Apolar Triads associated with the Nodal Cubic.

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Let the Nodal cubic have the equation

$$x = 6t^2,\ y = 6t,\ z = -(1 + t^3)$$

or,
$$x^3 + y^3 + 6xyz = 0.$$

Three points t_1, t_2, t_3 form an apolar triad to the curve when they satisfy an equation of the form

$$t^3 + 3p\,t^2 + 3q\,t + pq = 0.$$

Take two triads of apolar points as

$$t^3 + 3p\,t^2 + 3q\,t + pq = 0, \text{ and } t^3 + 3rt^2 + 3st + rs = 0, \ldots \text{(i and ii)}$$

then the third triad, in which the members of the pencil of cubics passing through these two triads and apolar to them cut the nodal cubic, is found to be

$$pqrst^3 + 3qst^2 + 3prt + 1 = 0,[1] \ldots\ldots\ldots\ldots\ldots \text{(iii)}$$

and the three triads are symmetrical in their properties.

Now three points on the cubic are collinear, when their parameters satisfy $t_1\,t_2\,t_3 = -1$; and six points lie on a conic when their parameters satisfy $t_1\,t_2\,t_3\,t_4\,t_5\,t_6 = 1$.

Consider the triads i with parameters $x, y, z,$
and ii ,, ,, $a, b, c,$
,, iii ,, ,, $u, v, w;$

and, as there is no confusion, denote the corresponding points on the cubic by the same letters.

Project x, y, z in turn from a, b, c on to the nodal cubic, again obtaining the points with parameters x', y', z'; x'', y'', z''; x''', y''', z'''.

It follows that $x'\,x''\,x'''$; $y'\,y''\,y'''$; $z'\,z''\,z'''$; are apolar triads, and so are $x'\,y'\,z'$; $x''\,y''\,z''$; $x'''\,y'''\,z'''$.

[1] W. P. Milne, *Proc. Edin. Math. Soc.*, **30** (1911–12.)

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The cubic determined by $x' y' z'$ is given by

$$t^3 - \frac{3t^2}{ap} + \frac{3t}{a^2 q} - \frac{1}{a^3 pq} = 0: \quad \ldots\ldots\ldots\ldots\ldots (\text{iv})$$

similarly, for $x'' y'' z''$,

$$t^3 - \frac{3t^2}{bp} + \frac{3t}{b^2 p} - \frac{1}{b^2 pq} = 0, \ldots\ldots\ldots\ldots\ldots (\text{v})$$

and for $x''' y''' z'''$,

$$t^3 - \frac{3t^2}{cp} + \frac{3t}{c^2 q} - \frac{1}{c^3 pq} = 0. \quad \ldots\ldots\ldots\ldots\ldots (\text{vi})$$

There is one binary cubic apolar to three independent binary cubics; and for (iv), (v), (vi) it is given by

$$t^3 pqrs + 3t^2 qs + 3tpr + 1 = 0,$$

or the third apolar triad which the pencil of cubics cut out on the nodal cubic.

It is clear from the process that the same binary cubic will be obtained as the apolar cubic for the three triads—$x' x'' x'''$; $y' y'' y'''$; $z' z'' z'''$;—obtained by projecting a, b, c from x, y, z. In a similar manner, by projecting the three points given by iii, namely u, v, w, from the points x, y, z in turn, the apolar cubic of these three triads is (ii).

(b) It is also true that if the lines joining[1] xy, yz, zx, meet the cubic again in $\lambda\mu\nu$, and the lines joining ab, bc, ca, meet the cubic in $\lambda'\mu'\nu'$, then by a similar process to the above, the projections of $\lambda\mu\nu$ from $\lambda'\mu'\nu'$ give three triads whose apolar binary cubic is the cubic whose parameters $\lambda''\mu''\nu''$ are the points in which uv, vw, wu, meet the cubic again, and similarly for the other sets. Certain other elementary properties of such triads on the nodal cubic will now be established.

(Use the same points as the above). Let the conics through $xyzab$, $xyzbc$, $xyzca$, meet the curve again in the points c', a', b' respectively. Then

$$xyzabc' = 1 = xyzacb' = xyzbca'.$$

[1] Saddler, *Proc. Lond. Math. Soc.*, **2, 26** (1926), 249-256—(in a similar connection).

Further, the cubic satisfied by $c' a' b'$ is given by

$$(pqrst)^3 + 3r\,(pqrst)^2 + 3s\,(pqrst) + rs = 0.$$

Hence $c' a' b'$ is also an apolar triad.

Similarly for the conics got by interchanging xyz with abc the points so obtained are c'', a'', b''. In this case the cubic is given by

$$(pqrst)^3 + 3p\,(pqrst)^2 + 3q\,(pqrst) + pq = 0.$$

Now take a pencil of cubics through the points, and apolar to them, on the nodal cubic given by c', a', b' and x, y, z; the third apolar triad determined by this pencil is

$$t^3 + 3t^2 qs + 3tpr\,(pqrs) + (pqrs)^2 = 0. \qquad \text{..........(vii)}$$

But the three points $\lambda''\mu''\nu''$ in which the lines uv, vw, wu, cut the cubic again are also given by this equation.

Similar theorems hold for the other sets of associated points.

Thus if three conics are drawn, each through three points of an apolar triad and the three pairs of other points of a second apolar triad—on the nodal cubic—the remaining points of intersection are again an apolar triad; and, if the pencil of cubics through these last three points and the first three and apolar to them be considered, the third triad of intersection will be the intersection of the three sides determined by the third triad cut out by the apolar cubics through the first two triads.

(d) Again it is known that every member of a pencil of conics through four points of a cubic cuts the cubic again in two points whose join passes through a fixed point on the cubic—"the point opposé." Take the theorem above for the nodal cubic. Two members of the pencil through $xyza$ pass through bc', cb'.

Hence the "point opposé" of this pencil passes through d given by

$$bc'd = -1, \; cb'd = -1;$$

thus, $d = -xyza$. Treat the other sets in a similar manner, namely $xyzb$ giving the point d', and $xyzc$ giving d''.

The points d', d'', d are found to satisfy the equation

$$t^3 + 3\,(pq)\,rt^2 + 3\,(pq)^2\,st + (pq)^3\,rs = 0$$

and are again an apolar triad. With these take the points in which

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the lines xy, yz, zx, meet the cubic again:—the points $\lambda\mu\nu$; which satisfy the equation

$$t^3 + \frac{3p}{pq}t^2 + \frac{3q}{(pq)^2}t + \frac{1}{(pq)^3} = 0.$$

The third triad associated with $\lambda\mu\nu$ and $d'\,d''\,d$ is found to be given by

$$pqrst^3 + 3qst^2 + 3prt + 1 = 0,$$

or again the residual of the first two xyz, abc.
Similar theorems hold for the other associated sets.

(*e*) The Hessian points of the apolar triad[1]

$$t^3 + 3pt^2 + 3qt + pq = 0$$

are given by $t^2 - q = 0$, and the tangents at these two points (say) h_1 and h_2, meet on the curve at a point X whose parameter is $-\dfrac{1}{q}$.

The pencil of lines joining the apolar triad to the node, being given by[1]

$$x^3 + 3px^2y + 3qxy^2 + pqy^3 = 0$$

has as its second polar with respect to the nodal tangents:—viz. $x = 0$, $y = 0$, the same line namely $x + py = 0$.
This line meets the cubic again in the point Y whose parameter is $-p$. The join of XY meets the cubic in the point Z whose parameter is $-\dfrac{q}{p}$; and the equation of the line joining Z to the node is

$$px + qy = 0. \quad\text{......................(viii)}$$

The line joining the two Hessian points h_1 and h_2 has the equation

$$x + yq^2 + 6zq = 0. \quad\text{....................(ix)}$$

The two lines (viii) and (ix) intersect in the point

$$\{6q, -6p, -(1-pq)\}. \quad\text{..................(x)}$$

Now the *osculant conic*[2] at the point t_1 of a rational cubic

$$x = f(t, t'),\ y = \phi(t, t'),\ z = \psi(t, t'),$$

[1] Milne. *loc. cit.*

[2] Winger. *Projective Geometry*, p. 387.

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(expressing the equation homogeneously in the parameters) has the equation

$$x=t_1\frac{\partial f}{\partial t}+t_1'\frac{\partial f}{\partial t'}\;;\;y=t_1\frac{\partial \phi}{\partial t}+t_1'\frac{\partial \phi}{\partial t'}\;;\;z=t_1\frac{\partial \psi}{\partial t}+t_1'\frac{\partial \psi}{\partial t'}\;;$$

and the *mixed osculant*[2] of $t_1 t_2$ is given by

$$x=\left(t_1\frac{\partial}{\partial t}\right)\left(t_2\frac{\partial}{\partial t}\right)f,$$

with similar expressions for y and z. Thus for the nodal cubic the *osculant conic* at a point t_1 has the equation

$$x=4tt_1+2t^2;\; y=4t+2t_1;\; z=-(1+t^2t_1),$$

and the mixed osculant of $t_1 t_2$ is the line

$$\begin{aligned} x&=2t(t_1+t_2)+2t_1t_2,\\ y&=2(t+t_1+t_2)\\ z&=-(1+tt_1t_2). \end{aligned}$$

This line is found to be the common tangent to the two osculant conics at t_1 *and* t_2, the other common tangents being the three flex tangents which are common to all osculant conics.

Now it is clear from the form, that the osculants of t_1t_2, t_2t_3, t_3t_1, meet in a point whose coordinates are

$$x:y:z=t_1t_2+t_2t_3+t_3t_1;\; t_1+t_2+t_3;\; -\tfrac{1}{2}(1+t_1t_2t_3) \;\ldots\ldots\text{(xi)}$$

This is the point of intersection of the three common tangents of the three osculant conics.

When the points $t_1t_2t_3$ are an apolar triad and satisfy the equation (i) or, $t^3+3pt^2+3qt+pq=0$, then the point (xi) is $\{6q,-6p,-(1-pq)\}$, or the point given by (x):—namely where the line, joining h_1h_2, meets the line joining Z to the node.

Hence, given the points $t_1t_2t_3$ of an apolar triad and the osculant conics of t_1t_2, the osculant conic of t_3 is geometrically constructible, since all the osculant conics touch the three inflexional tangents and the osculant conic of t_3 touches the tangent to the cubic at the point t_3, besides having the property determined by (xi).

In particular, when the three triads are given by (i), (ii), (iii), the three points corresponding to X, namely with parameters $-\frac{1}{q}$, $-\frac{1}{s}$, $-qs$ are collinear, while the three points corresponding

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to Y namely with parameters $-p$, $-r$, $-\frac{1}{pr}$ are also collinear: so also are the three points Z.

The Hessian points h_1 and h_2 are determined when X is given and hence given the triads (i) and (ii) and thus the points XX'; YY'; ZZ'; not only are the points of intersection of the three common tangents of the osculant conics of the triads given by (i) and (ii) uniquely determined, but so also is the point of intersection of these tangents to the osculant conics of the triad (iii).

(f) Again, since the points $\lambda\mu\nu$ were given by the cubic

$$t^3 + \frac{3pt}{pq} + \frac{3qt}{(pq)^2} + \frac{1}{(pq)^3} = 0,$$

by taking the harmonic of the lines joining $\lambda\mu\nu$ to the node, we obtain the points $\lambda_1\mu_1\nu_1$ which satisfy

$$t^3 - \frac{3pt}{pq} + \frac{3qt}{(pq)^2} - \frac{1}{(pq)^3} = 0,$$

The mixed osculants of $(\lambda_1\mu_1;\ \mu_1\nu_1;\ \nu_1\lambda_1)$ meet at the point whose coordinates are given by $\left\{\frac{3q}{p^2q^2},\ -\frac{3p}{pq},\ -\frac{1}{2}\left(1+\frac{1}{p^2q^2}\right)\right\}$.

The line, joining this point to the point of intersection of the mixed osculants of the pairs of points of (i) viz.

$t^3 + 3pt^2 + 3qt + pq = 0$, namely the point, M_1, whose coordinates are $\{3q,\ -3p,\ -\frac{1}{2}(1-pq)\}$; is the line joining the Hessian points $h_1 h_2$ of (i) (corresponding results hold for the other sets of points). Again consider the cubic given by (i), viz.

$$t^3 + 3pt^2 + 3qt^2 + pq = 0.$$

The cubic covariant of this cubic is given by

$$pt^3 + 3qt^2 + 3pqt + q^2 = 0.$$

Geometrically this is obtained by projecting the points given by (1) from the point with parameter $-\frac{1}{q}$, namely where the tangents at the two Hessian points $h_1 h_2$ meet on the curve.

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The mixed linear osculants at the three points of this new cubic will meet at a point M_2 on the line joining $h_1 h_2$ whose coordinates are given by $\{6pq, -6q, -p+q^2\}$. It is found that the two points $h_1 h_2$ will harmonically separate the pair of points given by M_1 and M_2. *Thus the Hessian pair of a triad of points forming an apolar triad to the curve harmonically separates the meet of the mixed linear osculants of the points of the triads and that of the mixed linear osculants of the cubic covariant of the triad.*

Similar theorems may be developed for the cubic envelope

$$3\,(l^3 + m^3) = lmn$$

where $l = \tau^2$, $m = \tau$ $n = 3\,(1 + \tau^3)$.

It is found that a triad of tangents will form an apolar triangle to this class cubic if their parameters satisfy an equation of the form

$$\tau^3 + 3p\tau^2 + 3q\tau + pq = 0.$$