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Corrigendum

Filtering free resolutions

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David Eisenbud, Daniel Erman and Frank-Olaf Schreyer

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Appendix A. Corrected proof of Corollary 4.2

An error in the definition $d^0 \ll d^1$ undermines our proof of Corollary 4.2, which is crucial to the proof of our main result. We are grateful to Amin Nematbakhsh and to Gunnar Fløystad for bringing this mistake to our attention. In this corrigendum, we add a condition to our definition $d^0 \ll d^1$ and use this to recover the various results from the paper.

Definition A.1. For two degree sequences d^0 and d^1 we say that $d^0 \ll d^1$ if $d^0 < d^1, d^0_2 \leqslant d^1_1$ and if $d^0_i \leqslant d^1_2 + i - 1$ for $i > 2$.

We remark that the conditions $d^0 < d^1$ and $d^0_2 \leqslant d^1_1$ imply that $d^0_i \leqslant d^1_2 + i - 1$ for $i \leqslant 2$ as well. Before correcting the proof of Corollary 4.2, we prove an auxiliary lemma.

Lemma A.2. *Let M be a module and let $\mathbf{f} = (f_0, \ldots, f_n) \in \mathbb{Z}^{n+1}$ be any sequence such that f_0 is greater than the maximal degree of a generator of M, and such that*

$$f_2 := \min\{j \mid j > f_1 \text{ and } \beta_{2,j}(M) \neq 0\}.$$

Let N be the cokernel of $\phi(\mathbf{f})^M$, as defined in Definition 2.1, and let K be the kernel of the natural surjection $N \to M$:

(i) $\mathrm{Tor}_i(K,k)$ *is generated in degree* $\geqslant f_2 + i - 1$;

(ii) *if* $e \leqslant f_2 + i - 2$, *then we get an injection* $\mathrm{Tor}_i(N,k)_e \to \mathrm{Tor}_i(M,k)_e$;

(iii) *if* $e < f_2 + i - 2$, *then we also get a surjection* $\mathrm{Tor}_i(N,k)_e \to \mathrm{Tor}_i(M,k)_e$.

Proof. By definition of N, we have that $\mathrm{Tor}_1(N,k) \subseteq \mathrm{Tor}_1(M,k)$, so the long exact sequence in Tor induces

$$\cdots \to \mathrm{Tor}_2(M,k) \to \mathrm{Tor}_1(K,k) \xrightarrow{0} \mathrm{Tor}_1(N,k) \to \cdots.$$

Let m be the minimal degree of a generator of $\mathrm{Tor}_1(K,k)$. Since K is generated in degree $\geqslant f_1$ (see Definition 2.1), it follows that $m > f_1$; the surjectivity of $\mathrm{Tor}_2(M,k) \to \mathrm{Tor}_1(K,k)$ then implies that $\beta_{2,m}(M) \neq 0$. Hence, $m \geqslant f_2$ and (i) follows immediately.

For (ii) and (iii), we consider

$$\cdots \to \mathrm{Tor}_i(K,k)_e \to \mathrm{Tor}_i(N,k)_e \to \mathrm{Tor}_i(M,k)_e \to \mathrm{Tor}_{i-1}(K,k)_e \to \cdots.$$

If $e \leqslant f_2 + i - 2$, then (i) implies that $\mathrm{Tor}_i(K,k)_e = 0$, which proves (ii). If $e < f_2 + i - 2$, then (i) implies $\mathrm{Tor}_{i-1}(K,k)_e = 0$, which yields (iii). □

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Corrected proof of Corollary 4.2. By Definition 3.1 and the fact that $d_2^0 \leqslant d_1^1$, we have $f_2 = d_2^1$. We remark that Lemma A.2 immediately implies the conclusion of the first paragraph of the proof. The gap in the proof occurred in the third paragraph, where we computed the degree sequences e that could conceivably contribute to the top strand of $\beta(N)$, so we now correct this argument. Let e_i be the minimal degree of a generator of the ith syzygy module of N, or ∞ if this syzygy module is zero. We claim that the degree sequence $e = (e_0, \ldots, e_n)$ is at least d^0. Since all first syzygies of N lie in degree d_1^0, this would imply that $e_1 = d_1^0$ or $e_1 = \infty$, thus correcting the proof.

For a contradiction, assume that $e_i < d_i^0$ for some $i \geqslant 2$. Definition A.1 then implies that

$$e_i < d_i^0 \leqslant d_2^1 + i - 1 = f_2 + i - 1.$$

The definition of e_i forces $\mathrm{Tor}_i(N,k)_{e_i}$ to be nonzero. But since $e_i \leqslant f_2 + i - 2$, Lemma A.2(ii) implies $\mathrm{Tor}_i(M,k)_{e_i}$ is also nonzero. Recalling that $e_i < d_i^0$, this contradicts the fact that d^0 was defined as the top strand of $\beta(M)$, yielding the claim. □

With Definition A.1 and the corrected proof of Corollary 4.2, the proof of Theorem 1.3 goes through as written. Although the resulting Theorem 1.3 is weaker (because Definition A.1 is more restrictive than previously), it is sufficiently strong to recover nearly all of the examples and corollaries that rely on Theorem 1.3. For instance, applying Definition A.1 to Example 1.7, we have

$$d^0 = (0,1,2,5) \ll d^1 = (0,2,3,5) \ll d^2 = (0,3,4,5),$$

and so the corrected Theorem 1.3 still applies to Example 1.7. There is, however, one application of Theorem 1.3 that now requires an additional argument: to correct the proof of Proposition 1.6, we will need to also incorporate an observation from Example 6.2.

Corrected proof of Proposition 1.6. Theorem 1.3 is invoked in the first sentence of third paragraph. Since we can no longer simply quote Theorem 1.3, we will directly prove the needed fact. We assume that $cD \in B_{\mathrm{mod}}$, and we must show that c is divisible by p. Let M be any module such that $\beta(M) = cD$. Let N be the cokernel of the submatrix of the presentation matrix of M containing all of the degree 1 and degree $\lfloor p/2 \rfloor$ columns (so we throw away the degree $p-2$ columns). Lemma A.2(ii) and (iii) then imply that $\beta_{i,j}(N) = \beta_{i,j}(M)$ for $i = 2, 3$ and $j < p + i - 3$.

In particular, the top strand of $\beta(N)$ is at least $(0,1,2,p)$, and so we may use the monotonicity principle to conclude that the first step of the Boij–Söderberg decomposition of $\beta(N)$ is given by $(c/p)\widetilde{\pi}_{d^0}$. The top strand of the resulting diagram $\beta(N) - (c/p)\widetilde{\pi}_{d^0}$ is then at least $(0, \lfloor p/2 \rfloor, \lceil p/2 \rceil, p)$, and an additional application of the monotonicity principle yields the full Boij–Söderberg decomposition of $\beta(N)$ to be

$$\beta(N) = \frac{c}{p}\widetilde{\pi}_{d^0} + \frac{c\alpha}{p}\widetilde{\pi}_{d^1} + \frac{c}{p}\pi_{(0,\infty,\ldots,\infty)}.$$

By Lemma 5.1, N splits off a free summand $S^{c/p}$, and hence p divides c. □

Remark A.3. In Example 6.2 we wrote 'The proof of Corollary 4.2 applies nearly verbatim to show $\beta(N) = \widetilde{\pi}_{(0,2,3,4,5,8)} + 2\widetilde{\pi}_{(0,2,3,5,6,8)} + 6\widetilde{\pi}_{(0)}$'. For completeness, we provide additional details for this claim. Since $f_2 = 4$, Lemma A.2 yields

$$\beta_{i,i+1}(N) = \beta_{i,i+1}(M) \quad \text{for } i = 2,3,4 \quad \text{and} \quad \beta_{5,7}(N) \leqslant \beta_{5,7}(M) = 0,$$

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and thus the top strand of $\beta(N)$ has degree at least $(0,2,3,4,5,8)$. We now use the monotonicity principle twice: first with $\beta_{3,4}/\beta_{4,5}$ applied to N, which yields that $\widetilde{\pi}_{(0,2,3,4,5,8)}$ is the first step of the Boij–Söderberg decomposition for N; and second with $\beta_{1,2}/\beta_{2,3}$ applied to $N - \widetilde{\pi}_{(0,2,3,4,5,8)}$ to conclude that the remainder of the Boij–Söderberg decomposition is $2\widetilde{\pi}_{(0,2,3,5,6,8)} + 6\widetilde{\pi}_{(0)}$.

We have also posted a version of the paper that incorporates these corrections on the arXiv.

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