NONNORMAL SMALL JUMP APPROXIMATION OF INFINITELY DIVISIBLE DISTRIBUTIONS

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Abstract

We study a type of nonnormal small jump approximation of infinitely divisible distributions. By incorporating compound Poisson, gamma, and normal distributions, the approximation has a higher order of cumulant matching than its normal counterpart, and, hence, in many cases a higher rate of approximation error decay as the cutoff for the jump size tends to 0. The parameters of the approximation are easy to fix, and its random sampling has the same order of computational complexity as the normal approximation. An error bound of the approximation in terms of the total variation distance is derived. Simulations empirically show that the nonnormal approximation can have a significantly smaller error than its normal counterpart.

Keywords: Infinitely divisible; normal approximation; compound Poisson approximation; gamma approximation; cumulant matching; sampling

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1. Introduction

The simulation of infinitely divisible (i.d.) random variables has many applications. In most cases, since exact simulation is either unavailable or computationally costly, good approximation methods are desired. The normal approximation of i.d. distributions, which was studied in [24] and later developed in [1] under the framework of the small jump approximation, has received much attention in the literature [2], [11], [12], [13], [17], [20], [22], [23], [33], [34].

Let X be a real-valued i.d. random variable, and let λ be its Lévy measure. Without loss of generality, we will always assume that X has no normal component. The normal (small jump) approximation starts with the decomposition $X = X_r + \Delta_r$ given $r > 0$, where X_r and Δ_r are independent and i.d. with Lévy measures $\lambda_r(\mathrm{d}x) = \mathbf{1}\{|x| < r\}\lambda(\mathrm{d}x)$ and $\lambda - \lambda_r$, respectively. As Δ_r is compound Poisson, its sampling is standard. The key is to approximate X_r by a normal random variable with the same mean and variance [1]. Thus, we can regard the approximation as relying on second-order cumulant matching. By a certain measure, the error of the approximation is bounded by

$$\frac{C|\kappa|_{3,X_r}}{\sigma_{X_r}^3}, \tag{1.1}$$

where C is a universal constant, σ_{X_r} is the standard deviation of X_r, and, for $j \geq 2$, $|\kappa|_{j,X_r} = \int |x|^j \lambda_r(\mathrm{d}x)$ [1]. Currently, the smallest available C seems to be 0.4748 [32]. In symmetric cases, since the third cumulants of X_r and the normal random variable are 0, the bound can be more or less replaced with $C|\kappa|_{4,X_r}/\sigma_{X_r}^4$ [1]. The pattern of the bound suggests that, if X_r

and some Y_r have the same cumulants of order $1, \ldots, q-1$ with $q \geq 5$, then X_r might be approximated by Y_r with the error being bounded by

$$\frac{C(r)(|\kappa|_{q,X_r} + |\kappa|_{q,Y_r})}{\sigma_{X_r}^q},$$

where $C(r)$ is nearly constant, at least for small r, or, most ideally, is a small universal constant. Since the qth cumulant of a normal random variable is 0, the bound is consistent with that for the normal approximation.

Even by rough Fourier analysis, there is good reason to expect the above bound to be true, and elementary calculations indicate that in many cases it vanishes at a strictly higher rate than the bound for the normal approximation as $r \to 0$. However, perhaps we first need to ask if an approximation based on higher-order cumulant matching can possibly be easily implemented. The motivation behind this question is twofold. First, the approximating distribution should be easy to identify and preferably i.d. Second, the approximating distribution should be easy to sample; preferably, the computational complexity of the sampling is of the same order as the normal approximation. If the answer to the question is positive then another question to ask is how large can q be. It can be anticipated that the larger q is, the faster the error of approximation vanishes as $r \to 0$. If both questions are answered then a wide range of available techniques can be potentially adapted to establish error bounds.

Clearly, cumulant matching is equivalent to moment matching. In fact, our proof of the above type of bound ultimately relies on moment matching. However, thanks to the Lévy–Khintchine representation, it is much more convenient to work on cumulants than moments. In Section 2.1 we present a simple way to construct approximating i.d. distributions with matching cumulants up to at least the fourth order, i.e. $q \geq 5$. In many important cases we obtain $q = 6$, and in symmetric cases, $q = 10$. Each approximating distribution is a convolution of compound Poisson and normal distributions, with the former made from gamma variables. Importantly, using standard algorithms [14], [21], the computational complexity of sampling for the approximation is universally bounded, and, hence, is of the same order as for the normal approximation.

We refer to the approximation as the Poisson-gamma-normal (PGN) approximation, although a term like 'compound Poisson-normal small jump approximation with gamma summands and higher-order cumulant matching' would be more accurate. In Section 2.2 we bound its error in terms of the total variation distance. The bound is nonasymptotic and of the desired type. In Section 2.3 we give some examples of the PGN approximation. The examples show that the bound yields a substantially higher rate of precision than the normal approximation as $r \to 0$. However, they also indicate that the bound may be far from being optimal or even practically useless. Therefore, in Section 3 we conduct large-scale simulations to show that, empirically, the PGN approximation can have a significantly smaller error than the normal approximation.

In Section 4 we prove the error bound by combining Fourier analysis, Lindeberg's method, and a device in [1]. Of course, on modern treatments of Poisson, compound Poisson, and normal approximations, there is now an extensive literature, and on gamma and other types of approximation, there is also a growing literature; see [3], [4], [7], [8], [18], [25], [26], [29], and the references therein. However, it appears that there has been little work on using convolutions of different types of simple distributions to improve approximation. We remark that while in this paper we are only concerned with the univariate case, accompanying results on the multivariate case have been reported elsewhere; see [10].

In the rest of this section we define notation and recall some basic facts. A Borel measure λ on $\mathbb{R}$ is the Lévy measure of an i.d. distribution if and only if $\lambda(\{0\}) = 0$ and $\int (u^2 \wedge 1)\,\lambda(\mathrm{d}u) < \infty$ [31, Theorem 8.1]. Denote by sppt(λ) the support of λ. If X is i.d. with Lévy measure λ, denote by $\Psi_X(t)$ and $\psi_X(t) = \exp\{-\Psi_X(t)\} = \mathbb{E}[\mathrm{e}^{\mathrm{i}tX}]$ its characteristic exponent and characteristic function, respectively, and define

$$\kappa_{j,X} = \frac{\mathrm{d}^j}{\mathrm{d}t^j} \ln \mathbb{E}[\mathrm{e}^{tX}]\bigg|_{t=0}, \qquad |\kappa|_{j,X} = \int |u|^j \lambda(\mathrm{d}u), \qquad j \in \mathbb{N},$$

and $\sigma_X = \kappa_{2,X}^{1/2}$. Here $\kappa_{j,X}$ is known as the jth cumulant of X. It is well defined if $\mathbb{E}[\mathrm{e}^{tX}] < \infty$ for all t in a neighborhood of 0. We refer to $|\kappa|_{j,X}$ as the jth absolute cumulant of X. For $a > 0$, $\mathbb{E}|X|^a < \infty$ if and only if $\int \mathbf{1}\{|u| > 1\}|u|^a \lambda(\mathrm{d}u) < \infty$ (see [31, Theorem 25.3 and Proposition 25.4]). If sppt(λ) is bounded then $\mathbb{E}[\mathrm{e}^{tX}] < \infty$ for all t [31, Theorem 25.17]. Since X has no normal component, $\kappa_{j,X} = \int u^j \lambda(\mathrm{d}u)$, $j > 1$, and var(X) = $\kappa_{2,X}$. If, in addition, $\Psi_X(t) = \int (1 + \mathrm{i}tu - \mathrm{e}^{\mathrm{i}tu}) \lambda(\mathrm{d}u)$ then $\kappa_{1,X} = \mathbb{E}X = 0$. If j is even or sppt(λ) $\subset \mathbb{R}_+ := [0, \infty)$, then $\kappa_{j,X} = |\kappa|_{j,X}$. Denote the total variation distance of X and the random variable Y by $d_{\mathrm{TV}}(X, Y) = \sup\{|\mathbb{P}\{X \in A\} - \mathbb{P}\{Y \in A\}|: A \text{ measurable}\}$ and their Kolmogorov–Smirnov (KS) distance by $d_{\mathrm{KS}}(X, Y) = \sup\{|\mathbb{P}\{X \le x\} - \mathbb{P}\{Y \le x\}|: x \in \mathbb{R}\}$.

Henceforth, we set X_r such that

$$\Psi_{X_r}(t) = \int (1 + \mathrm{i}tu - \mathrm{e}^{\mathrm{i}tu})\lambda_r(\mathrm{d}u).$$

Consequently, $\mathbb{E}X_r = 0$ and $\mathbb{E}|X_r|^p < \infty$ for all $p > 0$. Let $Z \sim N(0, 1)$ be independent of Δ_r. Then Δ_r and $\sigma_{X_r}Z + \Delta_r$ are known as the compound Poisson (CP) and normal approximations of X, respectively [1]. We refer to r as the cutoff for the jump size of approximations.

2. PGN approximation

2.1. Cumulant matching

In general, we can decompose $X = X_+ - X_- + X_s$, where $X_\pm$ and X_s are independent and i.d., with the Lévy measures of $X_\pm$ being supported on $\mathbb{R}_+$, and X_s being symmetric, i.e. $X_s \sim -X_s$. Indeed, the Lévy measure of X_s can be any symmetric Borel measure λ_s such that, for $A \subset \mathbb{R}_+$, $\lambda_s(A) \le \min(\lambda(A), \lambda(-A))$, and the Lévy measures $\lambda_\pm$ of $X_\pm$ are defined by $\lambda_\pm(A) = \lambda(\pm A) - \lambda_s(A)$. Although we can always set $\lambda_s = 0$, as seen below, it is useful to exploit $\lambda_s \ne 0$.

Without loss of generality, we will therefore only consider the asymmetric case where sppt(λ) $\subset \mathbb{R}_+$ and the symmetric case. First, let sppt(λ) $\subset \mathbb{R}_+$. Given $r > 0$ and $p \ge -1$, let Y_r be an i.d. random variable independent of Δ_r such that

$$\Psi_{Y_r}(t) = \int_0^\infty (1 + \mathrm{i}tu - \mathrm{e}^{\mathrm{i}tu})\gamma_r(\mathrm{d}u) \quad \text{with} \quad \gamma_r(\mathrm{d}u) = m(r)u^p \mathrm{e}^{-u/s(r)}\,\mathrm{d}u, \tag{2.1}$$

where $m(r) > 0$ and $s(r) > 0$ are constants to be determined. Then let

$$T_r = Y_r + \sigma(r)Z \quad \text{with } Z \sim N(0, 1) \text{ independent of } (Y_r, \Delta_r), \tag{2.2}$$

where $\sigma(r) > 0$ is a constant that also needs to be determined.

To use $T_r + \Delta_r$ to approximate X, first a comment on the sampling of T_r, which boils down to that of Y_r. Since $Y_r = U - \mathbb{E}U$, where $U \geq 0$ is i.d. with Lévy density $m(r)\,\mathbf{1}\{u > 0\}u^p e^{-u/s(r)}$ and $\mathbb{E}U = \Gamma(p+2)m(r)s(r)^{p+2}$, the sampling of Y_r is reduced to that of U. If $p = -1$ then $U \sim \mathrm{Gamma}(m(r), s(r))$, the gamma distribution with shape parameter $m(r)$ and scale parameter $s(r)$, whose sampling has universally bounded complexity (see [14, pp. 407–420]). If $p > -1$ then $U \sim \sum_{i=1}^{N} \xi_i$, where $N \sim \mathrm{Poisson}(a)$ with $a = \int_0^\infty m(r)u^p e^{-u/s(r)}\,\mathrm{d}u = \Gamma(p+1)m(r)s(r)^{p+1}$, and ξ_i are independent and identically distributed (i.i.d.) $\mathrm{Gamma}(p+1, s(r))$ random variables independent of N. The sampling of $\mathrm{Poisson}(a)$ has universally bounded complexity (see [15] or [21, pp. 228–241]). Then because, conditional on N, $U \sim \mathrm{Gamma}(N(p+1), s(r))$, the sampling of U, and, hence, that of T_r, has the same order of complexity as the sampling of $N(0, 1)$.

Owing to its Lévy–Khintchine representation, we refer to $T_r + \Delta_r$ as the PGN approximation of X with cutoff r. To match the cumulants of X_r and T_r, note that $\mathbb{E}T_r = \mathbb{E}Y_r = 0$ and

$$\kappa_{j,Y_r} = \Gamma(j+p+1)m(r)s(r)^{j+p+1}, \qquad \kappa_{j,T_r} = \kappa_{j,Y_r} + \mathbf{1}\{j=2\}\sigma(r)^2, \qquad j \geq 2. \quad (2.3)$$

We next show two results. The first result allows $r = \infty$ and, hence, applies to any i.d. random variable with finite fourth cumulant, subject to a mild constraint. However, it only attains fourth-order matching. The second result asserts that we can obtain fifth-order cumulant matching provided there exists a slowly varying Lévy density at $0+$.

Proposition 2.1. (Fourth-order cumulant matching.) *Let $0 < r \leq \infty$ and $0 < \kappa_{4,X_r} < \infty$. Suppose that λ_r is not concentrated on a single point. Then*

$$\frac{p+4}{p+3} < \frac{\kappa_{2,X_r}\kappa_{4,X_r}}{\kappa_{3,X_r}^2} \quad \text{for all large } p. \quad (2.4)$$

For any $p \geq -1$ satisfying (2.4), if

$$s(r) = \frac{\kappa_{4,X_r}}{(p+4)\kappa_{3,X_r}}, \qquad m(r) = \frac{\kappa_{3,X_r}}{\Gamma(p+4)s(r)^{p+4}}, \quad (2.5)$$

and if Y_r is defined by (2.1), then $\kappa_{2,X_r} > \kappa_{2,Y_r}$, and, by setting

$$\sigma(r) = (\kappa_{2,X_r} - \kappa_{2,Y_r})^{1/2}, \quad (2.6)$$

$\kappa_{j,X_r} = \kappa_{j,T_r}$ *for* $2 \leq j \leq 4$.

Remark. If $\lambda(\mathbb{R}_+) = \infty$ then, for any $r > 0$, $\lambda_r \neq 0$ and is not concentrated on a single point.

Proof of Proposition 2.1. Since λ_r is not concentrated on a single point and $0 < \kappa_{4,X_r} < \infty$, by Hölder's inequality, $0 < \kappa_{3,X_r}^2 < \kappa_{2,X_r}\kappa_{4,X_r} < \infty$, which implies (2.4). From (2.3), by setting $s(r)$ and $m(r)$ as in (2.5), $\kappa_{j,X_r} = \kappa_{j,Y_r}$ for $j = 3, 4$ and

$$\kappa_{2,Y_r} = \Gamma(p+3)m(r)s(r)^{p+3} = \Gamma(p+3)\frac{\kappa_{3,X_r}}{\Gamma(p+4)s(r)} = \frac{(p+4)\kappa_{3,X_r}^2}{(p+3)\kappa_{4,X_r}}.$$

Then, for $p \geq -1$ satisfying (2.4), $\kappa_{2,Y_r} < \kappa_{2,X_r}$. The rest of the result is then clear.

Proposition 2.2. (Fifth-order cumulant matching.) *Let* $\lambda(\mathrm{d}u) = \mathbf{1}\{u > 0\}u^{-\alpha-1}\ell(u)\,\mathrm{d}u$, *with* $\alpha \in (0, 2)$ *and* $\ell(u)$ *slowly varying at* $0+$. *Let* $p = p(r)$ *be defined by the equation*

$$\frac{1}{p+4} = \frac{\kappa_{3,X_r}\kappa_{5,X_r}}{\kappa_{4,X_r}^2} - 1.$$

Then, for all small $r > 0$, $p > -1$ *and satisfies* (2.4). *For any* $r > 0$ *with such* p, *set* $s(r)$ *and* $m(r)$ *as in* (2.5) *and* $\sigma(r)$ *by* (2.6). *Then* $\kappa_{j,X_r} = \kappa_{j,T_r}$, $2 \leq j \leq 5$.

    Proof. For $j \geq 3$, as $r \to 0+$, $\kappa_{j,X_r} = \int_0^r u^{j-1-\alpha}\ell(u)\,\mathrm{d}u \sim r^{j-\alpha}\ell(r)/(j-\alpha)$ [5, Theorem 1.5.11], so

$$\frac{1}{p+4} \sim \frac{(4-\alpha)^2}{(3-\alpha)(5-\alpha)} - 1 = \frac{1}{(3-\alpha)(5-\alpha)} \quad \Longrightarrow \quad p \sim \alpha^2 - 8\alpha + 11 > -1.$$

As a result, for all small $r > 0$, $p > -1$. Furthermore, as

$$\frac{\kappa_{2,X_r}\kappa_{4,X_r}}{\kappa_{3,X_r}^2} \sim \frac{(3-\alpha)^2}{(2-\alpha)(4-\alpha)} = 1 + \frac{1}{\alpha^2 - 6\alpha + 8},$$

combining with the previous display, it is not hard to obtain (2.4). By Proposition 2.1, it only remains to show that, given $r > 0$ such that $p > -1$ and satisfies (2.4), $\kappa_{5,X_r} = \kappa_{5,Y_r}$. However, this follows from $\kappa_{3,X_r}\kappa_{5,X_r}/\kappa_{4,X_r}^2 = (p+5)/(p+4) = \kappa_{3,Y_r}\kappa_{5,Y_r}/\kappa_{4,Y_r}^2$, where the second equality is due to (2.3). $\quad\blacksquare$

    Now consider the symmetric case. Let $X = X^{(1)} - X^{(2)}$, where $X^{(i)}$ are i.i.d. with Lévy measure λ supported in $\mathbb{R}_+$. Let $X_r = X_r^{(1)} - X_r^{(2)}$, and approximate it by $T_r = T_r^{(1)} - T_r^{(2)}$, where $T_r^{(i)}$ are i.i.d. defined in (2.2). Since all the odd-ordered cumulants of X_r and T_r are 0, it suffices to match their even-ordered cumulants. The next result asserts that in general one can match their cumulants up to the seventh order, and in some important cases up to the ninth order.

Proposition 2.3. (Symmetric case.) (i) *Let* $0 < r \leq \infty$ *and* $0 < \kappa_{4,X_r} < \infty$. *Suppose that* λ_r *is not concentrated on a single point. Then*

$$\frac{(p+5)(p+6)}{(p+3)(p+4)} < \frac{\kappa_{2,X_r}\kappa_{6,X_r}}{\kappa_{4,X_r}^2} \quad \text{for all large } p. \tag{2.7}$$

For any $p \geq -1$ *satisfying* (2.7), *if*

$$s(r) = \sqrt{\frac{\kappa_{6,X_r}}{(p+5)(p+6)\kappa_{4,X_r}}}, \qquad m(r) = \frac{\kappa_{4,X_r}}{2\Gamma(p+5)s(r)^{p+5}}, \tag{2.8}$$

and $Y_r = Y_r^{(1)} - Y_r^{(2)}$ *with* $Y_r^{(i)}$ *i.i.d. defined by* (2.1), *then* $\kappa_{2,X_r} > \kappa_{2,Y_r}$ *and, by setting* $\sigma(r)$ *as in* (2.6), $\kappa_{j,X_r} = \kappa_{j,T_r}$ *for* $2 \leq j \leq 7$.

(ii) *Let* $\lambda(\mathrm{d}u) = \mathbf{1}\{u > 0\}u^{-\alpha-1}\ell(u)\,\mathrm{d}u$, *with* $\alpha \in (0, 2)$ *and* $\ell(u)$ *slowly varying at* $0+$. *Then, for all small* $r > 0$, *there is a unique* $p = p(r) > 0$ *satisfying* (2.7) *and*

$$\frac{(p+7)(p+8)}{(p+5)(p+6)} = \frac{\kappa_{4,X_r}\kappa_{8,X_r}}{\kappa_{6,X_r}^2}. \tag{2.9}$$

Given $r > 0$ *with such* p, *set* $s(r)$ *and* $m(r)$ *as in* (2.8) *and* $\sigma(r)$ *as in* (2.6). *Then* $\kappa_{j,X_r} = \kappa_{j,T_r}$, $2 \leq j \leq 9$.

Proof. (i) By the assumption and Hölder's inequality, $0 < \kappa_{4,X_r}^2 < \kappa_{2,X_r}\kappa_{6,X_r} < \infty$, so (2.7) holds for all large p. Since, for even j, $\kappa_{j,Y_r} = 2\kappa_{j,Y_r^{(1)}} = 2\Gamma(j+p+1)m(r)s(r)^{j+p+1}$, it is easy to see that $\kappa_{4,X_r} = \kappa_{4,Y_r}$ and $\kappa_{6,X_r} = \kappa_{6,Y_r}$. On the other hand, for all odd j, $\kappa_{j,X_r} = \kappa_{j,Y_r} = 0$. Finally, by a similar argument as used in the proof of Proposition 2.1, $\kappa_{2,Y_r} < \kappa_{2,X_r}$, leading to $\kappa_{j,X_r} = \kappa_{j,T_r}$ for $2 \leq j \leq 7$.

(ii) Following the proof of Proposition 2.2, as $r \to 0+$,

$$\frac{\kappa_{4,X_r}\kappa_{8,X_r}}{\kappa_{6,X_r}^2} \sim \frac{(6-\alpha)^2}{(4-\alpha)(8-\alpha)} = 1 + \frac{4}{(4-\alpha)(8-\alpha)} := h(\alpha).$$

The function h is strictly increasing on $(0,2)$. On the other hand,

$$g(p) := \frac{(p+7)(p+8)}{(p+5)(p+6)} = \left(1 + \frac{2}{p+5}\right)\left(1 + \frac{2}{p+6}\right)$$

is strictly decreasing on $(-1,\infty)$, with $g(0) > h(2) > h(\alpha) > h(0) > 1 = g(\infty)$. Therefore, there is a unique $p > 0$ satisfying (2.9). We have to show that, for this $p = p(r)$, (2.7) is satisfied for all small $r > 0$. By continuity, it suffices to show, that for $p > 0$,

$$\frac{(p+7)(p+8)}{(p+5)(p+6)} = \frac{(6-\alpha)^2}{(4-\alpha)(8-\alpha)} \quad \Longrightarrow \quad \frac{(p+5)(p+6)}{(p+3)(p+4)} < \frac{(4-\alpha)^2}{(2-\alpha)(6-\alpha)}.$$

By calculation, the equality is equivalent to $2p^2 = 2p(\alpha^2 - 12\alpha + 21) + 13\alpha^2 - 156\alpha + 356$, while the inequality is equivalent to $2p^2 > 2p(\alpha^2 - 8\alpha + 5) + 9\alpha^2 - 72\alpha + 84$. Then, as $p > 0$ and $0 < \alpha < 2$, it is not hard to see that the implication holds. The rest of the proof then follows the proof of (i).

Propositions 2.2 and 2.3 immediately lead to the next result on i.d. distributions with truncated stable Lévy measures. It should be pointed out that simple exact sampling methods for stable distributions are well known [14] and i.d. distributions with truncated stable Lévy measures with $\alpha \in (0,1)$ can be sampled exactly but with high computational complexity [9].

Corollary 2.1. *Let* $\lambda(\mathrm{d}u) = c\,\mathbf{1}\{0 < u < r_0\}u^{-\alpha-1}\,\mathrm{d}u$, *where* $c > 0$, $0 < r_0 \leq \infty$, *and* $\alpha \in (0,2)$.

(i) *Let* X *have Lévy measure* λ. *If* $p = \alpha^2 - 8\alpha + 11$ *then* $p > -1$ *and, for all* $0 < r < r_0$, *by setting* $(s(r), m(r))$ *as in* (2.5), $\kappa_{2,X_r} > \kappa_{2,Y_r}$, *and, by setting* $\sigma(r)$ *as in* (2.6), $\kappa_{j,X_r} = \kappa_{j,T_r}$ *for* $2 \leq j \leq 5$.

(ii) *Let* $X = X^{(1)} - X^{(2)}$, *with* $X^{(i)}$ *i.i.d. with Lévy measure* λ. *If* p *is the (unique) solution to*

$$\frac{(p+7)(p+8)}{(p+5)(p+6)} = \frac{(6-\alpha)^2}{(4-\alpha)(8-\alpha)}, \qquad p \in (0,\infty),$$

then, for all $0 < r \leq r_0$, *by setting* $(s(r), m(r))$ *as in* (2.8), $\kappa_{2,X_r} > \kappa_{2,Y_r}$, *and, by setting* $\sigma(r)$ *as in* (2.6), $\kappa_{j,X_r} = \kappa_{j,T_r}$ *for* $2 \leq j \leq 9$.

2.2. Error bound for the PGN approximation

Define $C_0 = (\sin 1)^2/2 = 0.354\ldots$ Observe that, as $|\kappa|_{j+1,X_r} \leq r|\kappa|_{j,X_r}$, $j \geq 2$, for $s(r)$ defined in (2.5) or (2.8), $s(r) < r/(p+3)$. The main result is the following.

Theorem 2.1. *Fix $r \in (0, \infty)$ and $q \geq 5$. Let T_r be defined by (2.1)–(2.2) in the asymmetric case, and let $T_r = T_r^{(1)} - T_r^{(2)}$ in the symmetric case, where the $T_r^{(i)}$ are i.i.d. defined by (2.1)–(2.2). Suppose that $s(r) < r/(p+3)$ and $\sigma(r) > 0$. For $j \geq 1$, define*

$$Q_j(r) = \left[\frac{\Gamma(j + 1/2)}{2C_0^{j+1/2}} + \sigma_{X_r}^{2j+1} \int_{1/r}^{\infty} t^{2j} e^{-2L(t,r)} \, dt \right]^{1/2},$$

where $L(t, r) = t^2 \min\{C_0 \kappa_{2, X_{1/|t|}}, \sigma(r)^2/2\}$. Suppose that $\kappa_{j, X_r} = \kappa_{j, T_r}$ for $2 \leq j < q$. Then

$$d_{\mathrm{TV}}(X, T_r + \Delta_r) \leq \frac{|\kappa|_{q, X_r} + |\kappa|_{q, Y_r}}{q! \, \sigma_{X_r}^q} [q \, Q_{q-1}(r) + Q_q(r) + Q_{q+1}(r)]. \qquad (2.10)$$

Remarks. (i) The bound in (2.10) is on d_{TV} instead of the more commonly used d_{KS} [1], [24]. However, we have not been able to derive a Berry–Esseen type of bound $C(|\kappa|_{q, X_r} + |\kappa|_{q, Y_r})/\sigma_{X_r}^q$, with C a universal constant depending only on q. It appears that some key ingredients for the proof of the Berry–Esseen bound for the normal approximation cannot hold for higher-order approximations. Also, as seen later, the bound sometimes is quite suboptimal.

(ii) The bound will be proved by combining Fourier analysis, the Lindeberg method (cf. [6] for a modern application of it), and a device in [1] (cf. the proof of Theorem 25.18 of [31]). Although a bound on d_{KS} may be established solely based on Fourier analysis [24], our proof seems to be more transparent and suitable for generalization.

(iii) As seen later, in order for the right-hand side (RHS) of (2.10) to be finite, X_r must have a density, in particular, $\lambda(\mathbb{R}) = \infty$. The last condition implies that X is not compound Poisson and has no atom [31, Theorem 27.4]. It also excludes lattice distributions, to which the Poisson–Charlier approximation applies [3], [24]. Although the condition $\lambda(\mathbb{R}) = \infty$ is necessary for X to have a valid normal small jump approximation, it is not sufficient, which is a fact with several interesting and important implications [1], [12].

To evaluate the RHS of (2.10), we need to evaluate $\sigma(r)^2$ and $|\kappa|_{q, Y_r}$. Both can be expressed in terms of the cumulants of X_r. For the asymmetric case, the proof of Proposition 2.1 shows that $\kappa_{2, Y_r} = (p+4)\kappa_{3, X_r}^2/[(p+3)\kappa_{4, X_r}]$. Then, by (2.6),

$$\sigma(r)^2 = \kappa_{2, X_r} - \frac{(p+4)\kappa_{3, X_r}^2}{(p+3)\kappa_{4, X_r}}.$$

Similarly, for the symmetric case, using (2.3) and (2.8),

$$\sigma(r)^2 = \kappa_{2, X_r} - \frac{(p+5)(p+6)\kappa_{4, X_r}^2}{(p+3)(p+4)\kappa_{6, X_r}}.$$

Furthermore, if $\kappa_{j, X_r} = \kappa_{j, T_r}$ for $2 \leq j < q$ then $|\kappa|_{q, Y_r}$ can also be expressed in terms of the absolute cumulants of X_r. In the asymmetric case, by (2.3),

$$|\kappa|_{q, Y_r} = \kappa_{q, Y_r} = (q+p)s(r)\kappa_{q-1, Y_r} = (q+p)s(r)\kappa_{q-1, X_r}.$$

Similarly, in the symmetric case, if q is even then

$$|\kappa|_{q, Y_r} = \kappa_{q, Y_r} = (q+p)(q+p-1)s(r)^2\kappa_{q-2, Y_r} = (q+p)(q+p-1)s(r)^2\kappa_{q-2, X_r}.$$

In bound (2.10), the $Q_j(r)$ look rather technical. The next result gives their asymptotics as $r \to 0+$.

Proposition 2.4. *For* $b \in (0, 1)$ *and* $q \geq 3$, *there is* $M = M(b, q) > 0$ *such that if*

$$\limsup_{s \to 0} \frac{\kappa_{2, Y_s}}{\kappa_{2, X_s}} < b \quad and \quad \liminf_{s \to 0} \frac{\kappa_{2, X_s}}{s^2 \ln(1/s)} > M, \tag{2.11}$$

then, for any $2 \leq j \leq q + 1$,

$$Q_j(r)^2 = \frac{\Gamma(j + 1/2)}{2C_0^{j+1/2}} + o(1) \quad as \ r \to 0.$$

Here is a short proof. By (2.11), for small $r > 0$, $\sigma(r)^2 = \kappa_{2, X_r} - \kappa_{2, Y_r} > (1 - b)\kappa_{2, X_r}$. Then, from the increasing monotonicity of κ_{2, X_r} in r, there is a constant $c = c(b) > 0$ such that, for $t \geq 1/r$, $L(t, r) \geq ct^2 \kappa_{2, X_{1/t}}$. Consequently, if $M \geq (q + 2)/c$ then, by (2.11), for $t \geq 1/r$, $L(t, r) \geq Mc \ln t \geq (q + 2) \ln t$, and, hence, for all $2 \leq j \leq q + 1$,

$$\int_{1/r}^{\infty} t^{2j} e^{-2L(t, r)} \, dt \leq \int_{1/r}^{\infty} t^{2(q+1) - 2Mc} \, dt = o(1) \quad as \ r \to 0.$$

Since $\sigma_{X_r} = o(1)$ as $r \to 0$, the proof is complete.

2.3. Examples

Example 2.1. (*Truncated stable Lévy measure.*) Let $\lambda(du) = c \, \mathbf{1}\{0 < u < r_0\} u^{-\alpha - 1} \, du$ with $c > 0$, $0 < r_0 \leq \infty$, and $\alpha \in (0, 2)$. By Corollary 2.1, given $r \in (0, r_0)$, if $p = \alpha^2 - 8\alpha + 11$, and $s(r)$, $m(r)$, and $\sigma(r)$ are set as in (2.5)–(2.6), then $\kappa_{j, X_r} = \kappa_{j, T_r}$ for $2 \leq j < q = 6$. To apply (2.10), we need to know κ_{2, X_r}, κ_{6, X_r}, and κ_{6, Y_r}. For $j \geq 2$, $\kappa_{j, X_r} = cr^{j-\alpha}/(j - \alpha)$. Then

$$s(r) = \frac{\kappa_{4, X_r}}{(p + 4)\kappa_{3, X_r}} = \frac{(3 - \alpha)r}{(p + 4)(4 - \alpha)} = \frac{r}{(4 - \alpha)(5 - \alpha)},$$

$$\kappa_{2, Y_r} = \frac{\kappa_{3, Y_r}}{(p + 3)s(r)} = \frac{\kappa_{3, X_r}}{(p + 3)s(r)} = \frac{c(4 - \alpha)(5 - \alpha)r^{2-\alpha}}{(3 - \alpha)(\alpha^2 - 8\alpha + 14)},$$

and $\kappa_{6, Y_r} = (6 + p)s(r)\kappa_{5, Y_r} = (6 + p)s(r)\kappa_{5, X_r} = cA(\alpha)r^{6-\alpha}$, with $A(\alpha) = (\alpha^2 - 8\alpha + 17)/(4 - \alpha)(5 - \alpha)^2$. Therefore, by Theorem 2.1,

$$d_{\text{TV}}(X, T_r + \Delta_r) \leq (2 - \alpha)^3 \left[\frac{1}{6 - \alpha} + A(\alpha)\right] \frac{6Q_5(r) + Q_6(r) + Q_7(r)}{6!} \left(\frac{r^\alpha}{c}\right)^2. \tag{2.12}$$

Since $0 < \kappa_{2, Y_r}/\kappa_{2, X_r} < 1$ is a constant independent of r, and λ satisfies Orey's condition, $\liminf_{s \to 0} \kappa_{2, X_s}/s^{2-\alpha} > 0$ (see [28] and also [31, Proposition 28.3]), the conditions in (2.11) are satisfied no matter the value of M. Then, by Proposition 2.4, $d_{\text{TV}}(X, T_r + \Delta_r) = O(r^{2\alpha})$. This may be compared to the normal approximation in [1] and [24], where d_{KS} between X and its normal approximation is $O(r^{\alpha/2})$ when X is asymmetric. Specifically, by (1.1),

$$d_{\text{KS}}(X, \sigma_{X_r} Z + \Delta_r) \leq \frac{C(2 - \alpha)^{3/2}}{(3 - \alpha)} \left(\frac{r^\alpha}{c}\right)^{1/2}, \qquad C = 0.4748. \tag{2.13}$$

Furthermore, if $X = X^{(1)} - X^{(2)}$ is symmetric, where the $X^{(i)}$ are i.i.d. with Lévy measure λ, then by Corollary 2.1(ii), it can be seen that we can set $q = 10$ and get $d_{\text{TV}}(X, T_r + \Delta_r) = O(r^{4\alpha})$, whereas the d_{KS} between X and its normal approximation in this case is $O(r^\alpha)$ [1].
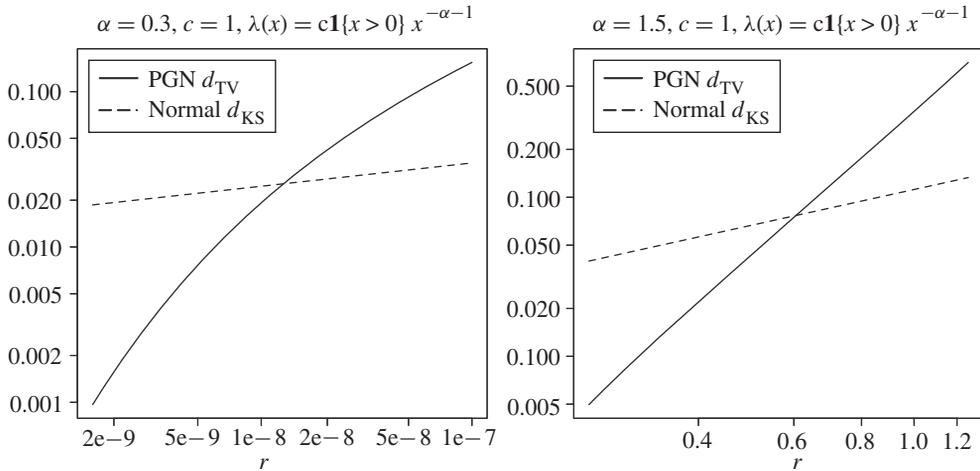
FIGURE 1: The bounds in (2.12) and (2.13) as functions of r, for $\alpha = 0.3$ and 1.5, respectively, with $\lambda(\mathrm{d}x) = \mathbf{1}\{u > 0\}u^{-\alpha-1}\,\mathrm{d}x$. The axes are plotted using a base-10 logarithmic scale.

Although the bound in (2.12) vanishes at a higher rate than that in (2.13) as $r \to 0$, the asymptotic result says little about how the bounds compare if r is not too small. This is especially the case when $\alpha < 1$. In Figure 1, for $c = 1$ and $r_0 = \infty$, the bounds are plotted as functions of r. The bound in (2.12) is evaluated numerically; see Appendix A for details. As can be seen from the plots, for $\alpha = 1.5$, the bound in (2.12) is smaller than that in (2.13) once $r < 0.6$, whereas, for $\alpha = 0.3$, this happens only if $r < 2 \times 10^{-8}$. Therefore, (2.12) may provide little evidence on whether the PGN approximation is better than the normal approximation in practice. To address this issue, we resort to numerical simulation in the next section.

Example 2.2. (*Tempered stable.*) Let $\lambda(\mathrm{d}u) = \mathbf{1}\{u > 0\}u^{-\alpha-1}\exp\{-u^\theta\}\,\mathrm{d}u$, where $\alpha \in (0, 2)$ and $\theta > 0$. Then, for $j \geq 2$,

$$\kappa_{j,X_r} = \int_0^r u^{j-\alpha-1}\exp\{-u^\theta\}\,\mathrm{d}u = \frac{1}{\theta}\int_0^{r^\theta} u^{(j-\alpha)/\theta-1}\mathrm{e}^{-u}\,\mathrm{d}u.$$

The last integral can be numerically evaluated as an incomplete gamma function [27]. However, it has no closed-form formulae. The following method can be used if we want to avoid this problem. Recall that, for any odd $n \geq 1$, $\mathrm{e}^{-u} \geq f_n(u)$ for $u \geq 0$, where $f_n(u) = \sum_{i=0}^n (-u)^i/i!$. Let $n = 2\lfloor\alpha/(2\theta)\rfloor + 1$, which is the smallest odd number greater than $\alpha/\theta - 1$. Let $F(u) = \mathbf{1}\{0 < u < r_0\}f_n(u^\theta)$, with $r_0 = \sup\{r > 0: f_n(u) > 0 \text{ for all } 0 \leq u < r^\theta\}$. Decompose $\lambda = \lambda_1 + \lambda_2$, where $\lambda_1(\mathrm{d}u) = \mathbf{1}\{u > 0\}u^{-\alpha-1}F(u)\,\mathrm{d}u$. It is easy to evaluate $\int_0^r u^j \lambda_1(\mathrm{d}u)$. Then we can apply the PGN approximation to λ_1, with all parameters set in closed form. Meanwhile, since $u^{-\alpha-1}[\exp\{-u^\theta\} - F(u)] = O(u^{(n+1)\theta-\alpha-1})$ as $u \to 0+$, λ_2 has finite mass, and, hence, corresponds to a compound Poisson random variable that can be sampled exactly. If X, X', and X'' denote i.d. random variables with Lévy measures λ, λ_1, and λ_2, respectively, and Δ_r and T_r are the i.d. random variables from the PGN approximation to X', then, by Proposition 2.2, $d_{\mathrm{TV}}(X, T_r + \Delta_r + X'') \leq d_{\mathrm{TV}}(X', T_r + \Delta_r) = O(r^{2\alpha})$.

Example 2.3. Let $\lambda(\mathrm{d}u) = c\,\mathbf{1}\{0 < u < 1\}u^{-1}\ln(1/u)\,\mathrm{d}u$. Since

$$\int_{u<r} u^2\,\lambda(\mathrm{d}u) = c\int_0^r u\ln\left(\frac{1}{u}\right)\,\mathrm{d}u = \frac{cr^2[2\ln(1/r) + 1]}{4},$$

by Proposition 2.1 of [1], the normal approximation is valid in the sense that its error in terms of d_{KS} tends to 0 as $r \to 0$. However, since, for $|t| \gg 1$, $L(t, r) = C_0 t^2 \int_{u < 1/|t|} u^2 \lambda(\mathrm{d}u) \sim c C_0 \ln |t|$, it can be seen that the RHS of (2.10) is finite only when c is large enough, and even in that case the RHS of (2.10) decreases to 0 very slowly as $r \to 0$.

3. A numerical study

As seen in Section 2.3, the bound (2.10) is sometimes a poor indicator of the precision of the PGN approximation in practice. To address this issue, we conducted a simulation study to empirically compare the errors of the PGN and normal approximations in terms of the KS distance. We exclusively considered the approximations to stable distributions for two reasons. First, besides gamma distributions, they are the only class of nonnormal i.d. distributions whose Lévy measures and distribution functions are both known (semi-)explicitly. Second, unlike gamma distributions, they have valid normal approximations [1]. The simulations were implemented in the R language (see http://www.R-project.org), with its `stabledist` package used for all computations involving stable distributions.

The Lévy measure of a stable distribution with exponent $\alpha \in (0, 2)$ was parametrized as

$$\lambda(\mathrm{d}u) = [M_+ \mathbf{1}\{u > 0\} + M_- \mathbf{1}\{u < 0\}]|u|^{-1-\alpha}\,\mathrm{d}u,$$

where $M_\pm \geq 0$ such that $M := M_+ + M_- > 0$. For simplicity, the stable distribution to be approximated ('target distribution') was centered in the sense that

$$\Psi_X(t) = \begin{cases} \gamma^\alpha |t|^\alpha \left[1 - \mathrm{i}\,\mathrm{sgn}(t)\,\beta \tan\left(\dfrac{\pi\alpha}{2}\right) \right], & \alpha \neq 1, \\[2ex] \gamma |t| \left[1 + \mathrm{i}\,\mathrm{sgn}(t)\left(\dfrac{2}{\pi}\right)\beta \ln |t| \right], & \alpha = 1, \end{cases}$$

where $\gamma > 0$ and $\beta \in [-1, 1]$ (see [30, p. 5]). Then, by [16, pp. 568–570],

$$\Psi_X(t) = \int \phi(t, u)\lambda(\mathrm{d}u) \quad \text{with} \quad \phi(t, u) = \begin{cases} \mathrm{e}^{\mathrm{i}tu} - 1, & 0 < \alpha < 1, \\ \mathrm{e}^{\mathrm{i}tu} - 1 - \mathrm{i}tu, & 1 < \alpha < 2, \\ \mathrm{e}^{\mathrm{i}tu} - 1 - \mathrm{i}t \sin u, & \alpha = 1, \end{cases}$$

and

$$\gamma = \left[\frac{M\pi}{2\Gamma(\alpha + 1)\sin(\pi\alpha/2)} \right]^{1/\alpha}, \qquad \beta = \frac{M_+ - M_-}{M}.$$

Given $r > 0$, Example 2.1 provides all the parameters for the PGN and normal approximations to X_r. The PGN approximation was sampled according to the description following (2.2). On the other hand, Δ_r was sampled by the representation

$$\Delta_r \sim d(r) + r \sum_{i=1}^{N} \varepsilon_i U_i^{-1/\alpha},$$

where $d(r)$ is a constant, and N, $\varepsilon_i \in \{\pm 1\}$, and $U_i \sim \mathrm{Unif}(0, 1)$, $i = 1, 2, \ldots$, are mutually independent, with N being Poisson distributed with mean $\int_{|u| \geq r} \lambda(\mathrm{d}u) = M\alpha^{-1}r^{-\alpha}$ and $\mathbb{P}\{\varepsilon_i = 1\} = M_+/M$. To determine $d(r)$, use $\Psi_X(t) = \Psi_{X_r}(t) + \Psi_{\Delta_r}(t)$. It follows that

$$\mathrm{i}d(r)t = \int \phi(t, u)\lambda(\mathrm{d}u) - \int_{-r}^{r} (\mathrm{e}^{\mathrm{i}tu} - 1 - \mathrm{i}tu)\lambda(\mathrm{d}u) - \int_{|u| \geq r} (\mathrm{e}^{\mathrm{i}tu} - 1)\lambda(\mathrm{d}u),$$

giving $d(r) = (M_+ - M_-)d_0(r)$, where

$$
d_0(r) = \begin{cases}
\dfrac{r^{1-\alpha}}{(1-\alpha)}, & \alpha \neq 1, \\[2mm]
\displaystyle\int_0^\infty u^{-2}(u\,\mathbf{1}\,\{u < r\} - \sin u)\,\mathrm{d}u, & \alpha = 1.
\end{cases}
$$

As in the previous section, $M_- = 0$ or $M_- = M_+$. Equivalently, $\beta = 1$ or $\beta = 0$. The simulations became quite numerically unstable for small α, so we started with $\alpha = 0.2$. Also, for $\alpha = 1.0$, the R routines provided by the `stabledist` package appeared to become numerically unstable in the asymmetric case but showed no serious problems in the symmetric case. Therefore, in the simulations for the asymmetric case, we chose $\alpha = 1.01$ instead of $\alpha = 1.0$, while, for the symmetric case, we chose $\alpha = 1.0$. With this exception, we let α range from 0.2 to 1.8 with a step size of 0.2.

For each value of α, at a given value of r, we sampled 10^6 triplets $(T_r, \sigma_{X_r}Z, \Delta_r)$. The paired sums $T_r + \Delta_r$ and $\sigma_{X_r}Z + \Delta_r$ formed samples from the PGN and normal approximations to X, respectively. Meanwhile, Δ_r formed a sample from the CP approximation of X. Denote by $\widehat{F}_{\mathrm{PGN}}$, $\widehat{F}_{\mathrm{Norm}}$, and $\widehat{F}_{\mathrm{CP}}$ the corresponding empirical distributions. For $\theta \in (0, 1)$, let x_θ be the (unique) quantile of X such that $\mathbb{P}\{X \leq x_\theta\} = \theta$. The empirical KS distance between the PGN approximation and the target distribution was defined as $\widehat{D}_{\mathrm{PGN}} = \max_\theta |\widehat{F}_{\mathrm{PGN}}(x_\theta) - \theta|$ with $\theta \in \{i/200 : i = 1, \ldots, 199\}$. Likewise, $\widehat{D}_{\mathrm{Norm}}$ and $\widehat{D}_{\mathrm{CP}}$ were calculated for the normal and CP approximations, respectively. This step was repeated 2500 times. The resulting 2500 triplets $(\widehat{D}_{\mathrm{PGN}}, \widehat{D}_{\mathrm{Norm}}, \widehat{D}_{\mathrm{CP}})$ were used to estimate $d_{\mathrm{KS}}(X, T_r + \Delta_r)$, $d_{\mathrm{KS}}(X, \sigma_{X_r}Z + \Delta_r)$, and $d_{\mathrm{KS}}(X, \Delta_r)$, respectively, and their pairwise ratios. The focus here was the ratio of $d_{\mathrm{KS}}(X, T_r + \Delta_r)$ to $d_{\mathrm{KS}}(X, \sigma_{X_r}Z + \Delta_r)$. However, to make sure that our implementation of the normal approximation was correct, we also estimated the ratio of $d_{\mathrm{KS}}(X, \sigma_{X_r}Z + \Delta_r)$ to $d_{\mathrm{KS}}(X, \Delta_r)$. In the following, all errors are in terms of the KS distance from target distribution.

We first compared the approximations with $M_+ = 1$ and the cutoff r fixed at 5. To compare with Example 2.1, we included $\alpha = 0.3$ and 1.5 in the simulations. The results are summarized in Table 1. The top half of the table displays the sample means of $\widehat{D}_{\mathrm{PGN}}$ ('PGN d_{KS}'), $\widehat{D}_{\mathrm{Norm}}$ ('Norm d_{KS}'), and $\widehat{D}_{\mathrm{CP}}$ ('CP d_{KS}'), respectively. The bottom half of the table displays $\widehat{D}_{\mathrm{PGN}}/\widehat{D}_{\mathrm{Norm}}$ ('P-N d_{KS} ratio'), the upper 99% t-confidence limit of $\mathbb{E}[\widehat{D}_{\mathrm{PGN}}/\widehat{D}_{\mathrm{Norm}}]$ ('CL$_{0.99}(R_{\mathrm{P\text{-}N}})$'), $\widehat{D}_{\mathrm{Norm}}/\widehat{D}_{\mathrm{CP}}$ ('N-C d_{KS} ratio'), and the upper 99% t-confidence limit of $\mathbb{E}[\widehat{D}_{\mathrm{Norm}}/\widehat{D}_{\mathrm{CP}}]$ ('CL$_{0.99}(R_{\mathrm{N\text{-}C}})$'), respectively. Since all of the standard errors are less than 1% of the corresponding sample means, they are omitted for brevity. To compare the performances of the approximations when the distribution is asymmetric ($\beta = 1$), and when the distribution is symmetric ($\beta = 0$), the sample means under these two conditions are displayed in pairs, with the results under the symmetric condition placed in parentheses. The results given in the table show that, generally speaking, except for small α, the error of the PGN approximation is significantly smaller than that of the normal approximation. For example, in the asymmetric case, for $\alpha = 0.3$ and $r = 5$, the sample mean of $\widehat{D}_{\mathrm{PGN}}$ is about $\frac{1}{3}$ of that of $\widehat{D}_{\mathrm{Norm}}$. This may be compared with Figure 1, which shows that the bound in (2.12) for the PGN approximation is smaller than that in (2.13) for the normal approximation only if r is extremely small. The results for $\alpha = 1.5$ in the table can also be compared with Figure 1. This indicates that the bound in (2.12) is quite conservative.

The results given in Table 1 also confirm that the normal approximation has a significantly smaller error than the CP approximation. In fact, from the confidence limits shown in the table, the ratio of reduction of error by the normal approximation compared to the CP approximation

TABLE 1: Errors of the approximations in terms of the KS distance from the target distribution at cutoff $r = 5$ with $\beta = 1$ (asymmetric) or 0 (symmetric) and $M_+ = 1$. The numbers in parentheses indicate values under the symmetric condition.

α	r	PGN d_{KS}	Norm d_{KS}	CP d_{KS}
0.2	5 (5)	3.84e−2 (8.71e−4)	3.99e−2 (8.72e−4)	5.72e−2 (9.34e−4)
0.3	5 (5)	2.82e−3 (8.31e−4)	8.75e−3 (8.53e−4)	7.00e−2 (8.49e−3)
0.4	5 (5)	5.14e−3 (8.39e−4)	2.08e−2 (1.64e−3)	1.50e−1 (3.63e−2)
0.5	5 (5)	7.45e−3 (8.44e−4)	3.33e−2 (3.54e−3)	2.35e−1 (8.36e−2)
0.6	5 (5)	8.96e−3 (8.66e−4)	4.47e−2 (6.13e−3)	3.05e−1 (1.41e−1)
0.8	5 (5)	9.60e−3 (9.23e−4)	6.14e−2 (1.16e−2)	4.10e−1 (2.51e−1)
1.0*	5 (5)	7.79e−3 (9.22e−4)	6.88e−2 (1.54e−2)	4.75e−1 (3.35e−1)
1.2	5 (5)	5.26e−3 (8.72e−4)	6.72e−2 (1.61e−2)	5.10e−1 (3.93e−1)
1.4	5 (5)	2.87e−3 (8.22e−4)	5.79e−2 (1.38e−2)	5.20e−1 (4.30e−1)
1.5	5 (5)	2.02e−3 (8.32e−4)	5.05e−2 (1.17e−2)	5.20e−1 (4.44e−1)
1.6	5 (5)	1.42e−3 (8.38e−4)	4.15e−2 (9.19e−3)	5.15e−1 (4.55e−1)
1.8	5 (5)	8.74e−4 (8.23e−4)	1.98e−2 (3.70e−3)	5.00e−1 (4.70e−1)

α	P-N d_{KS} ratio	$CL_{0.99}(R_{\text{P-N}})$	N-C d_{KS} ratio	$CL_{0.99}(R_{\text{N-C}})$
0.2	9.62e−1 (9.99e−1)	9.63e−1 (1.00e+0)	6.97e−1 (9.42e−1)	6.98e−1 (9.47e−1)
0.3	3.23e−1 (9.77e−1)	3.25e−1 (9.80e−1)	1.25e−1 (1.01e−1)	1.25e−1 (1.03e−1)
0.4	2.47e−1 (5.10e−1)	2.47e−1 (5.16e−1)	1.39e−1 (4.52e−2)	1.39e−1 (4.56e−2)
0.6	2.00e−1 (1.41e−1)	2.01e−1 (1.42e−1)	1.47e−1 (4.36e−2)	1.47e−1 (4.37e−2)
0.8	1.56e−1 (7.92e−2)	1.57e−1 (8.03e−2)	1.50e−1 (4.64e−2)	1.50e−1 (4.65e−2)
1.0*	1.13e−1 (6.00e−2)	1.14e−1 (6.08e−2)	1.45e−1 (4.58e−2)	1.45e−1 (4.59e−2)
1.2	7.83e−2 (5.41e−2)	7.86e−2 (5.49e−2)	1.32e−1 (4.10e−2)	1.32e−1 (4.11e−2)
1.4	4.96e−2 (5.95e−2)	4.99e−2 (6.04e−2)	1.11e−1 (3.21e−2)	1.11e−1 (3.22e−2)
1.5	4.01e−2 (7.11e−2)	4.04e−2 (7.21e−2)	9.71e−2 (2.64e−2)	9.72e−2 (2.64e−2)
1.6	3.42e−2 (9.13e−2)	3.46e−2 (9.27e−2)	8.06e−2 (2.02e−2)	8.06e−2 (2.02e−2)
1.8	4.42e−2 (2.24e−1)	4.48e−2 (2.27e−1)	3.96e−2 (7.88e−3)	3.96e−2 (7.91e−3)

* $\alpha = 1.01$ for $\beta = 1$.

is greater than that by the PGN approximation compared to the normal approximation. Also, the bound in (2.13) for the normal approximation is quite conservative compared to the numerical results. For example, in the asymmetric case, for $\alpha = 0.8$, the sample mean of $\widehat{D}_{\text{Norm}}$ is about 0.06, whereas the bound in (2.13) gives 0.54. In the other sets of simulations, the greater ratio of reduction of error by the normal approximation and the conservativeness of the bound in (2.13) were observed as well.

At cutoff $r = 5$, the error of the normal approximation varies with α. One question to ask is how the PGN approximation compares to the normal approximation when the error of the latter is fixed at a specified level. In the second set of simulations, we let r vary according to α, such that the empirical KS distance between the normal approximation and the target distribution was roughly 1%. The value of r was selected as follows. For each r, ten values of $\widehat{D}_{\text{Norm}}$ were sampled, each based on 10^6 observations from the normal approximation at cutoff r. Starting with a large r, we reduced r by half if the average of the ten sample values of $\widehat{D}_{\text{Norm}}$ was greater than 1.05%. We kept doing this until the average was within (0.95%, 1.05%) or was less than 0.95%. In the former case r was selected. In the latter case we got two values of r, one giving an average greater than 1.05%, the other giving an average smaller than 0.95%. Then a

TABLE 2: Errors of the approximations, with $\beta = 1$ (asymmetric) or 0 (symmetric), $M_+ = 1$, and r set so that the empirical KS distance between the normal approximation and the target distribution was roughly 1%. The numbers in parentheses indicate values under the symmetric condition.

α	r	PGN d_{KS}	Norm d_{KS}	CP d_{KS}
0.2	4.38e+1 (2.00e+3)	9.33e−3 (4.38e−3)	9.78e−3 (1.00e−2)	5.50e−2 (5.61e−2)
0.4	2.34e+0 (2.03e+1)	1.70e−3 (2.03e−3)	9.55e−3 (9.70e−3)	9.00e−2 (1.12e−1)
0.6	1.37e+0 (7.03e+0)	8.55e−4 (1.12e−3)	1.02e−2 (1.03e−2)	1.40e−1 (1.78e−1)
0.8	1.12e+0 (4.69e+0)	8.35e−4 (8.74e−4)	1.01e−2 (1.05e−2)	1.80e−1 (2.42e−1)
1.0*	1.07e+0 (3.91e+0)	1.47e−3 (8.37e−4)	1.06e−2 (1.01e−2)	2.25e−1 (3.00e−1)
1.2	1.03e+0 (3.71e+0)	8.21e−4 (8.29e−4)	9.83e−3 (9.67e−3)	2.55e−1 (3.54e−1)
1.4	1.12e+0 (4.10e+0)	8.24e−4 (8.32e−4)	1.02e−2 (9.86e−3)	3.10e−1 (4.10e−1)
1.6	1.37e+0 (5.08e+0)	8.24e−4 (8.32e−4)	1.00e−2 (9.42e−3)	3.75e−1 (4.56e−1)
1.8	2.54e+0 (9.38e+0)	8.36e−4 (8.38e−4)	1.01e−2 (9.70e−3)	4.65e−1 (4.90e−1)

α	P-N d_{KS} ratio	$CL_{0.99}(R_{\text{P-N}})$	N-C d_{KS} ratio	$CL_{0.99}(R_{\text{N-C}})$
0.2	9.54e−1 (4.37e−1)	1.05e+0 (4.81e−1)	1.78e−1 (1.78e−1)	1.89e−1 (1.87e−1)
0.4	1.78e−1 (2.09e−1)	1.79e−1 (2.11e−1)	1.06e−1 (8.69e−2)	1.06e−1 (8.70e−2)
0.6	8.41e−2 (1.08e−1)	8.52e−2 (1.10e−1)	7.27e−2 (5.78e−2)	7.28e−2 (5.79e−2)
0.8	8.27e−2 (8.34e−2)	8.39e−2 (8.46e−2)	5.62e−2 (4.33e−2)	5.63e−2 (4.33e−2)
1.0*	1.39e−1 (8.26e−2)	1.40e−1 (8.38e−2)	4.70e−2 (3.38e−2)	4.71e−2 (3.38e−2)
1.2	8.36e−2 (8.58e−2)	8.49e−2 (8.71e−2)	3.86e−2 (2.73e−2)	3.86e−2 (2.74e−2)
1.4	8.10e−2 (8.44e−2)	8.22e−2 (8.57e−2)	3.29e−2 (2.40e−2)	3.29e−2 (2.41e−2)
1.6	8.25e−2 (8.83e−2)	8.37e−2 (8.96e−2)	2.67e−2 (2.07e−2)	2.68e−2 (2.07e−2)
1.8	8.31e−2 (8.64e−2)	8.44e−2 (8.77e−2)	2.17e−2 (1.98e−2)	2.17e−2 (1.98e−2)

* $\alpha = 1.01$ for $\beta = 1$.

bisection search was used to get a value of r with the corresponding average of $\widehat{D}_{\text{Norm}}$ within $(0.95\%, 1.05\%)$.

After a value of r was selected, the simulations proceeded as those of Table 1. The results are summarized in Table 2, which also reports the selected values of r. The mean values of $\widehat{D}_{\text{Norm}}$ realized by the simulations are included to make sure that the values of r were selected appropriately. In general, the mean value of $\widehat{D}_{\text{Norm}}$ fell into the interval $(0.95\%, 1.05\%)$. However, due to random fluctuations, the mean value could fall outside of $(0.95\%, 1.05\%)$, even though during the selection of r, the average of the ten sampled values of $\widehat{D}_{\text{Norm}}$ fell into the interval. Similar to Table 1, except for small values of α, the error of the PGN approximation is significantly smaller than that of the normal approximation.

In the above simulations, $M_+ = 1$. In the last set of simulations, we set $M_+ = 0.1$ to see how the approximations performed. The results are summarized in Table 3. Again, we attempted to set r so that the empirical KS distance between the normal approximation and the target distribution was roughly 1%. However, although theoretically the approximation error vanishes as $r \to 0$, in our simulations, the numerical precision of the approximation deteriorated for small M_+, especially when α was small as well, and the minimum empirical KS distance for the normal approximation could be significantly larger than 1%. In this case, we set r so that the empirical KS distance was as small as possible. The results of Table 3 show that, for $\alpha \leq 0.4$, the empirical KS distance for the normal approximation sometimes could not reach 1%. When this happened, the empirical KS distance for the PGN approximation did

TABLE 3: Errors of the approximations, with $\beta = 1$ (asymmetric) or 0 (symmetric), $M_+ = 0.1$, and r set so that the empirical KS distance between the normal approximation and the target distribution was roughly 1%, or as low as possible if this could not be attained numerically. The numbers in parentheses indicate values under the symmetric condition.

α	r	PGN d_{KS}	Norm d_{KS}	CP d_{KS}
0.2	2.44e−1 (1.71e−1)	1.47e−1 (2.38e−2)	1.73e−1 (3.19e−2)	3.70e−1 (1.45e−1)
0.4	2.29e−2 (6.71e−2)	1.26e−2 (2.37e−3)	3.12e−2 (1.00e−2)	1.86e−1 (1.17e−1)
0.6	2.90e−2 (1.46e−1)	8.51e−4 (1.08e−3)	9.95e−3 (9.79e−3)	1.35e−1 (1.74e−1)
0.8	6.10e−2 (2.56e−1)	8.39e−4 (8.71e−4)	9.56e−3 (1.00e−2)	1.70e−1 (2.38e−1)
1.0*	1.04e−1 (3.91e−1)	1.47e−3 (8.37e−4)	9.60e−3 (1.01e−2)	2.15e−1 (3.00e−1)
1.2	1.53e−1 (5.37e−1)	8.35e−4 (8.33e−4)	1.00e−2 (9.42e−3)	2.55e−1 (3.52e−1)
1.4	2.20e−1 (7.81e−1)	8.38e−4 (8.26e−4)	1.04e−2 (9.64e−3)	3.10e−1 (4.09e−1)
1.6	3.17e−1 (1.27e+0)	8.28e−4 (8.28e−4)	9.75e−3 (1.03e−2)	3.70e−1 (4.59e−1)
1.8	6.84e−1 (2.73e+0)	8.33e−4 (8.29e−4)	9.74e−3 (1.04e−2)	4.60e−1 (4.91e−1)

α	P-N d_{KS} ratio	$CL_{0.99}(R_{P-N})$	N-C d_{KS} ratio	$CL_{0.99}(R_{N-C})$
0.2	8.53e−1 (7.32e−1)	8.54e−1 (7.39e−1)	4.66e−1 (2.19e−1)	4.66e−1 (2.20e−1)
0.4	3.60e−1 (2.39e−1)	3.69e−1 (2.43e−1)	1.66e−1 (8.59e−2)	1.68e−1 (8.62e−2)
0.6	8.56e−2 (1.10e−1)	8.67e−2 (1.11e−1)	7.37e−2 (5.63e−2)	7.38e−2 (5.64e−2)
0.8	8.79e−2 (8.69e−2)	8.92e−2 (8.82e−2)	5.62e−2 (4.20e−2)	5.63e−2 (4.21e−2)
1.0*	1.53e−1 (8.26e−2)	1.54e−1 (8.38e−2)	4.47e−2 (3.38e−2)	4.47e−2 (3.38e−2)
1.2	8.35e−2 (8.84e−2)	8.47e−2 (8.97e−2)	3.93e−2 (2.68e−2)	3.94e−2 (2.68e−2)
1.4	8.07e−2 (8.58e−2)	8.19e−2 (8.71e−2)	3.36e−2 (2.36e−2)	3.36e−2 (2.36e−2)
1.6	8.51e−2 (8.07e−2)	8.64e−2 (8.19e−2)	2.64e−2 (2.24e−2)	2.64e−2 (2.24e−2)
1.8	8.58e−2 (8.01e−2)	8.70e−2 (8.13e−2)	2.12e−2 (2.11e−2)	2.12e−2 (2.11e−2)

* $\alpha = 1.01$ for $\beta = 1$.

not reach 1% either. Since in our simulation each pair $(T_r + \Delta_r, \sigma_{X_r} Z + \Delta_r)$ shared the same sampled value of Δ_r, this suggests that the deterioration of the numerical precision might be largely due to the error in Δ_r. However, a thorough solution to the issue is beyond the scope of this paper. On the other hand, regardless of this issue, Table 3 again shows that the error of the PGN approximation can be significantly smaller than that of the normal approximation.

4. Technical details

4.1. Proof of Theorem 2.1

Denote by $\mathscr{S}$ the space of smooth and rapidly decreasing functions on $\mathbb{R}$. It is classical that the Fourier transform $h \to \widehat{h}(t) = \int e^{itx} h(x)\, dx$ is a homeomorphism of $\mathscr{S}$ onto itself (see [19, p. 103]). Let f_X be the probability density of X. If it exists then $\psi_X = \widehat{f_X}$. Let

$$\int_0^\infty t^{2(q+1)} e^{-2L(t,r)}\, dt < \infty. \tag{4.1}$$

Otherwise, $Q_{q+1} = \infty$ and (2.10) is trivial. We need two lemmas. Note that the second lemma does not require matching of cumulants.

Lemma 4.1. (i) *Let ξ be i.d. with $\Psi_\xi(t) = \int (1 + itu - e^{itu})\, \nu(du)$ and $\mathbb{E}|\xi|^j < \infty$ for all $j \geq 1$. Given $\varepsilon > 0$, let $Z \sim N(0, \varepsilon^2)$ be independent of ξ. Then $\psi_{\xi+Z} \in \mathscr{S}$.*

(ii) *Under condition (4.1), $f_{X_r} \in C^q(\mathbb{R})$, and, for $0 \le j \le q$, $f_{X_r}^{(j)}(x) \to 0$ as $|x| \to \infty$.*

Lemma 4.2. *Let T_r be defined as in Theorem 2.1 with $s(r) < 1/(p+3)$ and $\sigma(r) > 0$. Fix $\varepsilon > 0$. Given $A, B \ge 0$ with $A + B = 1$, let W be i.d. with $\Psi_W(t) = A\Psi_{X_r}(t) + B\Psi_{T_r}(t) + \varepsilon^2 t^2/2$. Let $\xi = W/v$, where $v = \sqrt{A\kappa_{2,X_r} + B\kappa_{2,T_r}}$. Then $f_\xi \in \mathcal{S}$ and, for $j \ge 1$,*

$$\int |f_\xi^{(j)}(x)| \, dx \le j I_{j-1}(r) + I_j(r) + \left(1 + \frac{\varepsilon^2}{v^2}\right) I_{j+1}(r),$$

where, for $j \ge 0$,

$$I_j(r) = v^{j+1/2} \left[\frac{\Gamma(j + 1/2)}{2D(r)^{2j+1}} + \int_{1/r}^{\infty} t^{2j} e^{-2H(t,r)} \, dt \right]^{1/2}$$

with $D(r) = \sqrt{2AC_0\kappa_{2,X_r} + B(C_0\kappa_{2,Y_r} + \sigma(r)^2)}$ and

$$H(t, r) = AC_0 t^2 \int_{u < 1/|t|} u^2 \, \lambda(du) + \frac{B\sigma(r)^2 t^2}{2}.$$

Assume that the lemmas are true for now. Since $d_{\mathrm{TV}}(X, T_r + \Delta_r) = d_{\mathrm{TV}}(X_r + \Delta_r, T_r + \Delta_r) \le d_{\mathrm{TV}}(X_r, T_r)$, to show Theorem 2.1, it suffices to show that, for any measurable $A \subset \mathbb{R}$,

$$\varrho(A) \le \frac{M}{q!}(|\kappa|_{q,X_r} + |\kappa|_{q,Y_r}), \tag{4.2}$$

where $\varrho(A) = |\mathbb{P}\{X_r \in A\} - \mathbb{P}\{T_r \in A\}|$ and $M = \sigma_{X_r}^{-q}[q Q_{q-1}(r) + Q_q(r) + Q_{q+1}(r)]$.

We start with smoothing X_r and T_r while maintaining the same order of cumulant matching. Let Z and Z' be i.i.d. $\sim N(0, 1)$ and independent of (X_r, T_r). Fix $\varepsilon > 0$. Let h be a measurable function with $\|h\|_\infty \le 1$. The goal now is to bound

$$\Delta_\varepsilon = \mathbb{E}[h(X_r + \varepsilon Z) - h(T_r + \varepsilon Z')].$$

For $n \ge 2$, let $U_i = U_{i,n}$ and $V_j = V_{j,n}$, $i, j = 1, \ldots, n+1$, be independent and i.d. with

$$\Psi_{U_i}(t) = n^{-1}\Psi_{X_r + \varepsilon Z}(t), \qquad \Psi_{V_i}(t) = n^{-1}\Psi_{T_r + \varepsilon Z'}(t).$$

For $k = 1, \ldots, n+1$, let

$$W_k = \sum_{1 \le j < k} V_j + \sum_{k < j \le n+1} U_j, \qquad g_k(x) = \mathbb{E}h(W_k + x).$$

Since $X_r + \varepsilon Z \sim W_1$ and $T_r + \varepsilon Z' \sim W_{n+1}$, then $\Delta_\varepsilon = g_1(0) - g_{n+1}(0)$, giving

$$|\Delta_\varepsilon| \le |\mathbb{E}[g_1(U_1) - g_{n+1}(V_{n+1})]| + |\mathbb{E}[g_1(U_1) - g_1(0)]| + |\mathbb{E}[g_{n+1}(V_{n+1}) - g_{n+1}(0)]|. \tag{4.3}$$

We bound the expectations on the RHS separately. By $W_k + V_k = W_{k+1} + U_{k+1}$,

$$h(W_1 + U_1) - h(W_{n+1} + V_{n+1}) = \sum_{k=1}^{n+1} [h(W_k + U_k) - h(W_k + V_k)].$$

For each k, since W_k, U_k, and V_k are independent, by conditioning, $\mathbb{E}h(W_k + U_k) = \mathbb{E}g_k(U_k)$ and $\mathbb{E}h(W_k + V_k) = \mathbb{E}g_k(V_k)$. Then taking the expectation on both sides of the displayed identity yields

$$\mathbb{E}[g_1(U_1) - g_{n+1}(V_{n+1})] = \sum_{k=1}^{n+1} \mathbb{E}[g_k(U_k) - g_k(V_k)]. \tag{4.4}$$

Define $\nu = \sigma_{X_r}$. Let $\xi_k = W_k/\nu$. By Lemma 4.1, $f_{\xi_k} \in \mathcal{S}$. As a result,

$$g_k(x) = \mathbb{E}[h(\nu\xi_k + x)] = \int h(\nu u) f_{\xi_k}\left(u - \frac{x}{\nu}\right) du \tag{4.5}$$

is smooth. By Taylor's expansion around 0,

$$g_k(U_k) - g_k(V_k) = \sum_{j=1}^{q-1} \frac{g_k^{(j)}(0)}{j!}(U_k^j - V_k^j) + \frac{1}{q!}[g_k^{(q)}(\theta(U_k)U_k)U_k^q - g_k^{(q)}(\theta(V_k)V_k)V_k^q],$$

where $\theta(x) \in [0, 1]$. By assumption, $\kappa_{j,X_r} = \kappa_{j,T_r}$ for $1 \le j < q$. Since $\kappa_{j,U_k} = n^{-1}\kappa_{j,X_r+\varepsilon Z} = n^{-1}(\kappa_{j,X_r} + \varepsilon^2 \mathbf{1}\{j = 2\})$, and likewise $\kappa_{j,V_k} = n^{-1}(\kappa_{j,T_r} + \varepsilon^2 \mathbf{1}\{j = 2\})$, then $\kappa_{j,U_k} = \kappa_{j,V_k}$ for $1 \le j < q$. As a result, $\mathbb{E}U_k^j = \mathbb{E}V_k^j$ for $1 \le j < q$ and, hence,

$$\mathbb{E}[g_k(U_k) - g_k(V_k)] = \frac{1}{q!}\mathbb{E}[g_k^{(q)}(\theta(U_k)V_k)U_k^q - g_k^{(q)}(\theta(V_k)V_k)V_k^q],$$

$$\implies \quad |\mathbb{E}[g_k(U_k) - g_k(V_k)]| \le \frac{\|g_k^{(q)}\|_\infty}{q!}[\mathbb{E}|U_k|^q + \mathbb{E}|V_k|^q]. \tag{4.6}$$

Since by (4.5) we have $g_k^{(q)}(x) = (-\nu)^{-q} \int h(\nu u) f_{\xi_k}^{(q)}(u - x/\nu)\, du$, then

$$\|g_k^{(q)}\|_\infty \le \nu^{-q} \int |f_{\xi_k}^{(q)}(u)|\, du < \infty. \tag{4.7}$$

By $\Psi_{W_k}(t) = (k-1)\Psi_{V_1}(t) + (n+1-k)\Psi_{U_1}(t)$,

$$\Psi_{W_k}(t) = \frac{n+1-k}{n}\Psi_{X_r}(t) + \frac{k-1}{n}\Psi_{T_r}(t) + \frac{\varepsilon^2 t^2}{2}.$$

Then we can apply Lemma 4.2 with $\nu^2 = \kappa_{2,X_r} = \kappa_{2,T_r}$, $A = (n+1-k)/n$, and $B = (k-1)/n$ therein. By the definitions of $D(r)$ and $H(t, r)$ in Lemma 4.2,

$$D(r)^2 = 2AC_0\nu^2 + B(C_0\kappa_{2,Y_r} + \sigma(r)^2) \ge C_0\nu^2$$

and

$$H(t, r) = AC_0 t^2 \int_{u<1/|t|} u^2\, \lambda(du) + \frac{B\sigma(r)^2 t^2}{2}$$

$$\ge (A + B)t^2 \min\left\{C_0 \int_{u<1/|t|} u^2\, \lambda(du),\ \frac{\sigma(r)^2}{2}\right\}$$

$$= L(t, r).$$

By the definition of $Q_j(r)$ in Theorem 2.1 and the definition of $I_j(r)$ in Lemma 4.2, $I_j(r) \leq Q_j(r)$. By condition (4.1), $Q_j(r) < \infty$ for $0 \leq j \leq q+1$. Thus, (4.7) and Lemma 4.2 give

$$\|g_k^{(q)}\|_\infty \leq \nu^{-q} \left[q\, Q_{q-1}(r) + Q_q(r) + \left(1 + \frac{\varepsilon^2}{\nu^2} \right) Q_{q+1}(r) \right] := M_\varepsilon < \infty.$$

Since M_ε is independent of k, by (4.4) and (4.6),

$$|\mathbb{E}g_1(U_1) - \mathbb{E}g_{n+1}(V_{n+1})| \leq \frac{M_\varepsilon}{q!} \sum_{k=1}^{n+1} (\mathbb{E}|U_k|^q + \mathbb{E}|V_k|^q).$$

Since the Lévy measure of X_r has bounded support, $\mathbb{E}|X_r + \varepsilon Z|^q < \infty$. Meanwhile, from (2.3), $\mathbb{E}|Y_r + \varepsilon Z|^q < \infty$. Then, by Lemma 3.1 of [1],

$$\sum_{k=1}^{n+1} \mathbb{E}|U_k|^q \to |\kappa|_{q, X_r + \varepsilon Z} = |\kappa|_{q, X_r}, \qquad \sum_{k=1}^{n+1} \mathbb{E}|V_k|^q \to |\kappa|_{q, T_r + \varepsilon Z'} = |\kappa|_{q, Y_r}.$$

Thus, for the first term on the RHS of (4.3),

$$\limsup_{n \to \infty} |\mathbb{E}g_1(U_1) - \mathbb{E}g_{n+1}(V_{n+1})| \leq \frac{M_\varepsilon}{q!} (|\kappa|_{q, X_r} + |\kappa|_{q, Y_r}). \tag{4.8}$$

To bound the other terms on the RHS of (4.3), first note that $|\mathbb{E}[g_1(U_1) - g_1(0)]| \leq \|g_1'\|_\infty \mathbb{E}|U_1|$. As in (4.7), $\|g_1'\|_\infty < \infty$. Since $g_1(x) = \mathbb{E}h(X_r + \varepsilon Z + x)$, $\|g_1'\|_\infty$ is independent of n. On the other hand, by $\mathbb{E}U_1 = 0$ and the Cauchy–Schwartz inequality, $\mathbb{E}|U_1| \leq \sigma_{U_1} = \sigma_{X_r + \varepsilon Z}/\sqrt{n}$, so $\mathbb{E}[g_1(U_1) - g_1(0)] \to 0$ as $n \to \infty$. Likewise, $\mathbb{E}[g_{n+1}(V_{n+1}) - g_{n+1}(0)] \to 0$. Together with (4.3) and (4.8), this implies that

$$|\mathbb{E}h(X_r + \varepsilon Z) - \mathbb{E}h(T_r + \varepsilon Z')| \leq \frac{M_\varepsilon}{q!} (|\kappa|_{q, X_r} + |\kappa|_{q, Y_r}).$$

Let G be the union of a finite number of (a_i, b_i) and $h(x) = \mathbf{1}\{x \in G\}$. By Lemma 4.1(ii), $\mathbb{P}\{X_r = a_i\}$ or b_i, some $i = 0$. Let $\varepsilon \to 0$. Then $h(X_r + \varepsilon Z) - h(X_r) \to 0$ almost surely. On the other hand, since T_r is the sum of Y_r and an independent nonzero normal random variable, by Lemma 4.1(i), $f_{T_r} \in \mathscr{S}$. As a result, $h(T_r + \varepsilon Z') - h(T_r) \to 0$ almost surely. Finally, $M_\varepsilon \to M$. Then, by dominated convergence, (4.2) holds for G.

Let A be measurable. Given $\delta > 0$, fix $R > 0$ such that $\mathbb{P}\{|X_r| \geq R\} + \mathbb{P}\{|T_r| \geq R\} < \delta$. Let $B = A \cap (-R, R)$. Then $\varrho(A) \leq \varrho(B) + \delta$. There is an open $G \supset B$ such that $\ell(G \setminus B) < \delta$, where ℓ is the Lebesgue measure. Here G is the union of at most countably many disjoint open intervals (a_i, b_i). Let $G_k = \bigcup_{i=1}^k (a_i, b_i)$. Then $\varrho(B) \leq \varrho(G_k) + \mathbb{P}\{X_r \in B \triangle G_k\} + \mathbb{P}\{T_r \in B \triangle G_k\}$. From the above paragraph, (4.2) holds for G_k. Next, $B \triangle G_k \subset (G \setminus G_k) \cup (G \setminus B)$, and $\mathbb{P}\{X_r \in G \setminus B\} \leq \|f_{X_r}\|_\infty \ell(G \setminus B)$ with a similar inequality holding for T_r. Then

$$\varrho(B) \leq \frac{M}{q!} (|\kappa|_{q, X_r} + |\kappa|_{q, Y_r}) + \mathbb{P}\{X_r \in G \setminus G_k\} + \mathbb{P}\{T_r \in G \setminus G_k\}$$
$$+ (\|f_{X_r}\|_\infty + \|f_{T_r}\|_\infty)\delta.$$

By Lemma 4.1, $\|f_{X_r}\|_\infty + \|f_{T_r}\|_\infty < \infty$. Letting $k \to \infty$ and then $\delta \to 0$, it is seen that (4.2) holds for A.

4.2. Proofs of Lemmas 4.1 and 4.2

We need the following elementary result for the proofs.

Lemma 4.3. (i) $1 - \cos x \geq C_0 x^2$ for $|x| \leq 1$.

(ii) $\inf_{p>0} (1/\Gamma(p)) \int_0^p u^{p-1} e^{-u} \, du = \frac{1}{2}$.

Proof. (i) For $x \in [0, 1]$, since $\sin x$ is concave, $\sin x \geq x \sin 1$, giving $1 - \cos x = 2[\sin(x/2)]^2 \geq 2[(x/2) \sin 1]^2 = C_0 x^2$. For $x \in [-1, 0]$, the proof follows from symmetry.

(ii) The inequality can be written as $\inf_{p>0} \mathbb{P}\{\xi_p \leq p\} = \frac{1}{2}$, where $\xi_p \sim \text{Gamma}(p, 1)$. By the central limit theorem, $\mathbb{P}\{\xi_p \leq p\} \to \frac{1}{2}$ as $p \to \infty$. Therefore, it suffices to show that, for every $p > 0$, $\mathbb{P}\{\xi_p \leq p\} > \frac{1}{2}$, or, equivalently, $\int_0^p u^{p-1} e^{-u} \, du > \int_p^\infty u^{p-1} e^{-u} \, du$. Applying the change of variable $u \leftarrow pu$ to the first integral and $u \leftarrow p/u$ to the second integral, the inequality is equivalent to $\int_0^1 u^{-p-1} [u^{2p} e^{-pu} - e^{-p/u}] \, du > 0$, which holds if, for all $u \in (0, 1)$, $u^{2p} e^{-pu} > e^{-p/u}$, or, equivalently, $2 \ln u + 1/u - u > 0$. The last inequality follows directly from calculus.

Proof of Lemma 4.1. (i) From the assumption, $\int |u|^j \lambda(du) < \infty$ for all $j \geq 2$. Then, by dominated convergence, $\Psi_\xi \in C^\infty(\mathbb{R})$ with $\Psi_\xi^{(j)}(t) = \int (\mathbf{1}\{j = 1\} - e^{itu})(iu)^j \nu(du)$ for $j \geq 1$. By $|1 - e^{ix}| \leq |x|$ for $x \in \mathbb{R}$, $|\Psi_\xi'(t)| \leq \kappa_{2,\xi} |t|$. Clearly, $|\Psi_\xi^{(j)}(t)| \leq |\kappa|_{j,\xi}$ for $j \geq 2$. Since $\psi_{\xi+Z}(t) = \exp\{-\Psi_\xi(t) - \varepsilon^2 t^2/2\}$, then, for $j \geq 0$, $\psi_{\xi+Z}^{(j)}(t) = P_j(\Psi_\xi'(t), \ldots, \Psi_\xi^{(j)}(t), t) \psi_\xi(t) \exp\{-\varepsilon^2 t^2/2\}$, where $P_j(z)$ is a multivariate polynomial in $z = (z_1, \ldots, z_{j+1})$ of order j. It follows that $|\psi_{\xi+Z}^{(j)}(t)| = O(|t|^j e^{-\varepsilon^2 t^2/2})$ and, hence, for any $p \geq 0$, $|t|^p |\psi_{\xi+Z}^{(j)}(t)| \to 0$ as $|t| \to \infty$, which completes the proof.

(ii) For $|t| \geq 1/r$,

$$\text{Re}[\Psi_{X_r}(t)] = \int_{|u|<r} (1 - \cos tu) \, \lambda(du) \geq \int_{|u|<1/|t|} (1 - \cos tu) \, \lambda(du).$$

Then, by Lemma 4.3, $1 - \cos tu \geq C_0 t^2 u^2$ for $0 \leq u < 1/|t|$ and, hence,

$$\text{Re}[\Psi_{X_r}(t)] \geq C_0 t^2 \int_{u<1/|t|} u^2 \, \lambda(du) \geq L(t, r).$$

On the other hand, by the Cauchy–Schwartz inequality,

$$\int_{|t|\geq 1/r} |t|^q |\psi_{X_r}(t)| \, dt \leq \left(\int \frac{dt}{1+t^2} \right)^{1/2} \left(\int_{|t|\geq 1/r} (1+t^2) t^{2q} |\psi_{X_r}(t)|^2 \, dt \right)^{1/2}$$

$$\leq \sqrt{\pi} \left(\int (1+t^2) t^{2q} e^{-2L(t,r)} \, dt \right)^{1/2}.$$

Then, by (4.1), $|t|^q |\psi_{X_r}(t)| \in L^1(\mathbb{R})$ and the proof follows from Proposition 28.1 of [31].

To prove Lemma 4.2, we need a type of inequality known in the literature (cf. [4, Lemma 11.6]). Since the expression of $(\widehat{f})^{(j)}$ becomes rapidly complicated as j increases, the following specific form is used to reduce the maximum order of the derivative involved.

Lemma 4.4. Let $f \in \mathcal{S}$ and $\psi = \widehat{f}$. Then, for $j \geq 1$,

$$\int |f^{(j)}| \leq \frac{1}{\sqrt{2}} \left[\left(\int |t^j \psi(t)|^2 \, dt \right)^{1/2} + j \left(\int |t^{j-1} \psi(t)|^2 \, dt \right)^{1/2} + \left(\int |t^j \psi'(t)|^2 \, dt \right)^{1/2} \right].$$

Proof. By the Cauchy–Schwartz and Minkowski inequalities,

$$\int |f^{(j)}| \le \left(\int \frac{dx}{1+x^2}\right)^{1/2} \left(\int |f^{(j)}(x)|^2 (1+x^2)\, dx\right)^{1/2}$$

$$\le \sqrt{\pi}\left[\left(\int |f^{(j)}(x)|^2\, dx\right)^{1/2} + \left(\int |xf^{(j)}(x)|^2\, dx\right)^{1/2}\right].$$

Then, by Plancherel's theorem and the fact that the Fourier transforms of $f^{(j)}(x)$ and $x^j f(x)$ are $(-it)^j \psi(t)$ and $(-i)^j \psi^{(j)}(t)$, respectively (see [19, pp. 100–102]),

$$\int |f^{(j)}| \le \frac{1}{\sqrt{2}}\left[\left(\int |t^j \psi(t)|^2\, dt\right)^{1/2} + \left(\int |(t^j \psi(t))'|^2\, dt\right)^{1/2}\right].$$

The proof is complete by applying Minkowski's inequality to the last integral.

Proof of Lemma 4.2. We only consider the case where sppt$(\lambda) \subset \mathbb{R}_+$. The proof for the symmetric case is similar. For brevity, write $f = f_\xi$, $\psi = \psi_\xi$, and $\Psi = \Psi_\xi$. By Lemma 4.1, $f, \psi \in \mathcal{S}$. Write $M = \varepsilon^2 + B\sigma(r)^2$. Then

$$\mathrm{Re}[\Psi(t)] = \mathrm{Re}\left[\Psi_W\left(\frac{t}{v}\right)\right] = \int \left(1 - \cos\left(\frac{tu}{v}\right)\right)[A\lambda_r(du) + B\gamma_r(du)] + \frac{Mt^2}{2v^2}.$$

If $|t| \le v/r$ then $|tu|/v \le 1$ for $0 \le u < r$, so, by Lemma 4.3, $1 - \cos(tu/v) \ge C_0 t^2 u^2 / v^2$. Consequently,

$$\mathrm{Re}[\Psi(t)] \ge \frac{C_0 t^2}{v^2} \int_0^r u^2 [A\lambda_r(du) + B\gamma_r(du)] + \frac{Mt^2}{2v^2}$$

$$= \frac{AC_0 \kappa_{2,X_r} t^2}{v^2} + \frac{BC_0 m(r) s(r)^{p+3} t^2}{v^2} \int_0^{r/s(r)} u^{p+2} e^{-u}\, du + \frac{Mt^2}{2v^2}.$$

Since $s(r) < r/(p+3)$, by Lemma 4.3(ii),

$$\int_0^{r/s(r)} u^{p+2} e^{-u}\, du \ge \int_0^{p+3} u^{p+2} e^{-u}\, du \ge \frac{\Gamma(p+3)}{2}.$$

Then, as $\Gamma(p+3) m(r) s(r)^{p+3} = \kappa_{2,Y_r}$,

$$\mathrm{Re}[\Psi(t)] \ge \frac{AC_0 \kappa_{2,X_r} t^2}{v^2} + \frac{BC_0 m(r) s(r)^{p+3} \Gamma(p+3) t^2}{2v^2} + \frac{Mt^2}{2v^2}$$

$$\ge \frac{AC_0 \kappa_{2,X_r} t^2}{v^2} + \frac{BC_0 \kappa_{2,Y_r} t^2}{2v^2} + \frac{B\sigma(r)^2 t^2}{2v^2}$$

$$= \frac{D(r)^2 t^2}{2v^2}.$$

If $|t| > v/r$ then $r > v/|t|$ and

$$\mathrm{Re}[\Psi(t)] \ge \frac{AC_0 t^2}{v^2} \int_{u < v/|t|} u^2 \lambda(du) + \frac{B\sigma(r)^2 t^2}{2v^2} = H\left(\frac{t}{v}, r\right).$$

Therefore, for $j \geq 0$,

$$
\begin{aligned}
\int |t^j \psi(t)|^2 \, dt &= 2 \int_0^\infty t^{2j} e^{-2 \operatorname{Re}[\Psi(t)]} \, dt \\
&\leq 2 \int_0^{v/r} t^{2j} e^{-D(r)^2 t^2 / v^2} \, dt + 2 \int_{v/r}^\infty t^{2j} e^{-2H(t/v, r)} \, dt \\
&\leq 2 \int_0^\infty t^{2j} e^{-D(r)^2 t^2 / v^2} \, dt + 2 v^{2j+1} \int_{1/r}^\infty t^{2j} e^{-2H(t,r)} \, dt \\
&\leq \frac{v^{2j+1} \Gamma(j+1/2)}{D(r)^{2j+1}} + 2 v^{2j+1} \int_{1/r}^\infty t^{2j} e^{-2H(t,r)} \, dt \\
&= 2 I_j(r)^2.
\end{aligned}
\tag{4.9}
$$

Next, $\psi'(t) = -\Psi'(t) \psi(t)$, with $\Psi'(t) = (i/v) \int (1 - e^{itu/v}) u \, [A \lambda_r(du) + B \gamma_r(du)] + Mt/v^2$. As $|1 - e^{ix}| \leq |x|$ for all $x \in \mathbb{R}$,

$$
\begin{aligned}
|\Psi'(t)| &\leq \frac{t}{v^2} \int u^2 \, [A \lambda_r(du) + B \gamma_r(du)] + \frac{Mt}{v^2} \\
&= \frac{A \kappa_{2, X_r} t}{v^2} + \frac{B \kappa_{2, Y_r} t}{v^2} + \frac{(\varepsilon^2 + B \sigma(r)^2) t}{v^2} \\
&= \frac{(A \kappa_{2, X_r} + B \kappa_{2, T_r}) t}{v^2} + \frac{\varepsilon^2 t}{v^2} \\
&= \left(1 + \frac{\varepsilon^2}{v^2} \right) t.
\end{aligned}
$$

As a result,

$$
\begin{aligned}
\int |t^j \psi'(t)|^2 \, dt &= \int |t^j \Psi'(t) \psi(t)|^2 \, dt \\
&\leq \left(1 + \frac{\varepsilon^2}{v^2} \right)^2 \int |t^{j+1} \psi(t)|^2 \, dt \\
&\leq 2 \left(1 + \frac{\varepsilon^2}{v^2} \right)^2 I_{j+1}(r)^2.
\end{aligned}
\tag{4.10}
$$

The proof is complete by combining Lemma 4.4, (4.9), and (4.10).

Appendix A

To evaluate the RHS of (2.12), we need to evaluate $Q_j(r)$, $j = 5, 6, 7$, which involves the integral of $t^{2j} e^{-2L(t,r)}$ over $t \in [1/r, \infty)$. One way to obtain good numerical precision is to employ incomplete gamma functions [27]. For $\lambda(du) = c \mathbf{1}\{u > 0\} u^{-\alpha-1} \, du$, $\kappa_{2, X_s} = \int_0^s u^2 \lambda(du) = c \int_0^s u^{1-\alpha} \, du = c s^{2-\alpha} / (2 - \alpha)$, $s > 0$. Then it is not hard to obtain $2L(t, r) = \min\{A t^2, B|t|^\alpha\}$, where $A = \sigma(r)^2$ and $B = 2 c C_0 / (2 - \alpha)$. Let $t_0 = (B/A)^{1/(2-\alpha)}$. Then $2L(t, r) = A t^2 \mathbf{1}\{|t| \leq t_0\} + B|t|^\alpha \mathbf{1}\{|t| > t_0\}$, and, hence, letting $t_1 = \max\{1/r, t_0\}$,

$$
\begin{aligned}
\int_{1/r}^\infty t^{2j} e^{-2L(t,r)} \, dt &= \int_{1/r}^{t_1} t^{2j} e^{-At^2} \, dt + \int_{t_1}^\infty t^{2j} e^{-Bt^\alpha} \, dt \\
&= \frac{1}{2 A^{j+1/2}} \int_{A/r^2}^{A t_1^2} u^{j-1/2} e^{-u} \, du + \frac{1}{\alpha B^{(2j+1)/\alpha}} \int_{B t_1^\alpha}^\infty u^{(2j+1)/\alpha - 1} e^{-u} \, du.
\end{aligned}
$$

The integrals on the last line can be expressed as incomplete gamma functions. For symmetric $\lambda(\mathrm{d}u) = c\,\mathbf{1}\{|u| > 0\}|u|^{-\alpha-1}\,\mathrm{d}u$, the formula is the same, except that $B = 4cC_0/(2-\alpha)$.

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