ON THE PERTURBATION TECHNIQUE FOR THE INFLUENCE OF MAGNETIC FIELD ON STELLAR OSCILLATIONS

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ABSTRACT. The modification of the Rayleigh's principle is developed for the determination of the perturbations of the eigenfrequencies due to a weak magnetic field. The perturbational procedure is outlined, including its generalization for a slowly rotating magnetic star.

INTRODUCTION

The problem of adiabatic oscillations of a magnetic star has been of considerable interest in many years. The variational principle and the method of tensor virial equations have been constructed for the study of the simplest modes of oscillations (Kovetz I966; Goossens I979; and references therein). For a weak magnetic field, its effect on the oscillation frequencies can be determined with the use of the perturbation technique, which has been constructed by Ledoux and Simon (I957) and in much more developed form by Goossens (I972) and Goossens et al. (I976). In the present paper somewhat different approach is developed, which may become more tractable for certain magnetic configurations. It is based on the modification of Rayleigh's principle, which is customary to use in the theory of free oscillations of the Earth (e.g. Backus and Gilbert I967; Zharkov et al. I968; Woodhouse I976).

THE MODIFIED RAYLEIGH'S PRINCIPLE

We consider here the oscillations of a magnetic configuration with equilibrium field continuous across the surface S , with pressure and density vanishing on S , so that the equilibrium configuration is in a state of "true equilibrium" (Roberts I955). It is assumed that the internal magnetic field tends to force-free field near the surface.

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The usual form of the equation of adiabatic oscillations is

$$\rho_0\omega^2\vec{u} = \nabla p_1 + \rho_1\nabla\psi_0 + \rho_0\nabla\psi_1 - \frac{1}{4\pi}\left[(\nabla\times\vec{H}_0)\times\vec{H}_1 + (\nabla\times\vec{H}_1)\times\vec{H}_0\right] , \qquad (1)$$

where the common time-dependent factor $\exp(i\omega t)$ is omitted, $\vec{u}$ is displacement field, ψ is gravitational potential, and

$$p_1 = -\Gamma_1 p_0 \nabla\cdot\vec{u} - \vec{u}\cdot\nabla p_0 , \qquad (2)$$

$$\rho_1 = -\nabla\cdot(\rho_0\vec{u}) , \qquad (3)$$

$$\nabla^2\psi_1 = 4\pi G\rho_1 , \qquad (4)$$

$$\vec{H}_1 = \nabla\times(\vec{u}\times\vec{H}_0) . \qquad (5)$$

Subscript 0 denotes the equilibrium value of the corresponding quantity, subscript 1 - its Eulerian perturbation. In the exterior vacuum $\vec{H}_1$ is described by its vector potential

$$\vec{H}_1^V = \nabla\times\vec{A}^V , \quad \nabla\times(\nabla\times\vec{A}^V) = 0 , \quad \nabla\cdot\vec{A}^V = 0 , \qquad (6)$$

$$\hat{n}\times\vec{A}^V = \hat{n}\times(\vec{u}\times\vec{H}_0) \quad \text{on } S \qquad (7)$$

(e.g. Kovetz 1966), where $\hat{n}$ is the unit normal to the surface. Denoting by $[\,]^+_-$ the discontinuity of the enclosed quantity across S, the boundary conditions for the eigenvalue problem (1) are known to be

$$[\psi_1]^+_- = 0 \quad \text{and} \quad [\hat{n}\cdot\nabla\psi_1 + 4\pi G\rho_0\hat{n}\cdot\vec{u}]^+_- = 0 \quad \text{on } S , \qquad (8)$$

$$\vec{H}_0\cdot[\vec{H}_1 + (\vec{u}\cdot\nabla)\vec{H}_0]^+_- = 0 \text{ and } (\hat{n}\cdot\vec{H}_0)[\vec{H}_1 + (\vec{u}\cdot\nabla)\vec{H}_0]^+_- = 0 \text{ on } S . \qquad (9^a)$$

If $\vec{H}_0$ is zero, the boundary condition (9^a) is replaced by the regularity condition

$$\Gamma_1 p_0 \nabla\cdot\vec{u} = 0 \quad \text{on } S . \qquad (9^b)$$

After the scalar multiplication of eq. (1) by $\vec{u}^*$, where the asterisk denotes complex conjugate, and integration over the volume of the star, the somewhat lengthy manipulations lead to

$$\int_E L\,dv + \frac{1}{4\pi}\int_S \hat{n}\times(\vec{u}^*\times\vec{H}_0)\cdot\left[\vec{H}_1\right]^+_- ds = 0 , \qquad (10)$$

$$\begin{aligned} L = {} & \Gamma_1 p_0(\nabla\cdot\vec{u}^*)(\nabla\cdot\vec{u}) + \tfrac{1}{2}\big[\rho_0(\vec{u}\cdot\nabla)(\vec{u}^*\cdot\nabla\psi_0) + \rho_0(\vec{u}^*\cdot\nabla)(\vec{u}\cdot\nabla\psi_0) + \\ & + (\nabla\cdot\vec{u}^*)(\vec{u}\cdot\nabla p_0) + (\nabla\cdot\vec{u})(\vec{u}^*\cdot\nabla p_0)\big] + \rho_0(\vec{u}\cdot\nabla\psi_1^* + \vec{u}^*\cdot\nabla\psi_1) + \\ & + (1/4\pi G)\nabla\psi_1^*\cdot\nabla\psi_1 - \rho_0\omega^2\vec{u}^*\cdot\vec{u} + \\ & + (1/4\pi)\vec{H}_1^*\cdot\vec{H}_1 - (1/8\pi)\big[(\nabla\times\vec{H}_0)\times\vec{H}_1\cdot\vec{u}^* + (\nabla\times\vec{H}_0)\times\vec{H}_1^*\cdot\vec{u}\big] , \end{aligned} \qquad (11)$$

where E is all of space. The eq. (10) was obtained without using the boundary condition (9^a). If the eigenfunctions satisfy (9^a), the surface integral in (10) vanish. In the external vacuum p_0, ρ_0 and $\nabla\times\vec{H}_0$ are identically zero, so that all but two terms in the Lagrangian (11) vanish there. The Lagrangian (11) has the symmetric form and differ from those corresponding to the non-magnetic problem (e.g. Backus and Gilbert 1967) only in that the last line in (11) has appeared. It contains the terms corresponding to the work done against the Lorentz forces.

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Now we can wright the variational equation

$$\int_E L\,(\psi_j\,,\psi_{j,i}\,,\,p_k\,,\lambda\,)\;dv = 0\;. \tag{I2}$$

The Lagrangian is homogeneous function of "fields" ψ_j and their space derivatives $\partial\psi_j/\partial x_i$ denoted by $\psi_{j,i}$. By "fields" in stellar interior we mean the components of displacement and the perturbation in the gravitational potential. In exterior vacuum, by "fields" we denote the perturbation in the gravitational potential and the components of the magnetic vector potential. Equilibrium ρ_o, p_o, ψ_o, Γ_1, the components of $\vec{H}_o$ and their space derivatives are denoted by "parameters" p_k. The eigenvalue λ is ω^2.

The perturbation of the integral in (I2) due to the perturbation in the fields alone is (to first order)

$$-\int_E\Big[\frac{\partial}{\partial x_i}\frac{\partial L}{\partial\psi_{j,i}}-\frac{\partial L}{\partial\psi_j}\Big]\delta\psi_j\;dv\;-\;\int_S n_i\Big[\frac{\partial L}{\partial\psi_{j,i}}\,\delta\psi_j\Big]^+_-\;ds\;. \tag{I3}$$

The summation convention is used here and will be used later. This perturbation may be shown to be zero, if the fields (but not necessary their perturbations) satisfy the eqs. (I,6) and boundary conditions (8,9). The volume integral in (I3) is zero because the eqs. (I,6) are equivalent to Euler-Lagrange equations

$$\frac{\partial}{\partial x_i}\frac{\partial L}{\partial\psi_{j,i}}\;-\frac{\partial L}{\partial\psi_j} = 0\;,\qquad j = I,2,\ldots \tag{I4}$$

The surface integral in (I3) is zero due to the boundary conditions (8,9). The perturbation in "fields" is not restricted by boundary conditions, so that the equation

$$\int_E L\;dv = 0 \tag{I5}$$

provides the basis for "unrestricted" variational principle, similar to those proposed by Kovetz (I966).

Now let us perturb the stellar model from non-magnetic to slowly magnetized configuration. We are left with the perturbation of the integral in (I2) due to the perturbation in the parameters and the eigenvalue, so that

$$\delta\lambda\int_E\frac{\partial L}{\partial\lambda}\,dv = -\int_E\frac{\partial L}{\partial p_k}\,\delta p_k\;dv\;, \tag{I6}$$

from which the perturbation of the eigenfrequency may be calculated when the eigenfunctions of non-magnetic spherically symmetric star are known. It is the form of Rayleigh's principle, generalized to include the effect of modification of boundary conditions.

Eq. (I6) gives the perturbation of the eigenfrequency as the sum of volume integrals over the spherical volume occupied by the non-magnetic star and one integral with $\vec{H}_1^*\cdot\vec{H}_1$ over the external vacuum. The latter is redused to the surface integral and determined with the use of (7).

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When the perturbations in the parameters (δp_o , $\delta \rho_o$, $\delta \psi_o$, $\delta \nabla p_o$, $\vec{H}_o$, $\nabla \times \vec{H}_o$) are given in terms of scalar and vector spherical harmonics, the volume integrals may be reduced to radial integrals and angular integrals containing the triplets of spherical harmonics. These angular integrals may be computed with the use of Vigner's 3-j symbols (e.g. Luh 1973).

The first-order effects of rotation may be easily included into the perturbation technique. In the co-rotating coordinate system in which $\vec{H}_o$ is stationary, these effects will be given by the additional term $2\rho_o i\omega\Omega\,\hat{z}\times\vec{u}$ in the right-hand side of eq. (1). This term describes the Coriolis force, $\hat{z}$ is unit vector along the axis of rotation. The corresponding additional term in the Lagrangian is $2\rho_o\omega\Omega\,\vec{u}^*\cdot(i\hat{z}\times\vec{u})$ and results in the additional integral in the right-hand side of eq. (16), which is

$$-\int_{V_o} 2\rho_o\omega\Omega\,\vec{u}^*\cdot(i\hat{z}\times\vec{u})\,dv\ .$$

If the axial symmetry is violated, the true zero-order eigenfunctions become linear combinations of $2l+1$ eigenfunctions with different m values. The coefficients may be found from the requirement that $\delta\lambda$ must be stationary to their variations. This requirement leads to the linear system of algebraic equations. If this situation includes rotation (e.g. an oblique magnetic rotator), then the field $i\hat{z}\times\vec{u}$ should be decomposed on to vector spherical harmonics (Vorontsov and Zharkov 1981).

The perturbational approach presented in this paper is restricted to the study of the perturbation of the eigenfrequencies. Quite different methods should be used to determine the perturbation of the eigenfunctions (Biront et al. 1982; Roberts and Soward 1983).

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