Glasgow Math. J. **56** (2014) 369–380.
doi:10.1017/S001708951300030X.

A DESCENT SPECTRAL SEQUENCE FOR ARBITRARY $K(n)$-LOCAL SPECTRA WITH EXPLICIT E_2-TERM

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(Received 2 October 2012; accepted 17 January 2013; first published online 13 August 2013)

Abstract. Let n be any positive integer and p be any prime. Also, let X be any spectrum and let $K(n)$ denote the nth Morava K-theory spectrum. Then we construct a descent spectral sequence with abutment $\pi_*(L_{K(n)}(X))$ and E_2-term equal to the continuous cohomology of G_n, the extended Morava stabilizer group, with coefficients in a certain discrete G_n-module that is built from various homotopy fixed point spectra of the Morava module of X. This spectral sequence can be contrasted with the $K(n)$-local E_n-Adams spectral sequence for $\pi_*(L_{K(n)}(X))$, whose E_2-term is not known to always be equal to a continuous cohomology group.

2010 *Mathematics Subject Classification.* Primary 55P42, 55T15, 55Q51.

1. Introduction. Given an integer $n \geq 1$ and any prime p, let $K(n)$ be the nth Morava K-theory spectrum and let E_n be the nth Lubin–Tate spectrum, with

$$\pi_*(E_n) = W(\mathbb{F}_{p^n})[\![u_1, \ldots, u_{n-1}]\!][u^{\pm 1}],$$

where $W(\mathbb{F}_{p^n})$ denotes the Witt vectors of the field $\mathbb{F}_{p^n}$, each u_i has degree zero, and the degree of u is -2. Also, let

$$G_n = S_n \rtimes \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$$

be the nth extended Morava stabilizer group.

Given a spectrum X, there is the Morava module $L_{K(n)}(E_n \wedge X)$ of X. Since G_n acts on E_n (by [**11**]), we can give X the trivial action and $E_n \wedge X$ the diagonal action, and hence, there is an induced G_n-action on the Morava module. In fact, for any closed subgroup K of the profinite group G_n, by [**3**, Section 9], $L_{K(n)}(E_n \wedge X)$ is a continuous K-spectrum and there is the homotopy fixed point spectrum

$$(L_{K(n)}(E_n \wedge X))^{hK}$$

formed with respect to the continuous K-action. Then, in this paper, we obtain the following result.

THEOREM 1.1. *Let X be any spectrum. There is a conditionally convergent descent spectral sequence $E_r^{*,*}$ that has the form*

$$E_2^{s,t} = H_c^s\Big(G_n; \pi_t\Big(\operatorname*{colim}_{N \lhd_o G_n} (L_{K(n)}(E_n \wedge X))^{hN}\Big)\Big) \Longrightarrow \pi_{t-s}(L_{K(n)}(X)),$$

where the E_2-term is the continuous cohomology of G_n with coefficients in a discrete G_n-module.

The spectral sequence $E_r^{*,*}$ is the descent spectral sequence for the homotopy fixed point spectrum

$$\Big(\operatorname*{colim}_{N \lhd_o G_n} (L_{K(n)}(E_n \wedge X))^{hN}\Big)^{hG_n} \simeq L_{K(n)}(X) \tag{1.2}$$

of the discrete G_n-spectrum $\operatorname{colim}_{N \lhd_o G_n} (L_{K(n)}(E_n \wedge X))^{hN}$ (see Theorem 3.4 for equivalence (1.2)).

The spectral sequence of Theorem 1.1 is in general different from the strongly convergent $K(n)$-local E_n-Adams spectral sequence $\widetilde{E}_r^{*,*}$ for $\pi_*(L_{K(n)}(X))$ (for the construction of this Adams spectral sequence, see, for example, [**10**, Appendix A]; for any X, this Adams spectral sequence is isomorphic to the descent spectral sequence for $(L_{K(n)}(E_n \wedge X))^{hG_n}$, by [**8**, Theorem 1.2]).

For any spectrum X, the E_2-term of spectral sequence $E_r^{*,*}$ is given by an explicit continuous cohomology group. By contrast, for arbitrary X it is not known if, in general, the E_2-term of spectral sequence $\widetilde{E}_r^{*,*}$ can be expressed in some way as a continuous cohomology group. This is known for $\widetilde{E}_r^{*,*}$ in certain cases, which include the following:

(a) By [**10**, Theorem 2,(ii)], if X is a finite spectrum, then

$$\widetilde{E}_2^{s,t} = H_c^s(G_n; \pi_t(E_n \wedge X)),$$

where $\pi_t(E_n \wedge X)$ is a profinite continuous $\mathbb{Z}_p[\![G_n]\!]$-module;

(b) by [**15**, Theorem 5.1], if $E_{n*}(X)$ is a flat $\pi_*(E_n)$-module then

$$\widetilde{E}_2^{s,t} = H_c^s(G_n; \pi_t(L_{K(n)}(E_n \wedge X))),$$

where again, in general, the coefficients of the continuous cohomology group need not be a discrete G_n-module;

(c) another well-known case is explained in [**13**, Proposition 7.4] (the details of which would take us too far afield in this introduction) and

(d) in the subtle case described in [**10**, Proposition 6.7], in which the continuous cohomology group has coefficients in a discrete G_n-module, we show in our next result that the two spectral sequences $E_r^{*,*}$ and $\widetilde{E}_r^{*,*}$ are the same.

To state this result, we need some notation. Let $E(n)$ be the Johnson–Wilson spectrum with $E(n)_* = \mathbb{Z}_{(p)}[v_1, \ldots, v_{n-1}][v_n^{\pm 1}]$, where each v_i has degree $2(p^i - 1)$, and let I_n be the ideal $(p, v_1, \ldots, v_{n-1})$ in $E(n)_*$. As implied above, the useful hypothesis in the following result comes from [**10**, Proposition 6.7].

THEOREM 1.3. *Let X be a spectrum such that, for each $E(n)$-module spectrum M, there exists an integer k with $I_n^k M_*(X) = 0$. Then spectral sequence $E_r^{*,*}$ is isomorphic to the strongly convergent $K(n)$-local E_n-Adams spectral sequence $\widetilde{E}_r^{*,*}$ that converges to $\pi_*(L_{K(n)}(X))$ from the E_2-terms onward.*

As in [**3**, Definition 2.3], let

$$F_n = \operatorname*{colim}_{N \lhd_o G_n} E_n^{dhN}, \tag{1.4}$$

where E_n^{dhN} is the spectrum constructed by Devinatz and Hopkins in [**10**] that behaves like an N-homotopy fixed point spectrum of E_n with respect to a continuous action of N. By construction, the spectrum F_n is a discrete G_n-spectrum. If $X = S^0$, then Theorem 1.1, together with Remark 3.6, yields the descent spectral sequence

$$H_c^s(G_n; \pi_t(F_n)) \Longrightarrow \pi_{t-s}(L_{K(n)}(S^0)),$$

which is a new tool for computing $\pi_*(L_{K(n)}(S^0))$ (the existence of this spectral sequence is also an immediate consequence of [**6**, last line of p. 254] and [**3**, Theorem 7.9]).

Given a spectrum X, the discrete G_n-spectrum $\operatorname{colim}_{N \lhd_o G_n}(L_{K(n)}(E_n \wedge X))^{hN}$, which appears in Theorem 1.1 and will be referred to here as $\mathcal{C}(X)$, is canonically associated to the Morava module $L_{K(n)}(E_n \wedge X)$: by Remark 2.2 and (2.5), $\mathcal{C}(X)$ is the output of a certain right adjoint from G_n-spectra to discrete G_n-spectra applied to $L_{K(n)}(E_n \wedge X)$. Also, by Remark 2.4 and (3.3), $\mathcal{C}(X)$ can be viewed as the homotopy limit in the category of discrete G_n-spectra of a diagram whose homotopy limit in the category of spectra is $L_{K(n)}(E_n \wedge X)$.

We point out that for any spectrum X, in Theorem 1.1 there is an isomorphism

$$E_2^{s,t} \cong \operatorname*{colim}_{N \lhd_o G_n} H^s\Big(G_n/N; \pi_t\Big((L_{K(n)}(E_n \wedge X))^{hN}\Big)\Big),$$

with each G_n/N a finite group, by [**18**, Proposition 8].

The proof of Theorem 1.1 is obtained by combining the equivalence

$$L_{K(n)}(X) \simeq (L_{K(n)}(E_n \wedge X))^{hG_n},$$

which is valid for any X (by [**8**, Theorem 1.1]), with (a version of) the fact that for any profinite group G, the homotopy fixed points of a continuous G-spectrum Z can always be obtained by taking the homotopy fixed points of a certain discrete G-spectrum that is closely related to Z. The proof of this last fact is given in Corollary 2.6, Theorem 2.3 gives the version that is needed for Theorem 1.1 and the remaining details of the proof of Theorem 1.1 are in Section 3. Section 4 contains the proof of Theorem 1.3.

We close this introduction with a comment about notation: We often use '$E_2^{s,t}$' to denote the E_2-term of different spectral sequences, but '$E_r^{*,*}$' only refers to the spectral sequence of Theorem 1.1.

2. Realizing the homotopy fixed points of a continuous G-spectrum by a discrete G-spectrum. Let G be any profinite group. In this section (and in the following two sections), we use the framework of continuous G-spectra that is developed in [**3**] and we refer the reader to this source for additional details.

Let Spt be the simplicial model category of Bousfield–Friedlander spectra and let Spt_G denote the simplicial model category of discrete G-spectra. We use holim and holim^G to denote the homotopy limit, as defined in [**12**, Definition 18.1.8], in Spt and Spt_G respectively. Let $G-\mathrm{Spt}$ be the category of G-spectra (the G-action is not required to be continuous in any sense) and G-equivariant maps of spectra: $G-\mathrm{Spt}$

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is the category of functors $\{*_G\} \to \mathrm{Spt}$, where $\{*_G\}$ is the groupoid associated to G (regarded as an abstract group).

The following definition is a special case of a map that is defined in [**4**, p. 146].

DEFINITION 2.1. Given any

$$\{X_i\}_i = \{\, X_0 \leftarrow X_1 \leftarrow \cdots \leftarrow X_i \leftarrow \cdots\}$$

in $\mathbf{tow}(\mathrm{Spt}_G)$, the category of towers in Spt_G, there is the canonical inclusion map

$$\widetilde{\mathbb{I}}_G\colon\ \operatorname*{holim}_i{}^G X_i \xrightarrow{\cong} \operatorname*{colim}_{N \triangleleft_o G} (\operatorname*{holim}_i X_i)^N \xrightarrow{\mathbb{I}_G} \operatorname*{holim}_i X_i$$

in $G-\mathrm{Spt}$, where the isomorphism is by [**4**, Theorem 2.3]. Note that the source of the map $\widetilde{\mathbb{I}}_G$ is a discrete G-spectrum.

Not surprisingly, the map $\widetilde{\mathbb{I}}_G$ need not be a weak equivalence in Spt: an example of this is given in Remark 3.6. Below we show that when a certain condition is satisfied, the map $\widetilde{\mathbb{I}}_G$, after taking fixed points, induces a weak equivalence between the homotopy fixed points of the source and target of $\widetilde{\mathbb{I}}_G$.

REMARK 2.2. If $Y \in \mathrm{Spt}_G$, then $Y \cong \mathrm{colim}_{N \triangleleft_o G}\, Y^N$ and this isomorphism makes it easy to see that the forgetful functor $\mathbb{U}_G\colon\ \mathrm{Spt}_G \to G-\mathrm{Spt}$ has a right adjoint

$$R_G\colon\ G-\mathrm{Spt} \to \mathrm{Spt}_G, \quad Z \mapsto R_G(Z) = \operatorname*{colim}_{N \triangleleft_o G} Z^N.$$

Thus, for the G-spectrum $\mathrm{holim}_i\, X_i$ of Definition 2.1, there is the isomorphism

$$\operatorname*{holim}_i{}^G X_i \cong R_G(\operatorname*{holim}_i X_i)$$

in Spt_G.

Let $(-)_{fG}\colon\ \mathrm{Spt}_G \to \mathrm{Spt}_G$ be a fibrant replacement functor so that given any Y in Spt_G there is a natural map $Y \xrightarrow{\simeq} Y_{fG}$ that is a trivial cofibration with fibrant target in Spt_G. Then we recall from [**3**] that (a) the homotopy fixed points of Y are defined by

$$Y^{hG} = (Y_{fG})^G;$$

(b) the object $\mathrm{holim}_i\, X_i$, where $\{X_i\}_i \in \mathbf{tow}(\mathrm{Spt}_G)$ with each X_i fibrant as a spectrum, is a *continuous G-spectrum*; and (c) given a continuous G-spectrum $\mathrm{holim}_i\, X_i$,

$$(\operatorname*{holim}_i X_i)^{hG} = \operatorname*{holim}_i (X_i)^{hG}.$$

The proof of the result below follows a script that was used in [**7**, p. 2807] in the context of delta-discrete G-spectra.

THEOREM 2.3. *Given $\{X_i\}_i \in \mathbf{tow}(\mathrm{Spt}_G)$, with each X_i fibrant in Spt_G, the map $(\widetilde{\mathbb{I}}_G)^G$ induces a weak equivalence*

$$(\operatorname*{holim}_i{}^G X_i)^{hG} \xrightarrow{\simeq} (\operatorname*{holim}_i X_i)^{hG}.$$

Proof. In Spt_G, since each X_i is fibrant, $\mathrm{holim}_i^G\, X_i$ is fibrant, and hence there exists (in Spt_G) a weak equivalence $(\mathrm{holim}_i^G\, X_i)_{fG} \xrightarrow{\simeq} \mathrm{holim}_i^G\, X_i$. Thus, there is the weak

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equivalence

$$(\operatorname{holim}_i{}^G X_i)^{hG} \xrightarrow{\simeq} (\operatorname{holim}_i{}^G X_i)^G \xrightarrow{\cong} \Big(\operatorname*{colim}_{N \lhd_o G} (\operatorname{holim}_i X_i)^N\Big)^G \xrightarrow[\cong]{(\mathbb{I}_G)^G} (\operatorname{holim}_i X_i)^G.$$

Also, there is the weak equivalence

$$(\operatorname{holim}_i X_i)^G \xrightarrow{\cong} \operatorname{holim}_i (X_i)^G \xrightarrow{\simeq} \operatorname{holim}_i ((X_i)_{fG})^G = (\operatorname{holim}_i X_i)^{hG}.$$

Composition of the above weak equivalences gives the desired conclusion. □

REMARK 2.4. Let $\{X_i\}_i$ be as in Theorem 2.3 and note that

$$\operatorname{holim}_i{}^G X_i \cong \operatorname*{colim}_{N \lhd_o G} \operatorname{holim}_i (X_i)^N.$$

Since N is open in G, each X_i is fibrant in Spt_N (by [**5**, Lemma 3.1]), and hence there is a weak equivalence

$$\operatorname{holim}_i (X_i)^N \xrightarrow{\simeq} \operatorname{holim}_i ((X_i)_{fN})^N = (\operatorname{holim}_i X_i)^{hN}$$

between fibrant objects in Spt, from the G/N-spectrum $\operatorname{holim}_i (X_i)^N$ to the spectrum $(\operatorname{holim}_i X_i)^{hN}$. Therefore, it is natural to make the identification

$$\operatorname{holim}_i{}^G X_i = \operatorname*{colim}_{N \lhd_o G} (\operatorname{holim}_i X_i)^{hN} \tag{2.5}$$

between the 'spectrum' on the right-hand side and the discrete G-spectrum on the left-hand side. To be more precise, the 'spectrum' on the right-hand side of (2.5) needs to be defined (since, a priori, the spectrum $((X_i)_{fN})^N$ has no G/N-action) and we have shown above that the left-hand side of (2.5) can be taken as its definition (this situation is discussed in more detail in [**6**, p. 260, top of p. 261]).

COROLLARY 2.6. *Let $\{X_i\}_i$ be any object in* **tow**(Spt_G)*. If* $\operatorname{holim}_i X_i$ *is a continuous G-spectrum, then there is a weak equivalence between its homotopy fixed points and those of the discrete G-spectrum* $\operatorname{holim}_i^G (X_i)_{fG}$:

$$(\operatorname{holim}_i X_i)^{hG} \xleftarrow{\simeq} (\operatorname{holim}_i{}^G (X_i)_{fG})^{hG}.$$

Proof. By the proof of Theorem 2.3, there is a weak equivalence

$$(\operatorname{holim}_i{}^G (X_i)_{fG})^{hG} \xrightarrow{\simeq} (\operatorname{holim}_i (X_i)_{fG})^G \xrightarrow{\cong} \operatorname{holim}_i (X_i)^{hG}.$$

□

3. The proof of Theorem 1.1.

We continue to let G be any profinite group. The following result is an immediate consequence of Corollary 2.6 and [**3**, Theorem 7.9].

THEOREM 3.1. *If G has finite virtual cohomological dimension and* $\operatorname{holim}_i X_i$ *is a continuous G-spectrum, then there is a conditionally convergent descent spectral sequence*

$$E_2^{s,t} = H_c^s(G; \pi_t(\operatorname{holim}_i{}^G (X_i)_{fG})) \Longrightarrow \pi_{t-s}((\operatorname{holim}_i X_i)^{hG}),$$

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where the E_2-term is the continuous cohomology of G with coefficients in the discrete G-module $\pi_t(\mathrm{holim}_i^G (X_i)_{fG})$.

REMARK 3.2. The descent spectral sequence of Theorem 3.1 is, in general, different from the 'usual' descent spectral sequence (of [**3**, Theorem 8.8]) for $(\mathrm{holim}_i X_i)^{hG}$. For example, if $\{\pi_t(X_i)\}_i$ satisfies the Mittag–Leffler condition for every integer t, then the latter descent spectral sequence has

$$E_2^{s,t} = H^s_{\mathrm{cts}}(G; \lim_i \pi_t(X_i)),$$

the cohomology of continuous cochains with coefficients in the topological G-module $\lim_i \pi_t(X_i)$, by [**3**, Definition 2.15, Theorem 8.8]. It is not known if, in general, the E_2-term of this latter spectral sequence can be expressed as continuous cohomology (this issue is discussed in [**2**]), whereas in the spectral sequence of Theorem 3.1 the E_2-term is always given by a continuous cohomology group.

Now let X be any spectrum. We consider Theorem 3.1 in the case of the continuous G_n-spectrum $L_{K(n)}(E_n \wedge X)$. As in [**14**, Proposition 4.22], let

$$M_0 \leftarrow M_1 \leftarrow \cdots \leftarrow M_i \leftarrow \cdots$$

be a tower of generalized Moore spectra (each of which is a finite spectrum) such that

$$L_{K(n)}(X) \simeq \mathrm{holim}_i\, (L_n(X) \wedge M_i)_f,$$

where $(-)_f \colon \mathrm{Spt} \to \mathrm{Spt}$ is a fibrant replacement functor. Then, as in [**3**, Lemma 9.1], the Morava module $L_{K(n)}(E_n \wedge X)$ is a continuous G_n-spectrum by the identification

$$L_{K(n)}(E_n \wedge X) = \mathrm{holim}_i\, (F_n \wedge M_i \wedge X)_{fG_n}, \tag{3.3}$$

where F_n is the discrete G_n-spectrum defined in (1.4) and each M_i has the trivial G_n-action.

We have

$$\begin{aligned} L_{K(n)}(X) &\simeq (L_{K(n)}(E_n \wedge X))^{hG_n} \\ &= (\mathrm{holim}_i\, (F_n \wedge M_i \wedge X)_{fG_n})^{hG_n} \\ &\simeq \Big(\operatorname*{colim}_{N \lhd_o G_n} (\mathrm{holim}_i\, (F_n \wedge M_i \wedge X)_{fG_n})^{hN}\Big)^{hG_n} \\ &= \Big(\operatorname*{colim}_{N \lhd_o G_n} (L_{K(n)}(E_n \wedge X))^{hN}\Big)^{hG_n}, \end{aligned}$$

where the first equivalence is by [**8**, Theorem 1.1], the two equalities just apply (3.3), and the second equivalence follows from Theorem 2.3 and (2.5). Thus, we have obtained the following result.

THEOREM 3.4. *For any spectrum X, there is an equivalence*

$$L_{K(n)}(X) \simeq \Big(\operatorname*{colim}_{N \lhd_o G_n} (L_{K(n)}(E_n \wedge X))^{hN}\Big)^{hG_n}.$$

REMARK 3.5. Suppose X is a finite spectrum. Then $E_n \wedge X$ is $K(n)$-local, and hence, there is the chain

$$\begin{aligned}\operatorname*{colim}_{N \lhd_o G_n} (L_{K(n)}(E_n \wedge X))^{hN} &= \operatorname*{colim}_{N \lhd_o G_n} (E_n \wedge X)^{hN} \simeq \operatorname*{colim}_{N \lhd_o G_n} (E_n^{hN} \wedge X) \\ &\simeq \big(\operatorname*{colim}_{N \lhd_o G_n} E_n^{dhN}\big) \wedge X = F_n \wedge X\end{aligned}$$

of equivalent discrete G_n-spectra, where the second step applies [**3**, Theorem 9.9] and the third step uses $E_n^{hN} \simeq E_n^{dhN}$ from [**1**, Theorem 8.2.1]. Therefore, Theorem 3.4 shows that whenever X is a finite spectrum,

$$(F_n \wedge X)^{hG_n} \simeq L_{K(n)}(X);$$

this special case of Theorem 3.4 was previously obtained in [**6**, p. 255].

REMARK 3.6. In Theorem 3.4, when $X = S^0$, Remark 3.5 implies that

$$\operatorname*{colim}_{N \lhd_o G_n} (L_{K(n)}(E_n \wedge X))^{hN} \simeq F_n.$$

Thus, Remark 2.4 shows that the map $\widetilde{\mathbb{1}}_{G_n}$ can be identified with the canonical G_n-equivariant map $F_n \to E_n$, which is not a weak equivalence, by [**3**, Lemma 6.7] (when $n = 1$, this can be seen explicitly from the fact that $\pi_{-1}(F_1) = \mathbb{Q}_p$). Since $\widetilde{\mathbb{1}}_{G_n}$ can be viewed as an inclusion, F_n can be regarded as a discrete G_n-spectrum that is a 'sub-G_n-spectrum' of the continuous G_n-spectrum E_n.

REMARK 3.7. Let X be any spectrum. The equivalence

$$L_{K(n)}(X) \simeq (L_{K(n)}\,(E_n \wedge X))^{hG_n},$$

which was used above, shows that $L_{K(n)}(X)$ is the homotopy fixed points of a continuous G_n-spectrum that is $K(n)$-local and formed from a tower of discrete G_n-spectra. By contrast, Theorem 3.4 says that $L_{K(n)}(X)$ can be realized as the homotopy fixed points of a single discrete G_n-spectrum that need not be $K(n)$-local (for example, F_n is not $K(n)$-local). By [**3**, Lemma 9.6], for each $N \lhd_o G_n$, $(L_{K(n)}(E_n \wedge X))^{hN}$ is $K(n)$-local, and hence $\operatorname{colim}_{N \lhd_o G_n}(L_{K(n)}(E_n \wedge X))^{hN}$ is $E(n)$-local. Thus, Theorem 3.4 shows that an arbitrary $K(n)$-local spectrum is the homotopy fixed points of a discrete G_n-spectrum that is always $E(n)$-local, but not necessarily $K(n)$-local.

Since G_n has finite virtual cohomological dimension, Theorems 3.1 and 3.4 immediately imply that there is a conditionally convergent descent spectral sequence

$$E_2^{s,t} = H_c^s\Big(G_n; \pi_t\Big(\operatorname*{colim}_{N \lhd_o G_n} (L_{K(n)}(E_n \wedge X))^{hN}\Big)\Big) \Longrightarrow \pi_{t-s}(L_{K(n)}(X)),$$

completing the proof of Theorem 1.1.

4. The proof of Theorem 1.3. Throughout this section, we let X be as in Theorem 1.3. By [**10**, Proposition 6.7], spectral sequence $\widetilde{E}_r^{*,*}$ has the form

$$\widetilde{E}_2^{s,t} \cong H_c^s(G_n; \pi_t(E_n \wedge X)) \Longrightarrow \pi_{t-s}(L_{K(n)}(X)),$$

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where

- $E_n \wedge X$ is $K(n)$-local (by [**10**, Lemma 6.11, (i)]);
- each $\pi_t(E_n \wedge X)$ is a discrete G_n-module (see [**10**, Remark 6.8]); and
- the abutment has the stated form since

$$L_{K(n)}(X \wedge E_n^{dhG_n}) \simeq L_{K(n)}(X \wedge L_{K(n)}(S^0)) \simeq L_{K(n)}(X),$$

which is obtained by applying [**10**, Theorem 1, (iii)] (for the first equivalence).

Since $E_n \wedge X$ is $K(n)$-local,

$$E_n \wedge X \simeq \operatorname*{holim}_i (F_n \wedge M_i \wedge X)_{fG_n} \tag{4.1}$$

is a continuous G_n-spectrum and

$$(E_n \wedge X)^{hK} = (L_{K(n)}(E_n \wedge X))^{hK}$$

for every closed subgroup K of G_n (as in the situation considered in [**3**, Remark 9.3]). By [**8**, Theorem 1.2] and [**1**, proof of Theorem 3.2.1], spectral sequence $\widetilde{E}_r^{*,*}$ is isomorphic to the descent spectral sequence

$$E_2^{s,t} = H_c^s(G_n; \pi_t(E_n \wedge X)) \Longrightarrow \pi_{t-s}((E_n \wedge X)^{hG_n}), \tag{4.2}$$

from the E_2-terms onward, where this descent spectral sequence is a special case of the homotopy spectral sequence of [**3**, Theorem 8.8]. (The identification of $E_2^{s,t}$ in (4.2) as continuous cohomology is due to the just-mentioned isomorphism of spectral sequences and the fact that [**10**, Proposition 6.7] identifies $\widetilde{E}_2^{s,t}$ as continuous cohomology.) Also, recall that descent spectral sequence $E_r^{*,*}$ of Theorem 1.1, which is a special case of the homotopy spectral sequence of [**3**, Theorem 7.9], has the form

$$E_2^{s,t} = H_c^s\Big(G_n; \pi_t\Big(\operatorname*{colim}_{N \lhd_o G_n} (E_n \wedge X)^{hN}\Big)\Big) \Longrightarrow \pi_{t-s}((E_n \wedge X)^{hG_n}). \tag{4.3}$$

REMARK 4.4. For each $i \geq 0$, the generalized Moore spectrum M_i has the property that there exists an ideal $I_i \subset BP_*$, such that $BP_*(M_i) \cong BP_*/I_i$ and I_i has the form $(p^{j(i)_0}, v_1^{j(i)_1}, \ldots, v_{n-1}^{j(i)_{n-1}})$, for some n-tuple $(j(i)_0, j(i)_1, \ldots, j(i)_{n-1})$ (see [**9**, p. 762] and [**16**, Proposition 3.7]). Then there are equivalences

$$\begin{aligned} E_n \wedge X &\simeq \operatorname*{colim}_i (E_n \wedge \Sigma^{-(n+\Sigma_{r=0}^{n-1} 2j(i)_r(p-1))} M_i \wedge X) \\ &\simeq \operatorname*{colim}_i (F_n \wedge \Sigma^{-(n+\Sigma_{r=0}^{n-1} 2j(i)_r(p-1))} M_i \wedge X) \end{aligned}$$

of spectra, each of which is G_n-equivariant, where the first equivalence applies [**10**, Lemma 6.11,(ii)]; the second equivalence follows from the fact that the G_n-equivariant map

$$E_n \wedge M_i \xleftarrow{\simeq} F_n \wedge M_i$$

is a weak equivalence of spectra, for each i (this is due to [**10**]; for an explicit proof, see [**3**, Corollary 6.5]); and the third spectrum in the above 'short chain' of equivalences is a discrete G_n-spectrum (since colimits in Spt_G are formed in Spt). Thus, the continuous G_n-spectrum $E_n \wedge X$ can also be regarded as a discrete G_n-spectrum. However, our argument below does not use this conclusion.

From the above observations (preceding Remark 4.4), it is not hard to see that to prove Theorem 1.3, it suffices to show that between the coefficient groups in the E_2-terms of the second and third spectral sequences referred to above (in (4.2) and (4.3) respectively), for each integer t, there is an isomorphism

$$\pi_t(\widetilde{\mathbb{1}}_{G_n})\colon \pi_t\Big(\operatorname*{colim}_{N\lhd_o G_n}(E_n\wedge X)^{hN}\Big)\xrightarrow{\cong}\pi_t(E_n\wedge X)$$

of discrete G_n-modules, where $E_n \wedge X$ is the continuous G_n-spectrum of (4.1).

Since G_n has finite virtual cohomological dimension, there is a cofinal collection $\{U\}$ of open normal subgroups of G_n such that the family $\{\mathrm{cd}(U)\}_U$ of cohomological dimensions are finite and uniformly bounded (for example, see [**3**, proof of Theorem 7.4]): thus, there is an integer r such that $H^s_c(V;P)=0$, whenever $V\in\{U\}$, for all $s>r$ and any discrete V-module P.

For each $N\lhd_o G_n$, there are equivalences

$$\begin{aligned}(E_n\wedge X)^{hN} &= (\operatorname*{holim}_i(F_n\wedge M_i\wedge X)_{fG_n})^{hN}\\ &\simeq \Big(\operatorname*{holim}_i\Big(\operatorname*{colim}_{V\in\{U\}}(E_n^{dhV}\wedge M_i\wedge X)\Big)_{fG_n}\Big)^{hN}\\ &\simeq \Big(\operatorname*{holim}_i\Big(\operatorname*{colim}_{V\in\{U\}}(L_{K(n)}(E_n^{dhV}\wedge X)\wedge M_i)\Big)_{fG_n}\Big)^{hN}\\ &\simeq L_{K(n)}\Big(\Big(\operatorname*{colim}_{V\in\{U\}}L_{K(n)}(E_n^{dhV}\wedge X)\Big)^{hN}\Big),\end{aligned}$$

where the equality (in the first line above) is by [**3**, Definition 9.2], the equivalence in the third line applies [**14**, Lemma 7.2] and the last equivalence is justified as in [**3**, Theorem 9.7], aided by the fact that $\mathrm{colim}_{V\in\{U\}}\, L_{K(n)}(E_n^{dhV}\wedge X)$, a discrete G_n-spectrum, is $E(n)$-local (and hence the spectrum

$$\Big(\operatorname*{colim}_{V\in\{U\}}L_{K(n)}(E_n^{dhV}\wedge X)\Big)^{hN}$$

is $E(n)$-local by [**1**, Theorem 3.2.1 and its proof; proof of Lemma 6.1.5, first sentence]).

To simplify our notation, we make the following conventions.

DEFINITION 4.5. If V is a member of the collection $\{U\}$, we set

$$E_n^{V,X}=L_{K(n)}(E_n^{dhV}\wedge X).$$

Also, '$\mathrm{colim}_U\, E_n^{U,X}$,' for example, means exactly '$\mathrm{colim}_{V\in\{U\}}\, E_n^{V,X}$'.

By [**10**, Proposition 6.7], there is a filtered system of $K(n)$-local E_n-Adams spectral sequences

$$\Big\{H^s_c(U;\pi_t(E_n\wedge X))\Longrightarrow\pi_{t-s}(E_n^{U,X})\Big\}_U.$$

By the uniform bound on $\{\mathrm{cd}(U)\}_U$ and [**17**, Proposition 3.3], the colimit of this diagram of spectral sequences is equal to the spectral sequence

$$E_2^{s,t}=\operatorname*{colim}_U H^s_c(U;\pi_t(E_n\wedge X))\Longrightarrow\pi_{t-s}\big(\operatorname*{colim}_U E_n^{U,X}\big).$$

https://doi.org/10.1017/S001708951300030X Published online by Cambridge University Press

Since $\pi_t(E_n \wedge X)$ is a discrete G_n-module and $\lim_U U = \{e\}$, [**18**, Proposition 8] implies that there is an isomorphism

$$\operatorname*{colim}_U H^s_c(U; \pi_t(E_n \wedge X)) \cong H^s(\{e\}; \pi_t(E_n \wedge X)).$$

Thus, the above colimit spectral sequence collapses, showing that there is an equivalence

$$\operatorname*{colim}_U L_{K(n)}(E_n^{dhU} \wedge X) \simeq E_n \wedge X. \tag{4.6}$$

REMARK 4.7. It will be helpful to be explicit about the equivalence in (4.6). Recall that for each closed subgroup K of G_n, there are equivalences $E_n^{dhK} \simeq E_n^{hK} \simeq L_{K(n)}((F_n)^{hK})$ by [**1**, Theorem 8.2.1] and [**3**, Theorem 9.7] respectively, and by [**5**, Lemma 3.1], a fibrant discrete G_n-spectrum is fibrant in Spt_N for each $N \lhd_o G_n$. Then the equivalence in (4.6) is given by

$$\begin{aligned}
&\operatorname*{colim}_U L_{K(n)}(E_n^{dhU} \wedge X) \simeq \operatorname*{colim}_U L_{K(n)}(E_n^{hU} \wedge X) \simeq \operatorname*{colim}_U L_{K(n)}\big((F_n)^{hU} \wedge X\big)\\
&\quad = \operatorname*{colim}_U L_{K(n)}\big(((F_n)_{fG_n})^U \wedge X\big) \to \operatorname*{colim}_U L_{K(n)}\Big(\big(((F_n)_{fG_n} \wedge X\big)_{fG_n}\big)^U\Big)\\
&\quad \leftarrow \operatorname*{colim}_U L_{K(n)}\Big(\big((F_n \wedge X)_{fG_n}\big)^U\Big) = \operatorname*{colim}_U L_{K(n)}\big((F_n \wedge X)^{hU}\big)\\
&\quad \simeq \operatorname*{colim}_U (L_{K(n)}(E_n \wedge X))^{hU} \cong \operatorname*{colim}_{N \lhd_o G_n} (E_n \wedge X)^{hN} \xrightarrow{\widetilde{\mathbb{I}}_{G_n}} E_n \wedge X,
\end{aligned}$$

where the first expression in the last row comes from applying [**3**, Theorem 9.7]. The above zigzag is useful because it shows that (4.6) is induced by the map $\widetilde{\mathbb{I}}_{G_n}$.

To continue, we need to recall some constructions that are useful in the theory of discrete G-spectra (where G is any profinite group).

DEFINITION 4.8. Given an abelian group A, let $\mathrm{Map}_c(G, A)$ denote the abelian group of continuous functions $G \to A$, where A is equipped with the discrete topology. Given any spectrum Y, let $\mathrm{Map}_c(G, Y)$ be the spectrum with l-simplices of the kth pointed simplicial set $\mathrm{Map}_c(G, Y)_k$ equal to $\mathrm{Map}_c(G, Y_{k,l})$, the set of continuous functions from G to the set $Y_{k,l}$ regarded as a discrete space, for each $k, l \geq 0$. Also, if m is any non-negative integer, then the spectrum $\mathrm{Map}_c(G^{m+1}, Y)$ has the G-action determined by

$$(g \cdot f)(g_1, \ldots, g_{m+1}) = f(g_1 g, g_2, g_3, \ldots, g_{m+1}), \quad f \in \mathrm{Map}_c(G^{m+1}, Y_{k,l}),$$

for $k, l \geq 0$ and $g, g_1, \ldots, g_{m+1} \in G$.

Note that for each $m \geq 0$ and every $N \lhd_o G_n$, there are isomorphisms

$$\begin{aligned}
&\pi_*\Big(\mathrm{Map}_c(G_n^{m+1}, \operatorname*{colim}_U L_{K(n)}(E_n^{dhU} \wedge X))^N\Big)\\
&\qquad \cong \textstyle\prod_{G_n/N} \mathrm{Map}_c\Big(G_n^m, \pi_*\Big(\operatorname*{colim}_U L_{K(n)}(E_n^{dhU} \wedge X)\Big)\Big)\\
&\qquad \cong \textstyle\prod_{G_n/N} \mathrm{Map}_c(G_n^m, \pi_*(E_n \wedge X))\\
&\qquad \cong \pi_*(\textstyle\prod_{G_n/N}(\underbrace{E_n \wedge E_n \wedge \cdots \wedge E_n}_{(m+1)\text{ times}} \wedge X)),
\end{aligned}$$

https://doi.org/10.1017/S001708951300030X Published online by Cambridge University Press

where the first isomorphism is justified as in [**3**, proof of Theorem 7.4], the second isomorphism applies (4.6) and the last isomorphism follows as in [**10**, proof of Proposition 6.7]. Since the spectrum $\prod_{G_n/N}(E_n \wedge E_n \wedge \cdots \wedge E_n \wedge X)$ in the last line above is $K(n)$-local (by [**10**, Lemma 6.11, (i)]), so is the spectrum

$$\mathrm{Map}_c(G_n^{m+1}, \operatorname*{colim}_U L_{K(n)}(E_n^{dhU} \wedge X))^N,$$

and hence it follows from [**3**, Remark 7.13] and the fact that the homotopy limit of a diagram of $K(n)$-local spectra is $K(n)$-local that the spectrum $(\operatorname{colim}_U E_n^{U,X})^{hN}$ is $K(n)$-local. Thus, there is an equivalence

$$(E_n \wedge X)^{hN} \simeq \left(\operatorname*{colim}_U E_n^{U,X}\right)^{hN}. \tag{4.9}$$

For each $N \lhd_o G_n$, there is a weak equivalence

$$\left(\operatorname*{colim}_U E_n^{U,X}\right)_{fG_n} \xrightarrow{\simeq} \left(\operatorname*{colim}_U E_n^{U,X}\right)_{fN}$$

between fibrant objects in Spt_N (the source is fibrant in Spt_N by [**5**, Lemma 3.1]), and hence (4.9) implies that

$$(E_n \wedge X)^{hN} \simeq \left(\left(\operatorname*{colim}_U E_n^{U,X}\right)_{fN}\right)^N \xleftarrow{\simeq} \left(\left(\operatorname*{colim}_U E_n^{U,X}\right)_{fG_n}\right)^N.$$

Therefore, we have

$$\begin{aligned}\operatorname*{colim}_{N \lhd_o G_n} (E_n \wedge X)^{hN} &\cong \operatorname*{colim}_{V \in \{U\}} (E_n \wedge X)^{hV} \simeq \operatorname*{colim}_{V \in \{U\}} \left(\left(\operatorname*{colim}_U E_n^{U,X}\right)_{fG_n}\right)^V \\ &\cong \left(\operatorname*{colim}_U E_n^{U,X}\right)_{fG_n} \xleftarrow{\simeq} \operatorname*{colim}_U E_n^{U,X} \simeq E_n \wedge X,\end{aligned}$$

where the last step uses (4.6). Recall that Remark 4.7 showed that the equivalence in (4.6) is induced by the map $\widetilde{\mathbb{1}}_{G_n}$. Thus, applying $\pi_*(-)$ to the above chain of equivalences completes the proof of Theorem 1.3.

Acknowledgements. The first author thanks Ethan Devinatz for a helpful conversation about the hypothesis of [**10**, Proposition 6.7], and Paul Goerss for a useful discussion on F_1. The second author was partially supported by NSF grant 0805833 and a fellowship from the Sloan Foundation. Both authors thank the referee for several helpful comments.

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