BULLETIN **13** (2), 1970

NOTES AND PROBLEMS

This department welcomes short notes and problems believed to be new. Solutions should accompany proposed problems.

Send all communications concerning this department to

NOTES ET PROBLÈMES

Cette section a pour but de présenter des notes brèves ainsi que des problèmes inédits. Les problèmes proposés doivent être accompagnés de leurs solutions.

Veuillez adresser les communications concernant cette section à

I. G. Connell
Department of Mathematics
McGill University
Montreal, Quebec

PROBLEMS FOR SOLUTION

This section is dedicated to the memory of Leo Moser who died on February 9, 1970. He was editor of this section during the first three years of the Bulletin (1958–1960) and always took an active interest in it.

The ten problems listed below were selected by him, when he visited McGill on February 3, from a huge collection he had formed over the years.

P.165. Prove that the set of squares of sides $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ can be placed without overlap in a square of side $\frac{5}{6}$ but in no smaller square.

P.166. Does there exist a real valued integrable function $f(x)$ which satisfies the conditions:

(i) $f(x) = (f(x))^2$;
(ii) $f(x) = f(x+1)$;
(iii) $\int_0^1 f(x) f(x+t)\, dx = \frac{1}{2}$ for all t?

P.167. Prove that for every positive integer n and every $\varepsilon > 0$ there exists an $N_0 = N_0(n, \varepsilon)$ such that for $N \geq N_0$ at least n hyperspheres of diameter $1-\varepsilon$ can be placed without overlap in an N-dimensional unit cube.

P.168. Prove that

$$\max_{x_1^2+x_2^2+x_3^2=1} \min_{\varepsilon_i=0,1,-1\,(\text{not all }0)} |\varepsilon_1 x_1+\varepsilon_2 x_2+\varepsilon_3 x_3| = \frac{1}{\sqrt{21}}$$

and generalize to the case of more variables.

P.169. What is the largest volume that can be obtained as the cross-section of a four-dimensional unit cube by a hyperplane through its center?

P.170. In the n-dimensional tic-tac-toe board let $f(n)$ be the maximal number of "squares" which can be entered without getting three in a line. Prove that $f(n) \geq c3^n/\sqrt{n}$. (Whether $f(n)=o(3^n)$ is unknown.)

P.171. Evaluate

$$\sum_{(a,b)=1} \frac{1}{a^{2m}b^{2n}}$$

for m and n positive integers. For which values of m and n is the sum rational?

P.172. Let $p(n)$ be the partition function. Prove that for every finite sequence of digits there is an n such that $p(n)$ contains the given sequence of digits in its decimal representation.

P.173. Let S be a measurable point set in the unit square. Suppose no two points of S are at a distance $1/100$ (exactly) apart. Prove or disprove that the measure of S is $<\frac{1}{5}$.

P.174. Prove that any measurable subset S of $[0, 1]$ having the property that $x, y \in S \Rightarrow (x+y)/2 \notin S$ has measure 0.

SOLUTIONS

P.156. Let G be a group with right invariant metric d_R. Suppose right multiplication is continuous, then

(i) inversion is continuous at the identity e.

(ii) if left multiplication is also continuous, then inversion is continuous everywhere (i.e. G is a topological group);

(iii) if G possesses a left invariant metric d_L equivalent to d_R then left multiplication is continuous and G is a topological group (equivalent means gives the same topology).

J. MARSDEN,
UNIVERSITY OF CALIFORNIA AT BERKELEY

https://doi.org/10.1017/S0008439500031660 Published online by Cambridge University Press

Solution by A. Finbow, Dalhousie University, Halifax, Nova Scotia. Let $\varepsilon > 0$ be a given real number, $x, y \in G$.

(i) $d_R(x, e) < \varepsilon \Rightarrow d_R(xx^{-1}, ex^{-1}) < \varepsilon \Rightarrow d_R(x^{-1}, e) < \varepsilon$, by right invariance of d_R, hence inversion is continuous at e.

(ii) Let $x_0 \in G$, then by continuity of left multiplication there is a real number $\delta > 0$ such that $d_R(x_0, x) < \delta \Rightarrow d_R(e, x_0^{-1}x) < \varepsilon$. But then by the right invariance of d_R, $d_R(x^{-1}, x_0^{-1}) < \varepsilon$. Hence inversion is continuous everywhere.

To show that G is a topological group, let $x_0, y_0 \in G$. Then by continuity of left multiplication, there is a real number $\delta > 0$ such that

$$d_R(y_0, y) < \delta \Rightarrow d_R(x_0y_0, x_0y) < \frac{\varepsilon}{2}.$$

Also by right invariance of d_R,

$$d_R(x_0, x) < \frac{\varepsilon}{2} \Rightarrow d_R(x_0y, xy) < \frac{\varepsilon}{2}.$$

Then

$$d_R(x_0y_0, xy) \leq d_R(x_0y_0, x_0y) + d_R(x_0y, xy) < \varepsilon.$$

(iii) $d_L(x_0, x) < \varepsilon \Rightarrow d_L(gx_0, gx) < \varepsilon$ for any $g \in G$. Thus since d_R and d_L are equivalent, left multiplication is continuous and by (ii) G is a topological group.

Remark. "Suppose right multiplication is continuous" is redundant since $d_R(x, y) < \varepsilon \Rightarrow d_R(xg, yg) < \varepsilon$ by right invariance of d_R.

P.162. Let G be a finite abelian group, written additively, and S a subset of G. S is said to be a *sum-free set* in G if $(S+S) \cap S = \varnothing$. Let $\lambda(G)$ denote the largest possible order of a sum-free set in G.

For which abelian group G does there exist a sum-free set S such that

(i) $|S| = \lambda(G)$

and (ii) $|S+S| = \dfrac{\lambda(G)\,[\lambda(G)+1]}{2}$?

A. P. Street,
University of Alberta

Solution by O. P. Lossers, Technological University, Eindhoven, Netherlands.

(a) Let C_l be the cyclic group of order l (represented as the integers mod l). We suppose $l \geq 4$. Then we can find a positive integer k such that $3(2k-1) < l \leq 3(2k+1)$ Now there exists a sum-free set S in C_l of $2k$ elements, namely

$$S = \{-(2k-1), -(2k-3), \ldots, -3, -1, 1, 3, \ldots, 2k-3, 2k-1\}.$$

Hence we deduce:

$$\frac{\lambda(C_l)}{|C_l|} \geq \frac{2k}{3(2k+1)} \geq \frac{4}{15} \quad \text{if } k \geq 2, \text{ i.e. } l \geq 10.$$

https://doi.org/10.1017/S0008439500031660 Published online by Cambridge University Press

Direct calculations show that for cyclic groups C_l of order $l \leq 9$ the same result holds.

CONCLUSION: $\lambda(G) \geq \frac{4}{15}|G|$ if G is cyclic.

(b) Let G be a finite abelian group which is not cyclic. We know that G is the direct sum $C \oplus H$ of a cyclic group $C(|C| \geq 2)$ and an abelian group H. For every sum-free set S_c in C the set $S_c \oplus H$ is sum-free in G, so that

$$\lambda(G) \geq \lambda(C) \cdot |H| = \frac{\lambda(C)}{|C|} \cdot |G|.$$

CONCLUSION:

$$(1) \qquad \lambda(G) \geq \frac{4}{15}|G| \quad \text{for all finite abelian groups } G.$$

(c) For each sum-free set S in a group G the following inequality holds trivially: $|S|+|S+S| \leq |G|$. In order that there exist in G a sum-free set S with the required properties (i), (ii), a necessary condition is that

$$(2) \qquad \lambda(G)+\frac{\lambda(G)[\lambda(G)+1]}{2} \leq |G|.$$

Substitution herein of (1) yields the condition $|G| \leq 16$.

(d) We consider groups G of order ≤ 16. If $|G|$ is even then $G = C_{2^j} \oplus H$ for some positive integer j and abelian group H. As $S=\{1, 3, \ldots, 2^j-1\}$ is sum-free in C_{2^j} we have

$$\frac{\lambda(G)}{|G|} \geq \frac{\lambda(C_{2^j})}{|C_{2^j}|} \geq \frac{2^{j-1}}{2^j} = \frac{1}{2}.$$

Now it follows from (2) that $G = C_2$. Indeed in C_2 the set $S=\{1\}$ has the required properties. The (cyclic) groups of order 11 and 13 do not have a sum-free subset with the required properties since in both $\{-3, -1, 1, 3\}$ is a sum-free set of 4 elements while $4+\frac{1}{2}\cdot 4\cdot 5 > 13$. The (cyclic) group of 15 elements $C_{15} = C_3 \oplus C_5$ has a sum-free set of 6 elements since $\lambda(C_5)=2$. Now (2) is not fulfilled: $6+\frac{1}{2}\cdot 6\cdot 7 > 15$. The only groups left to consider are C_3, C_5, C_7, C_9, and $C_3 \oplus C_3$. Direct calculations show:

In C_3:	$\lambda(C_3) = 1$,	take $S = \{1\}$	then $S+S = \{2\}$;
in C_5:	$\lambda(C_5) = 2$,	take $S = \{1, 4\}$	then $S+S = \{0, 2, 3\}$;
in C_7:	$\lambda(C_7) = 2$,	take $S = \{1, 3\}$	then $S+S = \{2, 4, 6\}$.

So C_3, C_5 and C_7 satisfy the conditions.

$\lambda(C_9)=3$. If there exists in C_9 a sum-free set S with the required properties the set $S+S$ must contain 0 since $3+\frac{1}{2}\cdot 3\cdot 4=9$. Without loss of generality we may assume $\{1, 8\}\subset S$. But then $S=\{1, 3, 8\}$ or $S=\{1, 6, 8\}$, and in both cases $|S+S|=5$: contradiction.

Finally, $\lambda(C_3\oplus C_3)\geq 3$ since $\lambda(C_3)=1$ and again for the sought after sum-free set S we must have $0\in S+S$. Hence there exist two opposite elements b and $-b$ in S. But $b+b=-b$: contradiction.

CONCLUSION: The only groups in which there exists a sum-free set S with the required properties are C_2, C_3, C_5, and C_7.

Also solved by N. Felsinger.