be an odd integer with $n \geq 3$. Using the standard presentation of D_n as given earlier, we see that the subgroup generated by the "reflection" t is malnormal, while the Frobenius kernel turns out to be the subgroup generated by the "rotation" r. Putting $h = 2$ and $k = n$ in (3.3) gives

$$
\mathrm{AM}_Z(D_n) = 1 + 3\Big(1 - \frac{1}{n}\Big)^2,
$$

just as we had before.

Example 3.6 (Affine groups of finite fields) Let $\mathbb{F}_q$ be a finite field of order q, where q is a prime power ≥ 3. The *affine group of* $\mathbb{F}_q$, which we shall denote by $\mathrm{Aff}(\mathbb{F}_q)$, is the set

$$
\big\{ \big(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\big) : a \in \mathbb{F}_q^\times, b \in \mathbb{F}_q \big\}
$$

equipped with the group structure it inherits from the usual matrix product and inversion. It is a metabelian group; more precisely, it is isomorphic to the semidirect product $\mathbb{F}_q \rtimes \mathbb{F}_q^\times$.

It is straightforward to check that the subgroup of $\mathrm{Aff}(\mathbb{F}_q)$ corresponding to the multiplicative group of $\mathbb{F}_q$ is a proper, non-trivial, malnormal subgroup; the Frobenius kernel turns out to be the normal subgroup of $\mathrm{Aff}(\mathbb{F}_q)$ corresponding to the additive group of $\mathbb{F}_q$. Both are abelian, so we can apply Theorem 3.4, which yields

$$
\mathrm{AM}_Z(\mathrm{Aff}(\mathbb{F}_q)) = 1 + 2 \cdot \frac{(q-1)^2 - 1}{q-1}\Big(1 - \frac{q-2}{q}\Big)\Big(1 - \frac{1}{q}\Big)
$$

$$
= 1 + 2 \cdot \frac{2q - q^2}{q-1} \cdot \frac{2}{q} \cdot \frac{q-1}{q}
$$

$$
= 1 + 4 \cdot \frac{q-2}{q} = 5 - \frac{8}{q}.
$$

Remark 3.7 One can also compute $\mathrm{AM}_Z(\mathrm{Aff}(\mathbb{F}_q))$ more directly from the character table of $\mathrm{Aff}(q)$, which is simple and well known and can be found in standard sources. In fact, the exact computation for these examples, which arose in other research of the authors related to [9], provided some of the motivation for Theorem 2.4.

Remark 3.8 For all odd primes p, $2 \leq AM(\text{Aff}(p)) \leq 5$, while $\text{Aff}(p)$ has an irreducible representation of dimension $p - 1$. This shows that the amenability constant of $Z\ell^1(G)$ cannot be bounded from below by an increasing function of $\max\{d_\chi : \chi \in \text{Irr}(G)\}$. (For context, see the remarks made at the end of Section 2.)

Example 3.9 ($a^2x + b$ groups) Let q be an odd prime power ≥ 5, and let $d = (q-1)/2$. Consider the following subgroup of $\text{Aff}(\mathbb{F}_q)$, sometimes referred to as the $a^2x + b$ group over $\mathbb{F}_q$:

$$G_q := \left\{ \left(\begin{smallmatrix} a^2 & b \\ 0 & 1 \end{smallmatrix} \right) : a \in \mathbb{F}_q^\times, b \in \mathbb{F}_q \right\}.$$

Recalling that $\mathbb{F}_q^\times$ is cyclic, pick a generator z, and let H be the subgroup of $\text{Aff}(\mathbb{F}_q)$ generated by $\left(\begin{smallmatrix} z^2 & 0 \\ 0 & 1 \end{smallmatrix} \right)$. One can check that H is malnormal, and so G_q is Frobenius. The Frobenius kernel K turns out to be the normal subgroup corresponding to the additive group of $\mathbb{F}_q$. So both K and H are abelian; the former has order q, while the latter has order d, so using (3.3) we get

$$AM_Z(G_q) = 1 + 2 \cdot \frac{d^2 - 1}{d} \left(1 - \frac{d-1}{q} \right) \left(1 - \frac{1}{q} \right)$$

$$= 1 + 2 \cdot \frac{q^2 - 2q - 3}{2(q-1)} \frac{q+3}{2q} \frac{q-1}{q},$$

which simplifies to

$$(3.4) \qquad AM_Z(G_q) = 1 + \frac{q+1}{2} \left(1 - \frac{9}{q^2} \right)$$

As a consistency check, when $q = 5$, equation (3.4) gives $AM_Z(G_5) = 73/25$. On the other hand, it is straightforward to check that G_5 is isomorphic to the dihedral group of order 10, and using our earlier formulas we have $AM_Z(D_5) = 73/25$.

Remark 3.10 Even though G_q is a (index 2) subgroup of $\text{Aff}(\mathbb{F}_q)$, it may have a larger ZL-amenability constant. Indeed, it is clear from the formulas obtained in Examples 3.6 and 3.9 that

$$\lim_{q \to \infty} AM_Z(\text{Aff}_q) = 5, \quad \text{while} \quad \lim_{q \to \infty} q^{-1} AM_Z(G_q) = \frac{1}{2}.$$

Example 3.9 shows that within the class of groups with two character degrees, we can obtain arbitrarily large ZL-amenability constants. It is natural to ask how small such constants can be. For Frobenius groups with abelian complement and kernel, we can obtain a complete answer.

Theorem 3.11 *Let G be a Frobenius group with abelian complement and kernel. Then* $AM_Z(G) \geq 7/3$, *with equality if and only if G is the dihedral group of order* 6.

Proof Let h be the order of the Frobenius complement of G, and k the order of its Frobenius kernel. Note that G is isomorphic to D_3 if and only if $h = 2$ and $k = 3$.

To reduce notational clutter, let $F(k, h)$ denote $\frac{1}{2}(\mathrm{AM}_Z(G) - 1)$. By Theorem 3.4,

$$F(k, h) = \frac{h^2 - 1}{h}\left(1 - \frac{h-1}{k}\right)\left(1 - \frac{1}{k}\right),$$

and it suffices to prove that $F(k, h) \geq 2/3$ with equality if and only if $(h, k) = (2, 3)$ (subject to h and k arising from a Frobenius group of the specified form).

Note that for fixed h, $F(\cdot, h)$ is a strictly increasing function. As observed above, h divides $k - 1$, so in particular $k \geq h + 1$; hence, $F(k, h) \geq F(h + 1, h)$, with equality if and only if $h = k - 1$. Direct calculation gives

$$F(h + 1, h) = \frac{h^2 - 1}{h} \cdot \frac{2}{h+1} \cdot \frac{h}{h+1} = \frac{2(h-1)}{h+1} = 2\left(1 - \frac{2}{h+1}\right),$$

and so $F(h + 1, h) \geq 2/3$, with equality if and only if $h = 2$. This completes the proof. ∎

If we consider more general groups with two character degrees, then there is an infinite family of such groups whose ZL-amenability constants are less than 2. This will be seen in the next and final subsection of the paper.

3.3 Extra-special p-groups

Definition 3.12 Fix a prime p. A finite group G is *p-extraspecial* if it has order p^{2n+1} for some integer n and has the following properties:

(i)   The centre $Z(G)$ and the derived subgroup G' both have order p.
(ii) The quotient $G/Z(G)$ is abelian, and each non-identity element in the quotient has order p.

Such groups do exist (for instance, the dihedral group of order 8 is 2-extraspecial), and their character tables and conjugacy classes turn out to be uniquely determined by these conditions. In particular, each non-linear irreducible group character of G is supported on $Z(G)$ and has degree p^n. This follows from, *e.g.,* [3, Chapter 5, Theorem 5.5], as pointed out to the second author by D. F. Holt. Alternatively, a short argument using some basic character theory is described by I. M. Isaacs in the appendix to [2].

Example 3.13 Let G be an extraspecial group of order p^{2n+1}, where p is a prime. We know every non-linear character has degree p^n, and we know G' has order p. To apply Theorem 2.4, we also need to know the sizes of the conjugacy classes. By some elementary group theory (see *e.g.,* the appendix of [2]), the conjugacy classes of G are either elements of the centre or the non-trivial cosets of the derived subgroup. Thus there are p conjugacy classes of size 1 and $p^{2n} - 1$ conjugacy classes of size p, and no others. Therefore,

$$\sum_{C \in \mathrm{Conj}(G)} |C|^2 = p \cdot 1^2 + (p^{2n} - 1) \cdot p^2 = p^{2n+2} - p^2 + p,$$

and so Theorem 2.4 gives

$$\mathrm{AM_Z}(G) = 1 + 2(p^{2n} - 1)\Big(1 - \frac{p^{2n+2} - p^2 + p}{p^{2n+2}}\Big)$$

$$= 1 + 2\Big(1 - \frac{1}{p^{2n}}\Big)\Big(1 - \frac{1}{p}\Big).$$

Remark 3.14 If G is an extraspecial group of order 2^{2n+1}, then $\mathrm{AM_Z}(G) = 2 - 2^{-2n}$. Thus we have an infinite family of finite groups G for which $1 < \mathrm{AM_Z}(G) < 2$.

Remark 3.15 Within the class of extra-special p-groups, the ZL-amenability constant is minimized when we take $p = 2$ and $n = 1$. This example is nothing but the dihedral group of order 8, whose amenability constant is $7/4$. This is the smallest ZL-amenability constant we have found for any non-abelian group.

Note added in proof After this paper was accepted for publication, the second author (YC) has since shown that the *ZL*-amenability constant of any finite non-abelian group is at least $7/4$. Details will appear in a forthcoming work.

3.4 Summary Information

We summarize the findings of this section in Table 1.

Ref.	G	$\lvert G \rvert$	$\lvert L \rvert$	c.d.	$\mathrm{AM_Z}(G) - 1$	min.
Ex. 3.6	$\mathrm{Aff}(\mathbb{F}_q)$, $q \geq 3$	$q(q-1)$	$q-1$	$q-1$	$4(1 - 2q^{-1})$	$4/3$
Ex. 3.9	$ax^2 + b$ of $\mathbb{F}_q$, $q \geq 5$	$\dfrac{q(q-1)}{2}$	$\frac{1}{2}(q-1)$	$q-1$	$\frac{1}{2}(q+1)(1 - 9q^{-2})$	$48/25$
Eq. (3.2)	D_n, n odd ≥ 3	$2n$	2	2	$3(1 - n^{-1})^2$	$4/3$
Eq. (3.1)	D_n, n even ≥ 4	$2n$	4	2	$3(1 - (2n)^{-1})^2$	$3/4$
Ex. 3.13	p-extraspecial	p^{2n+1}	p^{2n}	p^n	$2(1 - p^{-2n})(1 - p^{-1})$	$3/4$

Table 1: Summary table for some groups with two character degrees

- "Ref." gives the number of the relevant theorem, example, or equation.
- "L" is the set of linear characters.
- "c.d." stands for the character degree of the non-linear characters.
- "min." denotes the minimum value of $\mathrm{AM_Z}(G) - 1$ within the specified family of groups.

A Properties of Frobenius Groups

In this appendix we collect the facts about Frobenius groups that are needed to prove Proposition 3.3. Since we are interested only in the special case where both complement and kernel are abelian, it will sometimes be easier to give short proofs than to cite general results and then specialize. On the other hand, in some places we shall merely give appropriate references to the literature.

Throughout, $G = K \rtimes H$ is Frobenius, and k and h denote the orders of K and H respectively. By considering the permutation action of G on cosets of H, it follows easily from the malnormal property that h divides $k - 1$. (This does not need Frobenius's result that K is a group.)

Proof of Proposition 3.3(i) Throughout this proof x^G denotes the conjugacy class of x in G, and $C_G(x)$ denotes the centralizer of x in G. We repeatedly use the fact that each element of G can be written either as xb, where $x \in K$ and $b \in H$, or as ay, where $a \in H$ and $y \in K$. (This is immediate from the decomposition of G as a semidirect product of H and K.)

The first step is to identify the conjugacy classes of K inside G. Let $x \in K \setminus \{e\}$. By the malnormal property, $C_G(x) \cap H = \{e\}$. Since K is abelian and $G = HK$, it follows that $C_G(x) = K$. Therefore $|x^G| = |G|/|K| = h$.

For the second step, recall that by definition, $G \setminus K = \left(\bigcup_{g \in G} gHg^{-1} \right) \setminus \{e\}$. Since H is abelian and $G = KH$, we obtain $G \setminus K = \left(\bigcup_{x \in K} xHx^{-1} \right) \setminus \{e\}$. Now, by the malnormality of H inside G, and the fact that $H \cap K = \{e\}$, we see that

$$xHx^{-1} \cap yHy^{-1} = \{e\} \qquad \text{whenever } x, y \in K \text{ and } x \neq y.$$

Thus, the function $H \setminus \{e\} \to \mathrm{Conj}(G)$, $a \mapsto a^G$, is injective, and $|a^G| = k$ for each $a \in H \setminus \{e\}$. This gives us the required partition of $G \setminus K$ into $h - 1$ disjoint conjugacy classes, each of size k. ∎

To prove the second part of Proposition 3.3, we appeal to some general results on the character theory of Frobenius groups.

Proposition *Let $G = K \rtimes H$ be Frobenius. The following belong to* $\mathrm{Irr}(G)$:
- *the characters arising by composing irreducible characters of H with the quotient map $G \to G/K \cong H$;*
- *the characters arising by inducing an irreducible character of K up to G.*

Moreover, every irreducible character of G arises in this way.

Proof See, for example, [4, Theorem 6.34]. ∎

In the special case where H and K are abelian, it follows immediately that G has two character degrees. The irreducible characters of G that arise by inducing irreducible characters from K all have degree $|G{:}K| = h$; the remaining characters are all linear, arising from the irreducible characters of H, and there are precisely $|\widehat{H}| = |H| = h$ of them. This completes the proof of Proposition 3.3(ii). ∎

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