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AN INEQUALITY FROM APPLIED PROBABILITY

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Abstract

J. F. C. Kingman (1967) proved that if $\{\phi_n\}$, $n = 1, 2, \cdots, N$ is any set of Radon-Nikodym derivatives of λ_n w.r.t. μ_n, when λ_n, μ_n are the restrictions of λ, μ respectively to a sub σ-algebra G_n of G then

$$\int_X \prod_{n=1}^{N} \phi_n^{b_n}\, d\lambda \geqq \lambda(X)\left[\frac{\lambda(X)}{\mu(X)}\right]^B$$

where $b_n \geqq 0\,\forall_n$ and $B = \Sigma_{n=1}^N b_n$.

This paper simplifies Kingman's proof and finds upper and lower bounds for the integral when the b_n's can assume negative values.

Introduction

Let λ, μ be positive measures on a σ-algebra $\mathcal{G}$ of subsets of X and $\mathcal{G}' \subset \mathcal{G}$ be a sub σ-albegra of $\mathcal{G}$. Also define λ', μ' to be the respective restrictions of λ, μ to $\mathcal{G}'$. If λ' is absolutely continuous with respect to μ' then there exists a unique ϕ', the Radon-Nikodym derivative of λ' w.r.t. μ', with $\phi' - \mathcal{G}'$ measurable and

$$\lambda'(E) = \int_E \phi' d\mu' \quad \text{for all} \quad E \in \mathcal{G}'.$$

Kingman (1967) proved that if $\{\phi_n\}, n = 1, 2, \cdots, N$ is any set of Radon–Nikodym derivatives of λ_n w.r.t. μ_n where λ_n, μ_n are the restrictions of λ, μ respectively to a sub σ-algebra $\mathcal{G}_n$ of $\mathcal{G}$ then

$$I = \int_X \prod_{n=1}^{N} \phi_n^{b_n}\, d\lambda \geqq \lambda(X)\left[\frac{\lambda(X)}{\mu(X)}\right]^B$$

where $b_n \geqq 0$ for all n and $B = \Sigma_{n=1}^N b_n$.

In this paper we simplify Kingman's proof, especially for the case of equality and then find upper and lower bounds for I when the b_n's can assume negative values. The latter problem was pointed out to the author by Professor J. Darroch and Miss H. F. Wang of Flinders University who needed an upper bound to help solve a contingency table problem.

For convenience we define

$$B = \sum_n |b_n|, \qquad C_n = \frac{|b_n|}{B}, \qquad p = \sum_{n \geqq 0} C_n, \qquad \gamma_n = \begin{cases} \phi_n & \text{if} \quad b_n \geqq 0 \\ \phi_n^{-1} & \text{if} \quad b_n < 0 \end{cases}.$$

Kingman's inequality

If we assume that $b_n \geqq 0$ for all n and we use Holder's Inequality and the harmonic mean less than or equal to the geometric mean result, we have

$$(1) \qquad \int_X \prod_n \varphi_n^{b_n} \frac{d\lambda}{\lambda(X)} \geqq \int_X \left(\sum_n C_n \varphi_n^{-1}\right)^{-B} \frac{d\lambda}{\lambda(X)} \geqq \left[\int_X \sum_n C_n \varphi_n^{-1} \frac{d\lambda}{\lambda(X)}\right]^{-B}.$$

However, $d\lambda = \phi_n d\mu$ on $\mathscr{G}_n$ and $\Sigma_n C_n = 1$ and therefore we have

$$(2) \qquad I \geqq \lambda(X) \left[\frac{\lambda(X)}{\mu(X)}\right]^B.$$

To show when equality holds in this result, we need examine only (1) as the final step is actually an equality. We have equality in the first stage of (1) if and only if $\phi_n = f$ a.e. $[\lambda]$ for all n. [Mitrinović (1970) p. 76].

On substituting $\phi_n = f$ into (1) we obtain

$$\int_X f^B \frac{d\lambda}{\lambda(X)} \geqq \left[\int_X f^{-1} \frac{d\lambda}{\lambda(X)}\right]^{-B}.$$

From Holder's Inequality the only possibility remaining for equality (ignoring $b_n = 0$ for all n) is that $f =$ constant a.e. $[\lambda]$. By substituting $\phi_n = C$ into (2) it follows that equality holds if and only if

$$\phi_n = \frac{\lambda(X)}{\mu(X)} \text{ a.e.} [\lambda] \quad \text{for all } n.$$

Some bounds for I

To find bounds for I when $b_n \in R$ we assume that λ is absolutely continuous w.r.t. μ. Such an assumption means that the Radon–Nikodym derivative ϕ of λ w.r.t. μ exists and that unique R–N derivatives of λ w.r.t. μ exists for every sub σ-algebra of $\mathscr{G}$.

LEMMA. *Suppose φ, φ' are non-negative integrable functions such that φ' is $\mathscr{G}'$ measurable and*

$$\int_E \varphi' d\mu = \int_E \varphi \, d\mu \qquad \forall E \in \mathcal{G}'. \tag{3}$$

Let h be a convex function on $[0,\infty)$ such that $h(\varphi)$, $h(\varphi')$ are $\mathcal{G}$ measurable and summable on X. Then

$$\int_X h(\varphi')d\mu \leqq \int_X h(\varphi)d\mu. \tag{4}$$

The above inequality is more general than we need for our purposes. In fact we are interested only in the case $h(x) = x^r$. The general form given above was postulated and proved by Leon Simon of Stanford University to whom the author is very grateful.

PROOF. It suffices to prove the theorem subject to the assumption

$$|h(t) - h(\bar{t})| \leqq K\,|t - \bar{t}\,|, \quad t, \quad \bar{t} \in (0,\infty), \quad K \text{ constant} \tag{5}$$

because any convex function can be pointwise approximated by an increasing sequence of such functions.

First we choose a sequence of simple functions $\{s_n\}$ of the form $s_n = \sum_{k=1}^{N(n)} \lambda_k^{(n)} \chi_{E_k^{(n)}}$ where $\{E_k^{(n)}\} \subset \mathcal{G}'$ form a partition of X, χ_E is the characteristic function of the set E and

$$\int_X |s_n - \varphi'|\, d\mu \to 0 \qquad \text{as} \qquad n \to \infty \tag{6}$$

Now

$$\int_X h(s_n)d\mu = \sum_{k=1}^{N(n)} \int_{E_k^{(n)}} h(s_n)d\mu = \sum_{k=1}^{N(n)} h(\lambda_k^{(n)})\,\mu(E_k^{(n)}). \tag{7}$$

But by (5), we have

$$\begin{aligned} &\sum_{k=1}^{N(n)} \left| h(\lambda_k^{(n)}) - h\left\{[\mu(E_k^{(n)})]^{-1} \int_{E_k^{(n)}} \varphi' \, d\mu\right\} \right| \mu(E_k^{(n)}) \\ &\qquad \leqq K \sum_{k=1}^{N(n)} \left| \lambda_k^{(n)} \mu(E_k^{(n)}) - \int_{E_k^{(n)}} \varphi' \, d\mu \right| \\ &\qquad \leqq K \int_X |s_n - \varphi'|\, d\mu. \end{aligned} \tag{8}$$

Combining (6), (7) and (8) we then have

$$\int_X h(s_n)\, d\mu = \sum_{k=1}^{N(n)} [\mu(E_k^{(n)})]\; h\left\{[\mu(E_k^{(n)})]^{-1} \int_{E_k^{(n)}} \varphi' \, d\mu\right\} + \varepsilon_n,$$

where $\varepsilon_n \to 0$ as $n \to \infty$.

Then by (3) and Jensen's Inequality,

$$\int_X h(s_n)\,d\mu \leqq \sum_{k=1}^{N(n)} \int_{E_k^{(n)}} h(\varphi)\,d\mu + \varepsilon_n = \int_X h(\varphi)\,d\mu + \varepsilon_n.$$

Letting $n \to \infty$ we have (2).

We note that each of our Radon-Nikodym derivatives, ϕ_n, satisfy the preconditions of φ', and consequently qualify for this lemma.

THEOREM 1. *If $b_n \in R$ for all n, then,*

$$I \leqq \begin{cases} p\int_X \phi^B d\lambda + (1-p)\int_X \phi^{-B} d\lambda & \text{for} \quad B \geqq 1 \\ p\int_X \phi^B d\lambda + (1-p)\lambda(X)\left[\dfrac{\mu(X)}{\lambda(X)}\right]^B & 0 \leqq B \leqq 1. \end{cases}$$

PROOF.

$$\int_X \prod_n \phi_n^{b_n}\,d\lambda = \int_X \left(\prod_n \gamma_n^{C_n}\right)^B d\lambda = \int_X \lim_{t\to 0}\left(\sum_n C_n \gamma_n^t\right)^{B/t} d\lambda.$$

But for $0 \leqq t \leqq B$, $B/t \geqq 1$, and the power mean result (Mitrinović) gives

$$I \leqq \int_X \sum_n C_n \gamma_n^B\,d\lambda = \sum_{n\geqq 0} C_n \int_X \phi_n^B\,d\lambda + \sum_{n<0} C_n \int_X \phi_n^{-B}\,d\lambda.$$

Now because $d\lambda = \phi_n d\mu$ on $\mathscr{G}_n$ and $\phi_n^{1+B}, \phi_n^{1-B}$ are convex for $B \geqq 1$ the above Lemma gives the required result for $B \geqq 1$. For $0 \leqq B \leqq 1$, ϕ_n^{1+B} is still convex and

$$\int_X \phi_n^{-B} \frac{d\lambda}{\lambda(X)} \leqq \left[\int_X \phi_n^{-1} \frac{d\lambda}{\lambda(X)}\right]^B = \left[\frac{\mu(X)}{\lambda(X)}\right]^B. \qquad \text{Q.E.D.}$$

THEOREM 2. *If $b_n \in R$ for all n,*

$$I \geqq \begin{cases} [\lambda(X)]^2\left[(1-p)\int_X \phi^B d\lambda + p\lambda(X)\left[\dfrac{\mu(X)}{\lambda(X)}\right]^B\right]^{-1} & 0 \leqq B \leqq 1 \\ [\lambda(X)]^{1+B}\left[(1-p)\int_X \phi\,d\lambda + p\mu(X)\right]^{-B} & B \geqq 1. \end{cases}$$

PROOF.

$$\int_X \prod_n \phi_n^{b_n} \frac{d\lambda}{\lambda(X)} \geqq \lambda(X)\left[\int_X \prod_n \phi_n^{-b_n}\,d\lambda\right]^{-1}$$

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and utilising Theorem 1

$$(9)\quad I \geqq [\lambda(X)]^2 \left[(1-p)\int_X \phi^B d\lambda + p\int_X \phi^{-B} d\lambda\right]^{-1} \qquad \text{for} \qquad B \geqq 1$$

$$(10)\quad I \geqq [\lambda(X)^2 \left[(1-p)\int_X \phi^B d\lambda + p\lambda(X)\left[\frac{\mu(X)}{\lambda(X)}\right]^B\right]^{-1} \qquad 0 \leqq B \leqq 1.$$

However, an alternative lower bound can be found using the power mean inequality and Holder's Inequality as follows:

$$\begin{aligned}
\int_X \prod_n \phi_n^{b_n} \frac{d\lambda}{\lambda(X)} &= \int_X \left(\prod_n \gamma_n^{-C_n}\right)^{-B} \frac{d\lambda}{\lambda(X)} \\
&\geqq \int_X \left(\sum_n C_n \gamma_n^{-1}\right)^{-B} \frac{d\lambda}{\lambda(X)} \\
&\geqq \left[\int_X \sum_n C_n \gamma_n^{-1} \frac{d\lambda}{\lambda(X)}\right]^{-B} \\
&= \left[\sum_{n \geqq 0} C_n \int_X \phi_n^{-1} \frac{d\lambda}{\lambda(X)} + \sum_{n:n<0} C_n \int_X \phi_n \frac{d\lambda}{\lambda(X)}\right]^{-B}
\end{aligned}$$

and since the power is negative we use the upper bound for $\int_X \phi_n d\lambda$ from the lemma to obtain

$$(11)\qquad I \geqq [\lambda(X)]^{1+B}\left[(1-p)\int \phi d\lambda + p\mu(X)\right]^{-B}.$$

A simple comparison of (9), (10), (11) using Holder's inequality shows that for $0 \leqq B \leqq 1$ (10) gives a tighter lower bound whereas (11) is preferable for $B \geqq 1$. Q.E.D.

Conclusion

By an argument similar to that given in our proof of Kingman's inequality it can be shown that equality holds in (11), if and only if

$$\phi_n = 1 \text{ a.e. } [\lambda] \quad \text{for all } n \text{ and} \quad \int_X \phi d\lambda = \lambda(X) \qquad 0 < p < 1$$

$$\phi_n = \frac{\lambda(X)}{\mu(X)} \text{ a.e. } [\lambda] \quad \text{for all } n \text{ and} \quad \int_X \phi d\lambda = \frac{[\lambda(X)]^2}{\mu(X)} \qquad p = 0$$

$$\phi_n = \frac{\lambda(X)}{\mu(X)} \text{ a.e. } [\lambda] \quad \text{for all } n \qquad p = 1$$

Unfortunately cases of equality for (9), (10) and Theorem 1 are more difficult to analyse. At present we can say no more that if $p = 0, 1$ all bounds are

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attained when $\phi = \lambda(X)/\mu(X)$ a.e. $[\lambda]$ and if $0 < p < 1$ we must restrict $\phi = 1$ a.e. $[\lambda]$.

Finally, note that I is independent of $\mathscr{G}$ and ϕ. To obtain our best bounds for I we must, in each theorem, minimise $\int_X h(\phi)d\mu$ where h is some convex function on $(0, \infty)$. The lemma shows that this can be done by choosing $\mathscr{G}$ as the smallest σ-algebra satisfying the given conditions, i.e. $\mathscr{G}$ should be chosen as the smallest σ-algebra generated from $\bigcup_n \mathscr{G}_n$.

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