Compositio Mathematica **128:** 323–338, 2001.

Isomorphism Classes of A-Hypergeometric Systems

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(Received: 22 February 2000; accepted in final form: 19 September 2000)

Abstract. Given a finite set A of integral vectors and a parameter vector, Gel'fand, Kapranov, and Zelevinskii defined a system of differential equations, called an A-*hypergeometric* (or a *GKZ hypergeometric*) *system*. Classifying the parameters according to the D-isomorphism classes of their corresponding A-hypergeometric systems is one of the most fundamental problems in the theory. In this paper we give a combinatorial answer for the problem under the assumption that the finite set A lies in a hyperplane off the origin, and illustrate it in two particularly simple cases: the normal case and the monomial curve case.

Mathematics Subject Classifications (2000). 33C70, 14M25, 16S32.

Key words. A-hypergeometric systems, isomorphism classes, symmetry algebra.

1. Introduction

Given a finite set A of integral vectors and a parameter vector, Gel'fand, Kapranov, and Zelevinskii defined a system of differential equations, called an A-*hypergeometric* (or a *GKZ hypergeometric*) *system* ([5]). Many authors studied D-invariants of the A-hypergeometric systems: In the Cohen–Macaulay case, Gel'fand, Kapranov, and Zelevinskii determined the characteristic cycles ([6]) and proved the irreducibility of the monodromy representations for nonresonant parameters ([4]); Adolphson proved the rank of an A-hypergeometric system equals the volume of the convex hull of A in the semi-nonresonant case ([1]); the author, Sturmfels, and Takayama scrutinized the ranks in [13]; Cattani, D'Andrea, and Dickenstein determined rational solutions and algebraic solutions in the monomial curve case ([2]), and recently Cattani, Dickenstein, and Sturmfels in [3] considered when an A-hypergeometric system has a rational solution other than Laurent polynomial solutions.

The purpose of this paper is to classify A-hypergeometric systems with respect to D-isomorphisms. This is one of the most fundamental problems in the theory. Under the assumption that the finite set A lies in a hyperplane off the origin, we shall give a combinatorial answer for this problem, and illustrate it in two particularly simple cases: the normal case and the monomial curve case.

Throughout the paper, we consider the finite set A fixed. In Section 2, we define a finite set $E_\tau(\beta)$ associated to a parameter β and a face τ of the cone generated by A. Then our main theorem (Theorem 2.1) states that two A-hypergeometric systems corresponding to parameters β and β' are D-isomorphic if and only if $E_\tau(\beta)$ equals $E_\tau(\beta')$ for all faces τ. In Section 2, we prove the only-if-part of the theorem and state some basic properties of the set $E_\tau(\beta)$.

Sections 3 and 4 are devoted to the study of the algebra of contiguity operators, called the *symmetry algebra*. In Section 3, we summarize some known facts about the symmetry algebra. We introduce the *b-ideals* in Section 4 and prove their elements correspond to contiguity operators. Furthermore we describe each b-ideal in terms of the standard pairs of a certain monomial ideal. Using this description, we give the proof of the if-part of our main theorem at the end of Section 4.

In Sections 5 and 6, we illustrate our main theorem in the normal case and the monomial curve case respectively, since the theorem reduces to relatively simple forms in both cases.

2. Main Theorem

We work over a field $\mathbf{k}$ of characteristic zero. Let $A = (a_1, \dots, a_n) = (a_{ij})$ be a $d \times n$-matrix of rank d with coefficients in $\mathbf{Z}$. We assume that all a_j belong to one hyperplane off the origin in $\mathbf{Q}^d$. We denote by I_A the toric ideal in $\mathbf{k}[\partial] = \mathbf{k}[\partial_1, \dots, \partial_n]$, that is

$$I_A = \langle \partial^u - \partial^v \mid Au = Av,\ u, v \in \mathbf{N}^n \rangle \subset \mathbf{k}[\partial].$$

For a column vector $\beta = {}^t(\beta_1, \dots, \beta_d) \in \mathbf{k}^d$, let $H_A(\beta)$ denote the left ideal of the Weyl algebra

$$D = \mathbf{k}\langle x_1, \dots, x_n, \partial_1, \dots, \partial_n \rangle$$

generated by I_A and $\sum_{j=1}^n a_{ij}\theta_j - \beta_i$ $(i = 1, \dots, d)$ where $\theta_j = x_j \partial_j$. The quotient $M_A(\beta) = D/H_A(\beta)$ is called the *A-hypergeometric system with parameter β*.

We denote the set $\{a_1, \dots, a_n\}$ by A as well. Let τ be a face of the cone

$$\mathbf{Q}_{\geqslant 0}A = \left\{ \sum_{j=1}^n c_j a_j \mid c_j \in \mathbf{Q}_{\geqslant 0} \right\}. \tag{2.1}$$

We denote by $\mathbf{Z}(A \cap \tau)$ the $\mathbf{Z}$-submodule of $\mathbf{Z}^d$ generated by $A \cap \tau$, and by $\mathbf{k}(A \cap \tau)$ the $\mathbf{k}$-subspace of $\mathbf{k}^d$ generated by $A \cap \tau$. We agree that $\mathbf{k}(A \cap \tau) = \mathbf{Z}(A \cap \tau) = \{0\}$ when $\tau = \{0\}$. Let $\mathbf{N} = \{0, 1, 2, \dots\}$, and let $\mathbf{N}A$ denote the monoid generated by A. For a parameter $\beta \in \mathbf{k}^d$, we consider the following set:

$$E_\tau(\beta) := \{ \lambda \in \mathbf{k}(A \cap \tau)/\mathbf{Z}(A \cap \tau) \mid \beta - \lambda \in \mathbf{N}A + \mathbf{Z}(A \cap \tau) \}. \tag{2.2}$$

The following is the main theorem in this paper.

THEOREM 2.1. *The A-hypergeometric systems $M_A(\beta)$ and $M_A(\beta')$ are isomorphic as D-modules if and only if $E_\tau(\beta) = E_\tau(\beta')$ for all faces τ of the cone $\mathbf{Q}_{\geqslant 0}A$.*

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Before the proof, we recall the formal series solutions ϕ_v defined in [13]. For $v \in \mathbf{k}^n$, its *negative support* nsupp(v) is the set of indices i with $v_i \in \mathbf{Z}_{<0}$. When nsupp(v) is minimal with respect to inclusions among nsupp($v+u$) with $u \in \mathbf{Z}^n$ and $Au = 0$, v is said to have *minimal negative support*. For v with minimal negative support, we define a formal series

$$\phi_v = \sum_{u \in N_v} \frac{[v]_{u_-}}{[v+u]_{u_+}} x^{v+u}. \tag{2.3}$$

Here

$$N_v = \{\, u \in \mathbf{Z}^n \mid Au = 0,\ \mathrm{nsupp}(v) = \mathrm{nsupp}(v+u) \,\},$$

and $u_+, u_- \in \mathbf{N}^n$ satisfy $u = u_+ - u_-$ with disjoint supports, and $[v]_w = \prod_{j=1}^n v_j(v_j - 1)\cdots(v_j - w_j + 1)$ for $w \in \mathbf{N}^n$. Proposition 3.4.13 of [13] states that the series ϕ_v is a formal solution of $M_A(Av)$.

Proof. Here we prove the only-if-part of the theorem. The proof of the if-part will be given at the end of Section 4.

We suppose that $\lambda \in E_\tau(\beta) \setminus E_\tau(\beta')$ for some face τ, and we shall prove $M_A(\beta)$ and $M_A(\beta')$ are not isomorphic.

Represent λ as $\sum_{a_j \in \tau} l_j a_j$. Consider the direct product

$$R_{\tau,\lambda} := \prod_{u \in \mathbf{Z}^n,\ u_j \in \mathbf{N}\,(a_j \notin \tau)} \mathbf{k} x^{l+u}.$$

Here we put $l_j = 0$ for $a_j \notin \tau$. Note that $R_{\tau,\lambda}$ has a natural D-module structure. There exists $u \in \mathbf{Z}^n$ with $u_j \in \mathbf{N}$ $(a_j \notin \tau)$ such that $\beta = A(l+u)$ and $l+u$ has minimal negative support. Then the series $\phi_{l+u} \in R_{\tau,\lambda}$ is a formal solution of $M_A(\beta)$, and hence $\mathrm{Hom}_D(M_A(\beta), R_{\tau,\lambda}) \neq 0$. On the other hand, $\mathrm{Hom}_D(M_A(\beta'), R_{\tau,\lambda}) = 0$ since $A(l+u) \neq \beta'$ for any $u \in \mathbf{Z}^n$ with $u_j \in \mathbf{N}$ $(a_j \notin \tau)$. Therefore $M_A(\beta)$ and $M_A(\beta')$ are not isomorphic. $\square$

In the remainder of this section, we collect some properties of the sets $E_\tau(\beta)$. We call a face of $\mathbf{Q}_{\geqslant 0}A$ of dimension $d-1$, a facet. Recall that for a facet σ the linear form F_σ satisfying the following conditions is unique and called the *primitive integral support function*:

(1) $F_\sigma(\mathbf{Z}A) = \mathbf{Z}$,
(2) $F_\sigma(a_j) \geqslant 0$ for all $j = 1, \ldots, n$,
(3) $F_\sigma(a_j) = 0$ for all $a_j \in \sigma$.

PROPOSITION 2.2

(1) *Each $E_{\mathbf{Q}_{\geqslant 0}A}(\beta)$ consists of one element. The equality $E_{\mathbf{Q}_{\geqslant 0}A}(\beta) = E_{\mathbf{Q}_{\geqslant 0}A}(\beta')$ means $\beta - \beta' \in \mathbf{Z}A$.*
(2) *$E_{\{0\}}(\beta) = \{0\}$ or $\emptyset$. $E_{\{0\}}(\beta) = \{0\}$ if and only if $\beta \in \mathbf{N}A$.*
(3) *For a facet σ, $E_\sigma(\beta) \neq \emptyset$ if and only if $F_\sigma(\beta) \in F_\sigma(\mathbf{N}A)$.*

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(4) *For faces* $\tau \subset \sigma$, *there exists a natural map from* $E_\tau(\beta)$ *to* $E_\sigma(\beta)$. *In particular, if* $E_\tau(\beta) \neq \emptyset$, *then* $E_\sigma(\beta) \neq \emptyset$.

(5) *For any* $\chi \in \mathbf{N}A$, *there exists a natural inclusion from* $E_\tau(\beta)$ *to* $E_\tau(\beta + \chi)$.

Proof. All statements follow directly from the definition of $E_\tau(\beta)$. □

PROPOSITION 2.3.

(1) *For all* $\beta \in \mathbf{k}A = \mathbf{k}^d$,

$$|E_\tau(\beta)| \leqslant [(\mathbf{Q}(A \cap \tau)) \cap \mathbf{Z}A : \mathbf{Z}(A \cap \tau)], \tag{2.4}$$

where the right hand side is the index of $\mathbf{Z}(A \cap \tau)$ *in* $(\mathbf{Q}(A \cap \tau)) \cap \mathbf{Z}A$.

(2) *Assume* $(\mathbf{Q}(A \cap \tau)) \cap \mathbf{Z}A = \mathbf{Z}(A \cap \tau)$. *If* $\beta - \beta' \in \mathbf{Z}A$, *and if neither* $E_\tau(\beta)$ *nor* $E_\tau(\beta')$ *is empty, then* $E_\tau(\beta) = E_\tau(\beta')$.

Proof.

(1) Let $\lambda, \lambda' \in E_\tau(\beta)$. Then $\lambda - \lambda' \in (\mathbf{k}(A \cap \tau)) \cap \mathbf{Z}A$. By Cramer's formula, $(\mathbf{k}(A \cap \tau)) \cap \mathbf{Z}A = (\mathbf{Q}(A \cap \tau)) \cap \mathbf{Z}A$.

(2) Let $E_\tau(\beta) = \{\lambda\}$, $E_\tau(\beta') = \{\lambda'\}$. Since $\beta - \beta' \in \mathbf{Z}A$, there exist $\chi, \chi' \in \mathbf{N}A$ such that $\beta + \chi = \beta' + \chi'$. Then $\{\lambda\} = E_\tau(\beta + \chi) = E_\tau(\beta' + \chi') = \{\lambda'\}$ by Proposition 2.2 (5). □

EXAMPLE 2.4. Let

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$

There are four facets:

$$\sigma_{12} := \mathbf{Q}_{\geqslant 0}a_1 + \mathbf{Q}_{\geqslant 0}a_2, \tag{2.5}$$

$$\sigma_{23} := \mathbf{Q}_{\geqslant 0}a_2 + \mathbf{Q}_{\geqslant 0}a_3, \tag{2.6}$$

$$\sigma_{34} := \mathbf{Q}_{\geqslant 0}a_3 + \mathbf{Q}_{\geqslant 0}a_4, \tag{2.7}$$

$$\sigma_{14} := \mathbf{Q}_{\geqslant 0}a_1 + \mathbf{Q}_{\geqslant 0}a_4, \tag{2.8}$$

and four one-dimensional faces: $\mathbf{Q}_{\geqslant 0}a_1, \ldots, \mathbf{Q}_{\geqslant 0}a_4$. For all faces τ but σ_{14}, the indices $[(\mathbf{Q}(A \cap \tau)) \cap \mathbf{Z}A : \mathbf{Z}(A \cap \tau)]$ are one. Hence for $\beta \in \mathbf{N}A$, $E_\tau(\beta) = \{0\}$ for all faces $\tau \neq \sigma_{14}$. The quotient $(\mathbf{Q}(A \cap \sigma_{14})) \cap \mathbf{Z}A/\mathbf{Z}(A \cap \sigma_{14})$ has two elements and can be represented by 0 and ${}^t(1,1,0)$. Since $a_2 - {}^t(1,1,0) = a_3 - a_4$, and $a_3 - {}^t(1,1,0) = a_2 - a_1$, we obtain $E_{\sigma_{14}}(a_2) = E_{\sigma_{14}}(a_3) = \{0, {}^t(1,1,0)\}$. Proposition 2.2 (5) implies that for $\beta \in \mathbf{N}A$, $E_{\sigma_{14}}(\beta) = \{0\}$ if and only if $\beta \in \mathbf{N}a_1 + \mathbf{N}a_4$, otherwise $E_{\sigma_{14}}(\beta) = \{0, {}^t(1,1,0)\}$. Therefore $\mathbf{N}A$ splits into two isomorphism classes in this case.

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Recall that a parameter β is said to be *nonresonant* (respectively *semi-nonresonant*) if $\beta \notin \mathbf{Z}A + \mathbf{k}(A \cap \sigma)$ (respectively $\beta \notin (\mathbf{Z}A \cap \mathbf{Q}_{\geqslant 0}A) + \mathbf{k}(A \cap \sigma)$) for any facet σ, or equivalently, if $F_\sigma(\beta) \notin \mathbf{Z}$ (respectively $F_\sigma(\beta) \notin \mathbf{N}$) for any facet σ. Hence nonresonance implies semi-nonresonance.

PROPOSITION 2.5. *If β is semi-nonresonant, then $E_\tau(\beta) = \emptyset$ for all proper faces τ of $\mathbf{Q}_{\geqslant 0}A$.*

Proof. Semi-nonresonace clearly implies $E_\sigma(\beta) = \emptyset$ for all facets σ. Proposition 2.2 (4) finishes the proof. □

COROLLARY 2.6 *Let β and β' be semi-nonresonant. Then $M_A(\beta)$ and $M_A(\beta')$ are isomorphic if and only if $\beta - \beta' \in \mathbf{Z}A$.*

Recall that all elements of A lie on one hyperplane H off the origin. We normalize the volume of a polytope on H so that a simplex whose vertices affinely span the lattice $H \cap \mathbf{Z}A$ has volume one.

PROPOSITION 2.7. *If a parameter β satisfies*

$$E_\tau(\beta) = \emptyset \quad \text{for all proper faces } \tau, \tag{2.9}$$

then

(1) *for any $\chi \in \mathbf{N}A$, $M_A(\beta - \chi)$ is isomorphic to $M_A(\beta)$,*
(2) *the rank of $M_A(\beta)$ equals the volume of the convex hull of A.*

Proof. (1) By Proposition 2.2 (5), $E_\tau(\beta - \chi) = \emptyset$ for all proper faces τ. Hence by Proposition 2.2 (1), we deduce the statement from Theorem 2.1.

The proof of (2) is the same as that of Theorem 4.5.2 of [13] (p. 185). □

3. Symmetry Algebra

We consider the algebra of contiguity operators, called the symmetry algebra. It controls isomorphisms among A-hypergeometric systems with different parameters. We have investigated the symmetry algebra of normal A-hypergeometric systems in [11]. The proofs of some results in [11] remain valid without the normality condition. In this section, we summarize such results.

Let

$$\tilde{S} := \{ P \in D \mid I_A P \subset DI_A \}.$$

Then $\tilde{S}$ is an associative algebra and $\tilde{S} \cap DI_A = DI_A$ is a two-sided ideal. We call $S := \tilde{S}/DI_A$ the *symmetry algebra* of A-hypergeometric systems. The symmetry algebra S is nothing but the associative algebra $\mathrm{End}_D(D/DI_A)$.

In what follows, we denote simply by P, the element of D/DI_A represented by $P \in D$. For $\chi \in \mathbf{N}A$, all ∂^u with $Au = \chi$ represent the same element in D/DI_A. Hence we sometimes denote it by ∂^χ.

PROPOSITION 3.1.
(1) $\partial_1, \dots, \partial_n \in S$.
(2) $\sum_{j=1}^n a_{ij}\theta_j \in S$ *for all* $i = 1, \dots, d$.
(3) *The morphism from the polynomial ring* $\mathbf{k}[s] = \mathbf{k}[s_1, \dots, s_d]$ *to* S *mapping* s_i *to* $\sum_{j=1}^n a_{ij}\theta_j$ $(i = 1, \dots, d)$ *is injective.*

Proof. See Lemma 1.1 in [11] for (1) and (2), and Corollary 1.3 in [11] for (3). □

We consider that $\mathbf{Z}^d$, to which the vectors $a_1, \dots, a_n$ belong, is the character group of the algebraic torus $T = \{(t_1, \dots, t_d) \mid t_1, \dots, t_d \in \mathbf{k}^\times\}$. Let N be the dual group of $\mathbf{Z}^d$, and $s_1, \dots, s_d$ the basis of $\mathbf{k} \otimes_\mathbf{Z} N$ dual to the standard basis of $\mathbf{k}^d = \mathbf{k} \otimes_\mathbf{Z} \mathbf{Z}^d$. Under the identification of $\mathbf{k} \otimes_\mathbf{Z} N$ with the Lie algebra of T ([8]), each s_i equals $t_i\partial/\partial t_i$. Let H denote another algebraic torus

$$\left\{ (z_1, \dots, z_n) \in (\mathbf{k}^\times)^n \mid \prod_{j=1}^n z_j^{u_j} = 1 \quad \text{for all } u \in Z^n \quad \text{with } Au = 0 \right\}.$$

Then the character group and the Lie algebra of H coincide with $\mathbf{Z}A$ and $\sum_{i=1}^d \mathbf{k}(\sum_{j=1}^n a_{ij}\theta_j)$ respectively. The injective morphism in Proposition 3.1 (3) is induced from the differential of the morphism:

$$T \ni t \longmapsto (t^{a_1}, \dots, t^{a_n}) \in H. \tag{3.10}$$

We thus consider $\mathbf{k}[s]$ as a subspace of S and, accordingly, as a subspace of D/DI_A. For each $\chi \in \mathbf{Z}A$, we define the weight space S_χ with weight χ by

$$S_\chi := \{ P \in S \mid [s, P] = \chi(s)P \quad (\forall s \in N) \}.$$

Here the bracket $[P, Q]$ means $PQ - QP$.

Remark 3.2. Note that the multiplication by $P \in S_\chi$ from the right defines a D-homomorphism from $M_A(\beta + \chi)$ to $M_A(\beta)$. Hence $P(\psi_\beta)$ is a solution of $M_A(\beta + \chi)$ for any solution ψ_β of $M_A(\beta)$. In this sense, the operator P is a contiguity operator shifting parameters by χ.

THEOREM 3.2
(1) *The symmetry algebra* S *has no zero-divisors.*
(2) *The symmetry algebra* S *has the following weight space decomposition*:

$$S = \bigoplus_{\chi \in \mathbf{Z}A} S_\chi. \tag{3.11}$$

(3) *The weight space* S_0 *equals the polynomial ring* $\mathbf{k}[s]$.

(4) *For each* $\chi \in \mathbf{N}A$, *the weight space* $S_{-\chi}$ *equals* $\mathbf{k}[s]\partial^{\chi}$.

Proof. See Lemma 1.4, and Propositions 2.3, 2.4, 2.9 in [11]. □

The following proposition will be used in the next section.

PROPOSITION 3.4 (Proposition 2.6 in [11]). *The natural morphism*

$$D/DI_A \longrightarrow \mathbf{k}\langle x, \partial^{\pm}\rangle / \mathbf{k}\langle x, \partial^{\pm}\rangle I_A$$

is injective where $\mathbf{k}\langle x, \partial^{\pm}\rangle$ *is the algebra generated by* D *and elements* $\partial_1^{-1}, \ldots, \partial_n^{-1}$ *with relations* $[x_i, \partial_j^{-1}] = \delta_{ij}\partial_j^{-2}$ $(i, j = 1, \ldots, n)$.

4. *b*-Ideals

We have seen in Theorem 3.3 that the symmetry algebra S has a weight decomposition with respect to $\mathbf{Z}A$, and that each S_χ for $-\chi \in \mathbf{N}A$ is a free $\mathbf{k}[s]$-module of rank one with basis $\partial^{-\chi}$. Next we wish to compute the weight space S_χ for arbitrary χ. Suppose that $E \in S_\chi$ and $\chi = \chi_+ - \chi_-$ with $\chi_+, \chi_- \in \mathbf{N}A$. Then the operator $E\partial^{\chi_+}$ belongs to $S_{-\chi_-}$. Hence by Theorem 3.3 (4), there exists a polynomial $b \in \mathbf{k}[s]$ such that $E\partial^{\chi_+} = b\partial^{\chi_-}$. Such polynomials b form an ideal of $\mathbf{k}[s]$ as we vary $E \in S_\chi$. We shall define the *b-ideal* B_χ below to be such an ideal.

Fix any $\chi \in \mathbf{Z}A$, and define an ideal I_χ of $\mathbf{k}[\partial]$ by

$$I_\chi := I_A + M_\chi \tag{4.12}$$

where

$$M_\chi := \langle \partial^u \mid Au \in \chi + \mathbf{N}A \rangle. \tag{4.13}$$

Define the ideal B_χ of *b-polynomials* by

$$B_\chi := \mathbf{k}[s] \cap DI_\chi. \tag{4.14}$$

PROPOSITION 4.1. *Let* $\chi = \chi_+ - \chi_-$ *with* $\chi_+, \chi_- \in \mathbf{N}A$. *Given* $b \in B_\chi$, *there exists a unique operator* $E \in S_\chi$ *such that* $b\partial^{\chi_-} = E\partial^{\chi_+}$. *The operator* E *is independent of the expression* $\chi = \chi_+ - \chi_-$.

Moreover any operator in S_χ *can be obtained in this way.*

Proof. Since $b\partial^{\chi_-} \in DI_\chi\partial^{\chi_-} \subset DI_A + D\partial^{\chi_+}$, there exists an operator $E \in D$ such that $b\partial^{\chi_-} = E\partial^{\chi_+}$. The uniqueness, the independence, and the fact that $E \in S_\chi$ follow from the equality $E = b\partial^{\chi}$ in $\mathbf{k}\langle x, \partial^{\pm}\rangle$ and Proposition 3.4.

Let $E \in S_\chi$ and $\chi = \chi_+ - \chi_-$ with $\chi_+, \chi_- \in \mathbf{N}A$. Then $E\partial^{\chi_+} \in S_{-\chi_-}$. By Theorem 3.3 (4), there exists a polynomial $b \in \mathbf{k}[s]$ such that $E\partial^{\chi_+} = b\partial^{\chi_-}$. Then $b \in DI_\chi$ and thus $b \in B_\chi$. □

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We have the following algorithm for the operator $E \in S_\chi$ corresponding to $b \in B_\chi$, which generalizes Algorithm 3.4 in [12].

ALGORITHM 4.2. *Let $\chi = Au - Av$ and $u, v \in \mathbf{N}^n$.*
Input: a polynomial $b \in B_\chi$.
Output: an operator $E \in S_\chi$ with $E\partial^u = b\partial^v$.

(1) *For $i = 1, \ldots, n$, compute a Gröbner basis $\mathcal{G}_i$ of I_A with respect to any reverse lexicographic term order with lowest variable ∂_i.*
(2) *Expand $b(\sum_j a_{1j}\theta_j, \ldots, \sum_j a_{dj}\theta_j)\partial^v$ in $\mathbf{Q}\langle x, \partial\rangle$ into a $\mathbf{Q}$-linear combination of monomials $x^l\partial^m$.*
(3) *$i := 1$, $E :=$ the output of Step 2.*

While $i \leqslant n$, do

(a) *Reduce E modulo $\mathcal{G}_i$ in $\mathbf{Q}\langle x, \partial\rangle$.*
(b) *The output of Step 3-(a) has $\partial_i^{u_i}$ as a right factor. Divide it by $\partial_i^{u_i}$.*
(c) *$i := i + 1$, $E :=$ the output of Step $3-(b)$.*

The proof of the correctness is completely analogous to that of Algorithm 3.4 in [12].

We thus reduce the study of S_χ to that of B_χ, and for the study of $B_\chi = \mathbf{k}[s] \cap DI_\chi$, we study $\mathbf{k}[\theta] \cap DI_\chi$ first. Since M_χ is the largest monomial ideal in I_χ, we have by Lemma 4.4.4 in [13],

PROPOSITION 4.3.

$$\mathbf{k}[\theta] \cap DI_\chi = \widetilde{M_\chi} \tag{4.15}$$

where $\widetilde{M_\chi}$ is the distraction of M_χ, i.e., $\widetilde{M_\chi} = \mathbf{k}[\theta] \cap DM_\chi$.

For the study of $\widetilde{M_\chi}$, we recall the standard pairs of a monomial ideal. Let M be a monomial ideal of $\mathbf{k}[\partial]$. Then a pair (u, τ) with $u \in \mathbf{N}^n$ and $\tau \subset \{1, \ldots, n\}$ is called a *standard pair* of M if it satisfies the following conditions:

(1) $u_j = 0$ for all $j \in \tau$. (We abbreviate this to $u \in \mathbf{N}^{\tau^c}$, where ^{c} stands for the operation of taking the complement.)
(2) There exists no $v \in \mathbf{N}^\tau$ such that $\partial^{u+v} \in M$.
(3) For each $j \notin \tau$, there exists $v \in \mathbf{N}^{\tau \cup \{j\}}$ such that $\partial^{u+v} \in M$.

For algorithms of obtaining the set of standard pairs, see [7] and Algorithm 3.2.5 in [13]. Let $\mathcal{S}(M_\chi)$ denote the set of standard pairs of M_χ. By Corollary 3.2.3 in [13], the distraction $\widetilde{M_\chi}$ is described as follows:

$$\widetilde{M_\chi} = \bigcap_{(u,\tau) \in \mathcal{S}(M_\chi)} \langle \theta_i - u_i \mid i \notin \tau \rangle. \tag{4.16}$$

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LEMMA 4.4. *Fix any* $\chi \in \mathbf{Z}A$. *Let* (u, τ) *be a standard pair of* M_χ. *Then* $A\mathbf{Q}^\tau_{\geqslant 0} := \sum_{j\in\tau} \mathbf{Q}_{\geqslant 0} a_j$ *is a proper face of* $\mathbf{Q}_{\geqslant 0}A$, *and moreover* $\tau = \{\, i \mid a_i \in A\mathbf{Q}^\tau_{\geqslant 0}\,\}$.

Proof. Suppose that $A\mathbf{Q}^\tau_{\geqslant 0}$ is not contained in any facet of $\mathbf{Q}_{\geqslant 0}A$. Then there exists $\gamma \in A\mathbf{N}^\tau := \sum_{j\in\tau} \mathbf{N}a_j$ such that $F_\sigma(\gamma) > 0$ for all facets σ. Then $F_\sigma(Au + m\gamma) \gg 0$ for $m \gg 0$ and all facets σ. By Lemma 1 in the appendix of [14], $Au + m\gamma \in \chi + \mathbf{N}A$ for $m \gg 0$. This contradicts the assumption that (u, τ) is a standard pair of M_χ.

Next we claim $(A\mathbf{Q}^{\tau^c}_{\geqslant 0}) \cap (A\mathbf{Q}^\tau) = \{\,0\,\}$, which implies the lemma. Suppose $(A\mathbf{Q}^{\tau^c}_{\geqslant 0}) \cap (A\mathbf{Q}^\tau) \neq \{\,0\,\}$. Let $v \in \mathbf{N}^{\tau^c}$ be a nonzero element satisfying $Av \in A\mathbf{Z}^\tau$. Then there exists $w \in \mathbf{N}^\tau$ such that $Aw \in Av + A\mathbf{N}^\tau$. Since $A(u + mw) \notin \chi + \mathbf{N}A$ for any $m \in \mathbf{N}$, $(Au + A\mathbf{N}^{\tau\cup\tau'}) \cap (\chi + \mathbf{N}A) = \emptyset$ for $\tau' = \{i \mid v_i \neq 0\}$. This contradicts the assumption that (u, τ) is a standard pair of M_χ again. □

Thanks to Lemma 4.4, we regard the set τ of a standard pair (u, τ) as a proper face $A\mathbf{Q}^\tau$ of $\mathbf{Q}_{\geqslant 0}A$.

For an ideal I of $\mathbf{k}[s]$, we denote by $V(I)$ the zero set of I. Proposition 4.3 and equation (4.16) give the following prime decomposition of B_χ and irreducible decomposition of the zero set $V(B_\chi)$.

THEOREM 4.5.
(1)

$$B_\chi = \bigcap_{(u,\tau)\in\mathcal{S}(M_\chi)} \langle\, F_\sigma - F_\sigma(Au) \mid \sigma \textit{ facet} \supset \tau \rangle. \tag{4.17}$$

(2)

$$V(B_\chi) = \bigcup_{(u,\tau)\in\mathcal{S}(M_\chi)} (Au + \mathbf{k}(A \cap \tau)). \tag{4.18}$$

Proof. Using (4.16), we only need to show

$$\mathbf{k}[s] \cap \langle \theta_i - u_i \mid i \notin \tau\rangle = \langle F_\sigma - F_\sigma(Au) \mid \sigma \supset \tau\rangle. \tag{4.19}$$

First we have

$$V(\mathbf{k}[s] \cap \langle \theta_i - u_i \mid i \notin \tau\rangle) = Au + \mathbf{k}(A \cap \tau) = V(\langle F_\sigma - F_\sigma(Au) \mid \sigma \supset \tau\rangle). \tag{4.20}$$

Hence

$$\begin{aligned}\mathbf{k}[s] \cap \langle \theta_i - u_i \mid i \notin \tau\rangle &\supset \langle F_\sigma - F_\sigma(Au) \mid \sigma \supset \tau\rangle\\ &= I(V(\langle F_\sigma - F_\sigma(Au) \mid \sigma \supset \tau\rangle))\\ &= I(V(\mathbf{k}[s] \cap \langle \theta_i - u_i \mid i \notin \tau\rangle)),\end{aligned} \tag{4.21}$$

where I stands for the operation of taking the defining ideal. On the other hand, $J \subset I(V(J))$ is automatic for any ideal J. We therefore obtain (4.19). □

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PROPOSITION 4.6.
(1)

$$V(B_{\chi+\chi'}) \subset V(B_\chi) \cup (V(B_{\chi'}) + \chi) \qquad \textit{for } \chi, \chi' \in \mathbf{Z}A. \tag{4.22}$$

(2)

$$V(B_{\chi+\chi'}) = V(B_\chi) \cup (V(B_{\chi'}) + \chi) \qquad \textit{for } \chi, \chi' \in \mathbf{N}A. \tag{4.23}$$

Proof.

(1) Given $p_\chi \in B_\chi$ and $p_{\chi'} \in B_{\chi'}$, let $P_\chi \in S_\chi$ and $P_{\chi'} \in S_{\chi'}$ be the corresponding operators as in Proposition 4.1. Then

$$\begin{aligned} P_\chi P_{\chi'} \partial^{\chi'_+} \partial^{\chi_+} &= P_\chi p_{\chi'}(s) \partial^{\chi'_-} \partial^{\chi_+} \\ &= p_{\chi'}(s-\chi) P_\chi \partial^{\chi_+} \partial^{\chi'_-} \\ &= p_{\chi'}(s-\chi) p_\chi(s) \partial^{\chi_-} \partial^{\chi'_-}. \end{aligned} \tag{4.24}$$

Hence $p_{\chi'}(s-\chi)p_\chi(s) \in B_{\chi+\chi'}$.

(2) Given $p_{\chi+\chi'} \in B_{\chi+\chi'}$, let $P_{\chi+\chi'} \in S_{\chi+\chi'}$ be the corresponding operator as in Proposition 4.1. Then

$$p_{\chi+\chi'} = P_{\chi+\chi'} \partial^{\chi'} \cdot \partial^{\chi}.$$

Hence $p_{\chi+\chi'}(s) \in B_\chi$.
Furthermore

$$p_{\chi+\chi'}(s+\chi)\partial^\chi = \partial^\chi p_{\chi+\chi'}(s) = \partial^\chi P_{\chi+\chi'} \partial^{\chi'} \partial^\chi.$$

Hence $p_{\chi+\chi'}(s+\chi) = \partial^\chi P_{\chi+\chi'} \partial^{\chi'}$, which impies $p_{\chi+\chi'}(s+\chi) \in B_{\chi'}$. □

PROPOSITION 4.7. *Let $\chi \in \mathbf{Z}A$. Given $p_\chi \in B_\chi$ and $p_{-\chi} \in B_{-\chi}$, let $P_\chi \in S_\chi$ and $P_{-\chi} \in S_{-\chi}$ be the corresponding operators as in Proposition 4.1. Then*

$$P_{-\chi} P_\chi = p_\chi(s+\chi) p_{-\chi}(s). \tag{4.25}$$

Proof.

$$\begin{aligned} P_{-\chi} P_\chi \partial^{\chi_+} &= P_{-\chi} p_\chi(s) \partial^{\chi_-} \\ &= p_\chi(s+\chi) P_{-\chi} \partial^{\chi_-} \\ &= p_\chi(s+\chi) p_{-\chi}(s) \partial^{\chi_+}. \end{aligned} \tag{4.26}$$

Recall that the symmetry algebra S has no zero-divisors (Theorem 3.3). Divide equation (4.26) by ∂^{χ_+} to obtain the conclusion. □

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For $\chi \in \mathbf{Z}A$, define an ideal $B_{-\chi,\chi}$ by

$$B_{-\chi,\chi} := \langle p_\chi(s+\chi)p_{-\chi}(s) \mid p_\chi \in B_\chi, p_{-\chi} \in B_{-\chi} \rangle. \tag{4.27}$$

Then the following proposition is immediate from the definition of $B_{-\chi,\chi}$.

PROPOSITION 4.8.
(1)

$$V(B_{-\chi,\chi}) = (V(B_\chi) - \chi) \cup V(B_{-\chi}). \tag{4.28}$$

(2)

$$V(B_{-\chi,\chi}) = V(B_{\chi,-\chi}) - \chi. \tag{4.29}$$

THEOREM 4.9. *Let $\chi \in \mathbf{Z}A$. Assume $\beta \notin V(B_{-\chi,\chi})$. Then two A-hypergeometric systems $M_A(\beta)$ and $M_A(\beta+\chi)$ are isomorphic.*

Proof. First note that $\beta \notin V(B_{-\chi,\chi})$ is equivalent to $\beta+\chi \notin V(B_{\chi,-\chi})$ by Proposition 4.8. Take polynomials $p_\chi \in B_\chi$ and $p_{-\chi} \in B_{-\chi}$ such that $p_\chi(\beta+\chi)p_{-\chi}(\beta) \neq 0$. Let $P_\chi \in S_\chi$ and $P_{-\chi} \in S_{-\chi}$ be the corresponding operators to p_χ and $p_{-\chi}$ as in Proposition 4.1. Then by Proposition 4.7, we have the following equalities:

$$P_{-\chi}P_\chi = p_\chi(s+\chi)p_{-\chi}(s), \tag{4.30}$$
$$P_\chi P_{-\chi} = p_{-\chi}(s-\chi)p_\chi(s). \tag{4.31}$$

The multiplications by $P_{-\chi}, P_\chi$ respectively induce homomorphisms:

$$f : M_A(\beta) \longrightarrow M_A(\beta+\chi), \tag{4.32}$$
$$g : M_A(\beta+\chi) \longrightarrow M_A(\beta). \tag{4.33}$$

Then

$$g \circ f = p_\chi(\beta+\chi)p_{-\chi}(\beta) id_{M_A(\beta)} \tag{4.34}$$

and

$$f \circ g = p_{-\chi}((\beta+\chi)-\chi)p_\chi(\beta+\chi) id_{M_A(\beta+\chi)} \tag{4.35}$$
$$= p_{-\chi}(\beta)p_\chi(\beta+\chi) id_{M_A(\beta+\chi)}. \tag{4.36}$$

Hence f and g are isomorphisms. □

Now we are ready to prove the if-part of our main theorem.

Proof of the if-part of Theorem 2.1.

We suppose that $E_\tau(\beta) = E_\tau(\beta')$ for all faces. Let $\chi := \beta' - \beta$. We claim $\beta \notin V(B_{-\chi})$. Assume the contrary. Then by Theorem 4.5, there exists a standard pair $(u,\tau) \in \mathcal{S}(M_{-\chi})$ such that $\beta - Au \in \mathbf{k}(A \cap \tau)$. The equality $E_\tau(\beta) = E_\tau(\beta')$ implies that there exists $v \in \mathbf{N}^n$ such that $\beta - \beta' = A(u-v) + \mathbf{N}(A \cap \tau)$. Hence the intersection of

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$Au + \mathbf{N}(A \cap \tau)$ with $(\beta - \beta') + \mathbf{N}A$ is not empty. This contradicts the fact that (u, τ) is standard. We have thus proved $\beta \notin V(B_{-\chi})$. By symmetry we have $\beta' \notin V(B_\chi)$, which is equivalent to $\beta \notin V(B_\chi) - \chi$. Hence $\beta \notin V(B_{-\chi,\chi})$ by Proposition 4.8. From Theorem 4.9 we conclude $M_A(\beta)$ is isomorphic to $M_A(\beta')$. □

As a corollary of the proof of the if-part of Theorem 2.1, we obtain the following.

COROLLARY 4.10. *If $M_A(\beta)$ and $M_A(\beta')$ are isomorphic, then there exists an operator $P \in S_{\beta'-\beta}$ such that the multiplication by P from the right induces an isomorphism from $M_A(\beta)$ to $M_A(\beta')$.*

5. Normal Case

In this section, we consider the normal case:

$$\mathbf{N}A = \mathbf{Z}A \cap \mathbf{Q}_{\geqslant 0}A. \tag{5.37}$$

Many important examples are known to be normal, such as the Aomoto-Gel'fand systems, the A-hypergeometric systems corresponding to the univariate hypergeometric functions ${}_{p+1}F_p$, to Lauricella functions, etc. (see [9], [10]). It will turn out below that the parameter space can be classified in terms of the primitive integral support functions F_σ in the normal case.

LEMMA 5.1. *Assume A to be normal. Then we have the following.*
(1) *$(\mathbf{Q}(A \cap \tau)) \cap \mathbf{Z}A$ equals $\mathbf{Z}(A \cap \tau)$ for all faces τ.*
(2) *$F_\sigma(\mathbf{N}A) = \mathbf{N}$ for all facets σ.*
(3) *For a face τ,*

$$\mathbf{N}A + \mathbf{Z}(A \cap \tau) = \mathbf{Z}A \cap \bigcap_{\sigma \text{ facet } \supset \tau} (\mathbf{N}A + \mathbf{k}(A \cap \sigma)). \tag{5.38}$$

Proof.
(1) Let $\chi \in (\mathbf{Q}(A \cap \tau)) \cap \mathbf{Z}A$. Add a vector $\chi' \in \mathbf{N}(A \cap \tau)$ to χ so that $\chi + \chi' \in \mathbf{Q}_{\geqslant 0}(A \cap \tau)$. By normality, we see that $\chi + \chi' \in \mathbf{N}(A \cap \tau)$. Hence χ belongs to $\mathbf{Z}(A \cap \tau)$.
(2) Let $\chi \in \mathbf{Z}A$ satisfy $F_\sigma(\chi) = 1$. For any facet $\sigma' \neq \sigma$, there exists $a_j \in \sigma \setminus \sigma'$. Hence there exists $\chi' \in \mathbf{N}(A \cap \sigma)$ such that $F_{\sigma'}(\chi + \chi') \geqslant 0$ for all facets σ'. By normality, $\chi + \chi' \in \mathbf{N}A$. Since $F_\sigma(\chi + \chi') = 1$, we obtain $F_\sigma(\mathbf{N}A) = \mathbf{N}$.
(3) Let $\chi \in \mathbf{Z}A$ satisfy $F_\sigma(\chi) \geqslant 0$ for all facets containing τ. For a facet σ not containing the face τ, there exists $a_j \in \tau \setminus \sigma$. Hence there exists a vector $\chi' \in \mathbf{N}(A \cap \tau)$ such that $F_\sigma(\chi + \chi') \geqslant 0$ for all facets σ of the cone $\mathbf{Q}_{\geqslant 0}A$. By normality, $\chi + \chi' \in \mathbf{N}A$, and thus $\chi \in \mathbf{N}(A \setminus A \cap \tau) + \mathbf{Z}(A \cap \tau)$. □

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THEOREM 5.2. *Assume A to be normal. Let $\beta, \beta' \in \mathbf{k}^d$. Then $M_A(\beta)$ is isomorphic to $M_A(\beta')$ if and only if $\beta - \beta' \in \mathbf{Z}A$ and*

$$\{\sigma \text{ facet} \mid F_\sigma(\beta) \in \mathbf{N}\} = \{\sigma \text{ facet} \mid F_\sigma(\beta') \in \mathbf{N}\}. \tag{5.39}$$

Proof. By Proposition 2.2 (3), the only-if-part follows from Theorem 2.1.

Next we prove the if-part. Suppose $\beta - \beta' \in \mathbf{Z}A$ and (5.39). By Lemma 5.1 (1), (2), and Propositions 2.2, 2.3, we obtain $E_\sigma(\beta) = E_\sigma(\beta')$ for all facets. By Lemma 5.1 (3), the if-part follows from Theorem 2.1 □

EXAMPLE 5.3. Let

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}.$$

Let $\beta \in \mathbf{Z}A = \mathbf{Z}^d$. Then by Theorem 5.2, the A-hypergeometric system $M_A(\beta)$ is isomorphic to

$$\begin{array}{lll}
M_A({}^t(0,0,0)) & \text{if} & \beta_1 \geqslant 0, \beta_2 \geqslant 0, \beta_1+\beta_3 \geqslant 0, \beta_2+\beta_3 \geqslant 0, \\
M_A({}^t(-1,0,1)) & \text{if} & \beta_1 < 0, \beta_2 \geqslant 0, \beta_1+\beta_3 \geqslant 0, \beta_2+\beta_3 \geqslant 0, \\
M_A({}^t(0,-1,1)) & \text{if} & \beta_1 \geqslant 0, \beta_2 < 0, \beta_1+\beta_3 \geqslant 0, \beta_2+\beta_3 \geqslant 0, \\
M_A({}^t(0,1,-1)) & \text{if} & \beta_1 \geqslant 0, \beta_2 \geqslant 0, \beta_1+\beta_3 < 0, \beta_2+\beta_3 \geqslant 0, \\
M_A({}^t(1,0,-1)) & \text{if} & \beta_1 \geqslant 0, \beta_2 \geqslant 0, \beta_1+\beta_3 \geqslant 0, \beta_2+\beta_3 < 0, \\
M_A({}^t(-1,-1,1)) & \text{if} & \beta_1 < 0, \beta_2 < 0, \beta_1+\beta_3 \geqslant 0, \beta_2+\beta_3 \geqslant 0, \\
M_A({}^t(-1,0,0)) & \text{if} & \beta_1 < 0, \beta_2 \geqslant 0, \beta_1+\beta_3 < 0, \beta_2+\beta_3 \geqslant 0, \\
M_A({}^t(0,-1,0)) & \text{if} & \beta_1 \geqslant 0, \beta_2 < 0, \beta_1+\beta_3 \geqslant 0, \beta_2+\beta_3 < 0, \\
M_A({}^t(0,0,-1)) & \text{if} & \beta_1 \geqslant 0, \beta_2 \geqslant 0, \beta_1+\beta_3 < 0, \beta_2+\beta_3 < 0, \\
M_A({}^t(-2,-1,1)) & \text{if} & \beta_1 < 0, \beta_2 < 0, \beta_1+\beta_3 < 0, \beta_2+\beta_3 \geqslant 0, \\
M_A({}^t(-1,-2,1)) & \text{if} & \beta_1 < 0, \beta_2 < 0, \beta_1+\beta_3 \geqslant 0, \beta_2+\beta_3 < 0, \\
M_A({}^t(-1,0,-1)) & \text{if} & \beta_1 < 0, \beta_2 \geqslant 0, \beta_1+\beta_3 < 0, \beta_2+\beta_3 < 0, \\
M_A({}^t(0,-1,-1)) & \text{if} & \beta_1 \geqslant 0, \beta_2 < 0, \beta_1+\beta_3 < 0, \beta_2+\beta_3 < 0, \\
M_A({}^t(-1,-1,0)) & \text{if} & \beta_1 < 0, \beta_2 < 0, \beta_1+\beta_3 < 0, \beta_2+\beta_3 < 0.
\end{array}$$

6. Monomial Curve Case

In this section, we consider the case $d = 2$, called the monomial curve case. Let

$$A = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 0 & i_2 & i_3 & \cdots & i_{n-1} & i_n \end{pmatrix}$$

with $0 < i_2 < i_3 < \cdots < i_n$, relatively prime integers. In this case, there are only two facets: $\sigma_1 = \mathbf{Q}_{\geqslant 0}{}^t(1,0)$ and $\sigma_2 = \mathbf{Q}_{\geqslant 0}{}^t(1, i_n)$. Their primitive integral support functions are $F_{\sigma_1}(s) = s_2$ and $F_{\sigma_2}(s) = i_n s_1 - s_2$.

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We denote by $\mathcal{E}(A)$ the set of holes, i.e.,

$$\mathcal{E}(A) := ((\mathbf{N}A + \mathbf{Z}a_1) \cap (\mathbf{N}A + \mathbf{Z}a_n)) \setminus \mathbf{N}A \tag{6.40}$$
$$= \{ \beta \mid E_{\mathbf{Q}_{\geqslant 0}A}(\beta) = \{0\},\ E_{\sigma_1}(\beta) = \{0\},$$
$$E_{\sigma_2}(\beta) = \{0\},\ E_{\{0\}}(\beta) = \emptyset \}. \tag{6.41}$$

The rank of $M_A(\beta)$ is d or $d+1$, and it equals $d+1$ if and only if $\beta \in \mathcal{E}(A)$ (see [2], [13]).

In the monomial curve case, the assumption of Proposition 2.3 (2) is clearly satisfied:

LEMMA 6.1. *For any face* τ,

$$\mathbf{Z}A \cap (\mathbf{k}(A \cap \tau)) = \mathbf{Z}(A \cap \tau). \tag{6.42}$$

COROLLARY 6.2.

$$\mathcal{E}(A) = \{ \beta \in \mathbf{Z}A \mid F_{\sigma_1}(\beta) \in F_{\sigma_1}(\mathbf{N}A),\ F_{\sigma_2}(\beta) \in F_{\sigma_2}(\mathbf{N}A) \} \setminus \mathbf{N}A. \tag{6.43}$$

Proof. This is immediate from Lemma 6.1 □

Theorem 2.1 reads as follows in the monomial curve case.

THEOREM 6.3. *Let* $\beta, \beta' \in \mathbf{k}^d$.

(1) *Suppose* $\beta \notin \mathcal{E}(A)$. *Then* $M_A(\beta')$ *is isomorphic to* $M_A(\beta)$ *if and only if* $\beta - \beta' \in \mathbf{Z}A$, $\beta' \notin \mathcal{E}(A)$, *and* $\{\sigma_i \mid F_{\sigma_i}(\beta) \in F_{\sigma_i}(\mathbf{N}A)\} = \{\sigma_i \mid F_{\sigma_i}(\beta') \in F_{\sigma_i}(\mathbf{N}A)\}$.

(2) *Suppose* $\beta \in \mathcal{E}(A)$. *Then* $M_A(\beta')$ *is isomorphic to* $M_A(\beta)$ *if and only if* $\beta' \in \mathcal{E}(A)$.

Proof. (2) directly follows from Theorem 2.1.

The only-if-part of (1) follows from Theorem 2.1 by Proposition 2.2 (3). Next suppose that $\beta - \beta' \in \mathbf{Z}A$, $\beta, \beta' \notin \mathcal{E}(A)$, and that $\{\sigma_i \mid F_{\sigma_i}(\beta) \in F_{\sigma_i}(\mathbf{N}A)\} = \{\sigma_i \mid F_{\sigma_i}(\beta') \in F_{\sigma_i}(\mathbf{N}A)\}$. Then by Lemma 6.1, Proposition 2.2 (3), and Proposition 2.3 (2), we have $E_{\sigma_i}(\beta) = E_{\sigma_i}(\beta')$ for $i = 1, 2$. Moreover we know $E_{\{0\}}(\beta), E_{\{0\}}(\beta') = \emptyset$ from Proposition 2.2 (2). Hence $M_A(\beta)$ and $M_A(\beta')$ are isomorphic by Theorem 2.1. □

EXAMPLE 6.4. Example 4.2.2 in [13]) Let

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 4 & 7 & 9 \end{pmatrix}.$$

Then

$$F_{\sigma_1}(\mathbf{N}A) = \{0, 2, 4, 6, 7, 8, 9, \ldots\}, \tag{6.44}$$

and

$$F_{\sigma_2}(\mathbf{N}A) = \{0, 2, 4, 5, 6, 7, 8, 9, \ldots\}, \tag{6.45}$$

Parameters in $\mathbf{Z}A = \mathbf{Z}^2$ are decomposed into five parts according to the isomorphism classes of their corresponding A-hypergeometric systems:

(1) $\mathbf{N}A$,
(2) $\{{}^t(\beta_1, \beta_2) \mid \beta_2 \in F_{\sigma_1}(\mathbf{N}A),\ 9\beta_1 - \beta_2 \notin F_{\sigma_2}(\mathbf{N}A)\}$,
(3) $\{{}^t(\beta_1, \beta_2) \mid \beta_2 \notin F_{\sigma_1}(\mathbf{N}A),\ 9\beta_1 - \beta_2 \in F_{\sigma_2}(\mathbf{N}A)\}$,
(4) $\{{}^t(\beta_1, \beta_2) \mid \beta_2 \notin F_{\sigma_1}(\mathbf{N}A),\ 9\beta_1 - \beta_2 \notin F_{\sigma_2}(\mathbf{N}A)\}$,
(5) $\mathcal{E}(A) = \{{}^t(2, 10), {}^t(2, 12), {}^t(3, 19)\}$: the set of holes.

7. Final Remark

Thanks to Theorem 2.1, all D-invariants of A-hypergeometric systems can be described in terms of $E_\tau(\beta)$; the characteristic cycles (in particular, the rank), the monodromy representations, etc. One of the most recent results is given by Tsushima ([15]) on Laurent polynomial solutions. He has proved that the vector space of Laurent polynomial solutions of $M_A(\beta)$ has a basis consisting of canonical series whose negative supports correspond to faces τ of $\mathbf{Q}_{\geqslant 0}A$ such that $\dim \tau = |\{a_j \mid a_j \in \tau\}|$, and that $0 \in E_\tau(\beta)$ but $0 \notin E_{\tau'}(\beta)$ for any proper face τ' of τ. In particular, the dimension of the vector space of Laurent polynomial solutions equals the cardinality of the set of such faces. This is a generalization of the corresponding result by Cattani, D'Andrea and Dickenstein ([2]) in the monomial curve case.

Acknowledgement

The author is very grateful to Professor Nobuki Takayama for his fruitful comments.

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