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On a nonlinear differential-integral equation for ecological problems

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In this paper we consider a nonlinear differential-integral equation which arises in the problems of ecology and study the questions of convergent solutions and asymptotic equivalence of the solutions.

1. Introduction

Ecology is the study of various species in relation to their environment, competition for resources within and among the species, and predator-prey relations among them. At times the environment may be poisoned or polluted by metabolic actions of the species. In all these situations, since timerates of changes of population sizes are involved, it is natural that the mathematical modelling be given by differential equations, systems of differential equations or differential-integral equations. Problems of this type have gained increasing significance in recent years and many interesting results have been accumulated. The reader is referred to the works [3], [4], [5], [9], [10], [12], [13], for motivation and for reference to earlier literature. In animal population the accumulation of metabolic products may cause inconvenience to the whole population and may ultimately result in a fall of the birth rate while the death rate is increased. Volterra assumed that the total toxic effect on birth and death rates be expressed by the following differential-integral equation

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$$(1)\quad x'(t) = ax(t) - bx^2(t) - cx(t)\int_0^t K(t-s)x(s)ds\ ,\quad 0 \le s \le t < \infty\ ,$$

where $' = \frac{d}{dt}$, a, b , and c are constants. The kernel function K is due to the residual action and will be a decreasing function vanishing after a certain value.

Consider the differential system

$$(2)\qquad x'(t) = A(t)x(t) + (Gx)(t)\ ,$$

where $x \in R^n$, the euclidean n-space, A is an $n \times n$ matrix function, and G is a linear or non-linear operator. Note that equation (1) may be regarded as a special case of (2) with

$$A(t) \equiv a\ ,\quad (Gx)(t) \equiv -bx^2(t) - cx(t)\int_0^t K(t-s)x(s)ds\ .$$

In [10] equation (1) is discussed when $K(t) \equiv 1$. In many situations, for example, populations governed by a logistic equation

$$x'(t) = ax(t) - bx^2(t)$$

for $0 \le x \le a/b$, this growth rate decreases as population increases. Also equations of this type may arise in the study of problems of communicable diseases.

In studying the behaviour of solutions of (2) we are interested in the questions of existence and uniqueness of convergent solutions, whether (2) has solutions which converge to any *a priori* given vector. Also we present results on asymptotic behaviour of solutions for (2). The hypotheses we give here will be appropriate for many population problems, and also permit us to include a large variety of perturbations in (2).

2. Convergent solutions

In the following discussion J stands for the half real line $[0, \infty)$. Let C_L denote the space of all n-vector functions φ defined and continuous on J , and such that $\varphi(t)$ admits a finite limit as $t \to \infty$. It follows that C_L is a Banach space (see [1]), with the norm of $\varphi \in C_L$ given by

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$$\|\varphi\| = \sup_{t \in J} |\varphi(t)| ,$$

where $|\cdot|$ is the euclidean norm on R^n. The same symbol $|\cdot|$ will also be used to denote a convenient matrix norm. By $x(t, t_0, x_0)$, we mean a solution of the initial value problem associated with (2).

THEOREM 1. *Assume the following:*

(i) A *is an* $n \times n$ *matrix function defined and continuous on* J *and* $\int_0^\infty |A(s)|ds < \infty$ *;*

(ii) $G(0) \equiv 0$ *;*

(iii) there exists a function $g : J \to J$*, which is continuous on* J*, such that* $|(G\varphi_1)(t)-(G\varphi_2)(t)| \le g(t)|\varphi_1(t)-\varphi_2(t)|$ *and* $\int_0^\infty g(s)ds < \infty$.

Then all the solutions of (2) *belong to* C_L *. Further, given any* $\eta \in R^n$ *, there exists a unique solution* x *of* (2) *such that* $\lim_{t\to\infty} x(t) = \eta$.

Proof. A solution $x(t, t_0, x_0)$ of (2) is given by

$$x(t) = x_0 + \int_0^t A(s)x(s)ds + \int_0^t (Gx)(s)ds , \quad t \ge 0 .$$

In view of the hypotheses *(i)*-*(iii)*, it follows that

$$|x(t)| \le |x_0| + \int_0^t |A(s)||x(s)|ds + \int_0^t g(s)|x(s)|ds .$$

Applying the integral inequality of Bellman (see [2]),

$$|x(t)| \le |x_0|\exp\left(\int_0^t [|A(s)|+g(s)]ds\right) . \tag{3}$$

From *(i)*, *(ii)*, and *(iii)* we see that x is bounded, and by taking limits on both sides as $t \to \infty$, the convergence follows.

Now let $\eta \in R^n$ and K be such that $\|\eta\| < K$. *(i)* and *(ii)* imply

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that there exists a $T \geq 0$ such that

$$\|\eta\| + K \int_t^\infty |A(s)|ds + K \int_t^\infty g(s)ds \leq K \tag{4}$$

for all $t \geq T$. Set

$S_K = \{\varphi : \varphi$ is an n-vector function defined and continuous on $[T, \infty)$ and also such that $\sup_{t \geq T} |\varphi(t)| \leq K\}$.

Define an operator $T : S_K \to S_K$ by the relation

$$(T\varphi)(t) = \eta - \int_t^\infty A(s)\varphi(s)ds - \int_t^\infty (G\varphi)(s)ds . \tag{5}$$

From (4) and (5) it is easy to conclude that the operator T is a contraction and has a unique fixed point in S_K, for all $t \geq T$, and has all the requirements. By continuity the same can be extended to the whole of the interval J and the proof is complete.

REMARK 1. Theorem 1 is an existence and uniqueness type result for the solutions of (2) to be in C_L and convergent to any *a priori* given member of R^n. The following theorem extends Theorem 1 at the expense of giving up uniqueness.

THEOREM 2. *Let the condition (i) of Theorem* 1 *hold. Let* G *satisfy the following hypotheses: there exists a continuous function* $g : J \to J$ *such that* $|(G\varphi)t| \leq g(t)|\varphi(t)|$ *and* $\int_0^\infty g(s)ds < \infty$. *Then all the solutions of* (2) *belong to* C_L *and further, given any* $\eta \in R^n$, *there exists a solution* x *of* (2) *such that* $x(t)$ *converges to* η *as* $t \to \infty$.

3. Asymptotic behaviour

Throughout we denote by $BC = BC[0, \infty)$ the class of all bounded continuous n-vector valued functions defined on J. Let $LL_1(J)$ denote the class of all measurable functions which are integrable on compact subsets of J. It is easy to see that the initial value problem associated with (2) may be regarded as a perturbed system of the linear system

$$y'(t) = A(t)y(t) \ , \quad y(t_0) = x_0 \ . \tag{6}$$

Now we present an asymptotic equivalence type result for the solutions between the systems (2) and (6).

Concerning the perturbations G in (2) we assume that G is a continuous mapping of BC into $LL_1(J)$, and that there exists a measurable function λ with $\lambda(t) \geq 0$ and $\int_0^\infty \lambda(s)ds < \infty$, such that

$$|(G\varphi)(t)| \leq (1+|\varphi|)\lambda(t) \ , \text{ for } t \in J \ . \tag{7}$$

The hypothesis (7) allows us to include a variety of perturbations in G, for example,

$$(Gx)(t) = \int_0^t B(t, s)x^\sigma(s)ds \ ,$$

where B is a matrix and $\sigma > 0$.

Let Y be the fundamental matrix solution of (6). Then a solution $x(t, t_0, x_0)$ of (2) may be expressed as

$$x(t) = Y(t)x_0 + \int_0^t Y(t)Y^{-1}(s)(Gx)(s)ds \ . \tag{8}$$

THEOREM 3. *Suppose that* $A \in LL_1(J)$ *and that* Y *satisfies the following conditions:*

$$\int_0^t |Y(t)Y^{-1}(s)|ds < \infty \quad (0 \leq t < \infty) \ ,$$

and for each fixed $T > 0$,

$$\lim_{t\to\infty} \int_0^T |Y(t)Y^{-1}(s)|ds = 0 \ .$$

Let the perturbations G *satisfy* (7).

Then given a solution $y(t, t_0, x_0)$ *in* $BC[0, \infty)$ *of the system* (6),

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there exists a solution $x(t, t_0, x_0)$ *of the system* (2) *such that* $\lim_{t\to\infty} (x(t)-y(t)) = 0$; *and conversely.*

The proofs of Theorems 2 and 3 crucially depend on the Schauder-Tychonoff fixed point theorem. For the specific version of the theorem the reader is referred to [7]. Results parallel to our theorems may be found in [11], [13].

In this paper we have not discussed the stability and other qualitative behaviour of (2). Also it will be a matter of interest when the perturbations in (2) are impulsive. However these results are under study and will be reported elsewhere. Further, the authors strongly feel that the results of this exposition can be carried over to more general Banach spaces and dynamical systems without much difficulty.

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