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EQUALIZING THE COEFFICIENTS IN A PRODUCT OF POLYNOMIALS

BY
R. A. MACLEOD AND F. D. K. ROBERTS

1. **Introduction.** In 1959, Moser [4] posed the following problem: how should a pair of n-sided dice be loaded (identically) so that, on throwing the dice, the frequency of the most frequently occurring sum is as small as possible? This can be recast in the following form: determine for each $n(\geq 1)$, the polynomial $P_n(x)$ which minimizes the maximum coefficient in the polynomial $P_n^2(x)$ subject to the conditions that the coefficients of $P_n(x)$ are nonnegative and sum to unity.

In this paper we discuss various problems related to the problem of Moser, and also provide counter-examples to a conjectured solution of Moser's problem due to Clements [2]. The problems considered are the following:

The Conjugacy Problem. Given a polynomial $P_n(x)=p_0+p_1x+\cdots+p_nx^n$, with $\sum_{i=0}^n p_i=1$, determine the polynomial $Q_n(x)=q_0+q_1x+\cdots+q_nx^n$, with $\sum_{i=0}^n q_i=1$, so that the coefficients of the polynomial $R_{2n}(x)=P_n(x)Q_n(x)=r_0+r_1x+\cdots+r_{2n}x^{2n}$ are as nearly equal as possible.

The Minimum Conjugate Pair Problem. Given integers m, $n\geq 1$, determine the polynomials $P_m(x)$ and $Q_n(x)$ with $\sum_{i=0}^m p_i=\sum_{i=0}^n q_i=1$, so that the coefficients of the polynomial $R_{m+n}(x)=P_m(x)Q_n(x)$ are as nearly equal as possible.

The Minimum Square Problem. Given an integer $n\geq 1$, determine the polynomial $P_n(x)$ with $\sum_{i=0}^n p_i=1$ so that the coefficients of the polynomial $R_{2n}(x)=P_n^2(x)$ are as nearly equal as possible.

For each of these three problems, if we denote by k the degree of the product polynomial $R_k(x)$, then since the sum of the coefficients of $R_k(x)$ is also unity, we wish $R_k(x)$ to be close to the polynomial $I_k(x)=1/(k+1)\{1+x+\cdots+x^k\}$. We consider the following three criteria:

$$l_1\text{: minimize} \sum_{j=0}^{k} \left| r_j-\frac{1}{k+1}\right|$$

$$l_2\text{: minimize} \left\{\sum_{j=0}^{k} \left(r_j-\frac{1}{k+1}\right)^2\right\}^{1/2}$$

$$l_\infty\text{: minimize} \max_j \left| r_j-\frac{1}{k+1}\right|.$$

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Existence of solutions to these problems can be demonstrated by a standard compactness argument. However, the solutions are not all necessarily unique (the l_1 and l_∞ norms for example are not strict).

In the following three sections we obtain some analytic results and also discuss some numerical techniques for solving these problems. In the last section, Moser's problem is formulated as a nonlinear programming problem. A numerical technique is developed to obtain local minima for the problem. The minimum values produced by this technique for $n=3, 5, 6, 7, 8, 9$, and 10 are less than the minima suggested by Clements, thus disproving his conjecture.

2. **The conjugacy problem.** Given a polynomial $P_n(x)=p_0+p_1x+\cdots+p_nx^n$, with $\sum_{i=0}^n p_i=1$, we wish to determine a polynomial $Q_n(x)=q_0+q_1x+\cdots+q_nx^n$, with $\sum_{i=0}^n q_i=1$, so that the coefficients of the polynomial $R_{2n}(x)=P_n(x)Q_n(x)=r_0+r_1x+\cdots+r_{2n}x^{2n}$ are as nearly equal as possible (i.e. we wish $R_{2n}(x)$ to be close to the polynomial $I_{2n}(x)=1/(2n+1)\{1+x+\cdots+x^{2n}\}$).

We shall say that $Q_n(x)$ is *conjugate* to the polynomial $P_n(x)$. We show that the determination of the coefficients of $Q_n(x)$ using the l_1 and l_∞ criteria can be accomplished by linear programming, whereas the l_2 problem can be reduced to the solution of a system of linear equations.

We first note that the coefficients of $R_{2n}(x)$ are given in terms of the coefficients of $P_n(x)$ and $Q_n(x)$ by the following matrix equation:

$$\begin{bmatrix} r_0 \\ r_1 \\ \cdot \\ \cdot \\ \cdot \\ r_n \\ \cdot \\ \cdot \\ \cdot \\ r_{2n-1} \\ r_{2n} \end{bmatrix} = \begin{bmatrix} p_0 & 0 & 0 & \cdots & 0 & 0 \\ p_1 & p_0 & 0 & \cdots & 0 & 0 \\ & & & & & \\ & & & & & \\ & & & & & \\ p_n & p_{n-1} & p_{n-2} & \cdots & p_1 & p_0 \\ & & & & & \\ & & & & & \\ & & & & & \\ 0 & 0 & 0 & \cdots & p_n & p_{n-1} \\ 0 & 0 & 0 & \cdots & 0 & p_n \end{bmatrix} \begin{bmatrix} q_0 \\ q_1 \\ \cdot \\ \cdot \\ \cdot \\ q_n \end{bmatrix}.$$

We re-write this as $\mathbf{r}=A_p\mathbf{q}$.

The l_1 criterion. In the l_1 case we wish to minimize $\sum_{j=0}^{2n}|r_j-1/(2n+1)|$ subject to the linear constraints $\mathbf{r}=A_p\mathbf{q}$, $\sum_{i=0}^n q_i=1$. Following Barrodale and Roberts [1], we introduce nonnegative variables u_j, v_j, s_i, t_i, and put $r_j-1/(2n+1)=u_j-v_j$ for $j=0, 1, \ldots, 2n$, and $q_i=s_i-t_i$ for $i=0, 1, \ldots, n$. The minimization may be accomplished by solving the following linear programming problem: minimize $\sum_{j=0}^{2n}(u_j+v_j)$ subject to

$$\mathbf{u}-\mathbf{v}+\frac{1}{2n+1}\mathbf{e} = [A_p \quad -A_p]\begin{bmatrix}\mathbf{s}\\ \mathbf{t}\end{bmatrix}; \qquad \sum_{i=0}^n (s_i-t_i) = 1; \qquad u_j, v_j, s_i, t_i \geq 0.$$

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Here, $\mathbf{u}$ and $\mathbf{v}$ are $(2n+1)$-dimensional column vectors, and $\mathbf{s}$, $\mathbf{t}$ and $\mathbf{e}$ are $(n+1)$-dimensional vectors, $\mathbf{e}$ being a vector of ones.

The l_2 criterion. In the l_2 case, the minimization of $\{\sum_{j=0}^{2n}(r_j-1/(2n+1))^2\}^{1/2}$ is equivalent to the minimization of $\sum_{j=0}^{2n} r_j^2$, since $\sum_{j=0}^{2n} r_j=1$. The constraints are the same as in the previous case, namely $\mathbf{r}=A_p\mathbf{q}$, $\sum_{i=0}^{n} q_i=1$. This is a quadratic programming problem which can be solved by introducing Lagrange multipliers $\mu_0, \mu_1, \ldots, \mu_{2n}$, and λ, and considering the auxiliary function F defined by

$$F=\sum_{j=0}^{2n} r_j^2+\sum_{j=0}^{2n}\mu_j(r_j-A_p^j\mathbf{q})+\lambda\sum_{i=0}^{n}(q_i-1),$$

where A_p^j is the j-th row of A_p. If we set the partial derivatives of F equal to zero and eliminate $\mu_0, \mu_1, \ldots, \mu_{2n}$, the problem reduces to that of solving the system of linear equations

$$\begin{bmatrix} B & \mathbf{e} \\ \mathbf{e}^T & 0 \end{bmatrix}\begin{bmatrix} \mathbf{q} \\ \lambda \end{bmatrix}=\begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix},$$

where $B=A_p^TA_p$ and $\mathbf{e}$ and $\mathbf{0}$ are the $(n+1)$-dimensional column vectors of ones and zeros respectively. We note the matrix A_p is of full rank $n+1$.

The l_∞ criterion. In the l_∞ case we wish to minimize $w=\max_{0\le j\le 2n}|r_j-1/(2n+1)|$ subject to $\mathbf{r}=A_p\mathbf{q}$ and $\sum_{i=0}^{n} q_i=1$. Putting $r_j-1/(2n+1)=u_j-v_j$, $j=0, 1, \ldots, 2n$, and $q_i=s_i-t_i$, $i=0, 1, \ldots, n$, we obtain the following linear programming problem: minimize w subject to $u_j-v_j\le w$ and $-u_j+v_j\le w$ for $j=0, 1, \ldots, 2n$;

$$\mathbf{u}-\mathbf{v}+\frac{1}{2n+1}\mathbf{e}=[A_p \quad -A_p]\begin{bmatrix} \mathbf{s} \\ \mathbf{t} \end{bmatrix}; \qquad \sum_{i=0}^{n}(s_i-t_i)=1; \quad \text{and} \quad u_j, v_j, s_i, t_i, w\ge 0.$$

In practice, it is more efficient computationally to solve the dual formulation of this linear programming problem.

The problem of minimizing the maximum coefficient of $R_{2n}(x)$ subject to the additional constraints $q_i\ge 0$, $i=0, 1, \ldots, n$, can similarly be recast as a linear programming problem. Also, these methods can be used for polynomials $P_m(x)$ and $Q_n(x)$ of differing degrees. Finally, we note that, given polynomials $P_m(x)$ and $R_{m+n}(x)$, the problem of determining the polynomial $Q_n(x)$ so that $P_m(x)Q_n(x)$ best approximates $R_{m+n}(x)$ can also be solved by these techniques.

3. **The minimum conjugate pair problem.** Given integers m, $n\ge 1$, we wish to determine polynomials $P_m(x)$ and $Q_n(x)$, with $\sum_{i=0}^{m} p_i=\sum_{i=0}^{n} q_i=1$, so that the coefficients of the polynomial $R_{m+n}(x)=P_m(x)Q_n(x)$ are as nearly equal as possible. In this problem we wish to determine the coefficients of both $P_m(x)$ and $Q_n(x)$ so that the product is close to the polynomial $I_{m+n}(x)=1/(m+n+1)\{1+x+\cdots+x^{m+n}\}$.

As is well known, any polynomial with real coefficients can be factored into linear and quadratic polynomials with real coefficients. If n and m are both even, we can factor $I_{m+n}(x)$ into $(m+n)/2$ quadratic polynomials, and by selecting any $m/2$

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of these factors as $P_m(x)$, and the remainder as $Q_n(x)$ we have an exact factorization. We can similarly obtain an exact factorization if m is odd and n is even, or vice versa. Thus, the only real problem arises if m and n are both odd. In this case it suffices to consider $m=1$ and $n=2k+1$, since the general case can be obtained from this by transferring quadratic factors from $Q_{2k+1}(x)$ to $P_1(x)$. Thus, we consider the product $(p_0+p_1x)(q_0+q_1x+\cdots+q_{2k+1}x^{2k+1})$.

The l_2 problem is to minimize $\{\sum_{j=0}^{2k+2}(r_j-1/(2k+3))^2\}^{1/2}$, or equivalently to minimize $\sum_{j=0}^{2k+2} r_j^2$, subject to the constraints $r_0=p_0q_0$; $r_j=p_0q_j+p_1q_{j-1}$, $j=1, 2, \ldots, 2k+1$; $r_{2k+2}=p_1q_{2k+1}$; $p_0+p_1=1$; $q_0+q_1+\cdots+q_{2k+1}=1$. This can be accomplished by introducing Lagrange multipliers λ and μ, and considering the auxiliary function F given by

$$F = p_0^2q_0^2+\sum_{j=1}^{2k+1}(p_0q_j+p_1q_{j-1})^2+p_1^2q_{2k+1}^2+\lambda(p_0+p_1-1)+\mu(q_0+q_1+\cdots+q_{2k+1}-1).$$

By setting the partial derivatives of F with respect to p_0 and p_1 equal to zero, we obtain

$$2p_0\sum_{j=0}^{2k+1}q_j^2+2p_1\sum_{j=1}^{2k+1}q_{j-1}q_j+\lambda = 0$$

and

$$2p_0\sum_{j=1}^{2k+1}q_{j-1}q_j+2p_1\sum_{j=0}^{2k+1}q_j^2+\lambda = 0,$$

which, together with the equation $p_0+p_1=1$, imply that $p_0=p_1=\frac{1}{2}$. (It is easy to show that $\sum_{j=0}^{2k+1} q_j^2\neq\sum_{j=1}^{2k+1} q_{j-1}q_j$).

If we set the partial derivatives of F with respect to $q_0, q_1, \ldots, q_{2k+1}$ equal to zero, and include the constraint $\sum_{j=0}^{2k+1} q_j=1$, we obtain the linear system

$$\begin{bmatrix} 2 & 1 & 0 & \cdots & 0 & 0 & 0 & 1 \\ 1 & 2 & 1 & \cdots & 0 & 0 & 0 & 1 \\ 0 & 1 & 2 & \cdots & 0 & 0 & 0 & 1 \\ & & & \cdots & & & & \\ 0 & 0 & 0 & \cdots & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 2 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ \cdots \\ q_{2k} \\ q_{2k+1} \\ 2\mu \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \cdots \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

The solution to these equations is given by

$$q_{2j} = \frac{k+1-j}{(k+1)(k+2)}, \qquad q_{2j+1} = \frac{j+1}{(k+1)(k+2)}, \qquad j = 0, 1, \ldots, k,$$

$$\mu = -\frac{2k+3}{2(k+1)(k+2)}$$

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Thus the solution to the l_2 problem is given by

$$P_1(x) = \tfrac{1}{2}+\tfrac{1}{2}x,$$

$$Q_{2k+1}(x) = \frac{(k+1)+1\cdot x+k\cdot x^2+2\cdot x^3+(k-1)\cdot x^4+3\cdot x^5+\cdots+1\cdot x^{2k}+(k+1)\cdot x^{2k+1}}{(k+1)(k+2)},$$

$$R_{2k+2}(x) = \frac{1}{2k+4}+\frac{x}{2k+2}+\frac{x^2}{2k+4}+\frac{x^3}{2k+2}+\frac{x^4}{2k+4}+\frac{x^5}{2k+2}+\cdots+\frac{x^{2k+1}}{2k+2}+\frac{x^{2k+2}}{2k+4},$$

$$\left\{\sum_{j=0}^{2k+2}\left(r_j-\frac{1}{2k+3}\right)^2\right\}^{1/2} = \frac{1}{2}\left\{\frac{1}{(k+1)(k+2)(2k+3)}\right\}^{1/2}.$$

We note that $Q_{2k+1}(-1)=0$, and hence $Q_{2k+1}(x)$ has a factor $(x+1)$. However, there does not appear to be a convenient analytic technique for obtaining the quadratic factors of $Q_{2k+1}(x)$ in general. A numerical solution may be obtained by using Bairstow's method (see for example Fröberg [3]).

In the l_1 and l_∞ cases, when either m or n is even, we can obtain an exact factorization of $I_{m+n}(x)$ as in the l_2 case. A possible numerical technique when m and n are both odd is to select the coefficients of $P_m^{(0)}(x)$ arbitrarily, and then compute the polynomial $Q_n^{(0)}(x)$ which is conjugate to $P_m^{(0)}(x)$. The polynomial $P_m^{(1)}(x)$ is then chosen to be conjugate to $Q_n^{(0)}(x)$ and the process repeated until the coefficients of the polynomials $P_m^{(k)}(x)$ converge. This algorithm has been tried and used successfully. The disadvantage of the method is that it can converge to local minima and hence a global minimum cannot be guaranteed.

4. **The minimum square problem.** Given an integer n, the minimum square problem is to determine the polynomial $P_n(x)$ with $\sum_{i=0}^n p_i=1$ so that the coefficients of $P_n^2(x)$ are as nearly equal as possible. The following iterative technique has been tried for each of the three criteria. The coefficients $p_0, p_1, \ldots, p_n$ are selected arbitrarily, subject to the condition $\sum_{i=0}^n p_i=1$. We denote this polynomial by $P_n^{(0)}(x)$. The polynomial $Q_n^{(0)}(x)$, which is conjugate to $P_n^{(0)}(x)$, is then computed using the methods of §2. The polynomial $P_n^{(1)}(x)$ is then computed by

$$P_n^{(1)}(x) = \tfrac{1}{2}(P_n^{(0)}(x)+Q_n^{(0)}(x)).$$

This process is repeated until a polynomial $P_n^*(x)$ is produced which is conjugate to itself.

In the l_2 case, this algorithm has proved successful. The method was tried using several starting values for various values of n. The convergence of the method appears to be linear, and the convergence rate is approximately $\tfrac{1}{2}$. In Table I, we list the coefficients of $P_n^*(x)$ (to 6 d.p.) obtained by this method for $n=1, 2, \ldots, 6$. In the l_1 and l_∞ cases the method may not converge for all starting values, and can also converge to local minima which are not necessarily global minima.

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n	Coefficients of $P_n^*(x)$	$\sum_{j=0}^{2n}\left(r_j-\frac{1}{2n+1}\right)^2$
1	0.500000, 0.500000	0.041667
2	0.373439, 0.253122, 0.373439	0.028015
3	0.310460, 0.189540, 0.189540, 0.310460	0.020644
4	0.271252, 0.157514, 0.142469, 0.157514, 0.271252	0.016264
5	0.243883, 0.137479, 0.118638, 0.118638, 0.137479, 0.243883	0.013393
6	0.223398, 0.123472, 0.103654, 0.098952, 0.103654, 0.123472, 0.223398	0.011375

TABLE I. Coefficients of the l_2 minimum square polynomials.

In the l_1 case, we have the following theorem, the proof of which uses a generalization of a method of Moser [5]. This theorem yields a lower bound which is valid for all polynomials of fixed degree.

THEOREM. *For $n=1, 2, \ldots$, and any choice of $p_i \geq 0$, $i=0, 1, \ldots, n$, we have*

$$\sum_{j=0}^{2n}\left|r_j-\frac{1}{2n+1}\right| > \frac{1}{\left(1+\frac{\pi}{2}+\frac{\pi}{4n}\right)^2}.$$

Proof. Writing $\delta_j=r_j-1/(2n+1)$, we have

$$\begin{aligned} P_n^2(x) &= \sum_{j=0}^{2n} r_j x^j = \sum_{j=0}^{2n}\left(\delta_j+\frac{1}{2n+1}\right)x^j \\ &= \sum_{j=0}^{2n}\delta_j x^j+\frac{1}{2n+1}\,\frac{1-x^{2n+1}}{1-x}, \qquad x\neq 1. \end{aligned}$$

Now, let $x_0=e^{2\pi i/(2n+1)}$, $x_k=e^{4\pi ik/(2n+1)}$, $k=1, 2, \ldots, n$. Then

$$P_n^2(x_k) = \sum_{j=0}^{2n}\delta_j x_k^j, \qquad k = 0, 1, \ldots, n,$$

so that

$$(1) \qquad |P_n^2(x_k)| = |P_n(x_k)|^2 \leq \sum_{j=0}^{2n}|\delta_j|, \qquad k = 0, 1, \ldots, n.$$

Also,

$$\begin{aligned} P_n(x_0) &= \sum_{j=0}^{n} p_j x_0^j = \sum_{j=0}^{n} p_j e^{2\pi ji/(2n+1)} \\ &= \sum_{j=0}^{n} p_j \cos\frac{2\pi j}{2n+1}+i\sum_{j=0}^{n} p_j \sin\frac{2\pi j}{2n+1}. \end{aligned}$$

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Writing $\theta=2\pi/(2n+1)$,

$$|P_n(x_0)|^2 = \left(\sum_{j=0}^{n} p_j \cos j\theta\right)^2 + \left(\sum_{j=0}^{n} p_j \sin j\theta\right)^2,$$

and

$$|P_n(x_0)| \geq \left|\sum_{j=0}^{n} p_j \sin j\theta\right|, \qquad 0 \leq j\theta \leq \frac{2\pi n}{2n+1} < \pi$$

or

$$|P_n(x_0)| \geq \sum_{j=0}^{n} p_j \sin j\theta = \sum_{j=0}^{n} p_j\, |\sin j\theta|.$$

Similarly,

$$|P_n(x_k)| \geq \left|\sum_{j=0}^{n} p_j \cos 2kj\theta\right|, \qquad k = 1, 2, \ldots, n.$$

Consider now the Fourier series

$$1 = \frac{\pi}{2}\,|\sin \phi| + 2\sum_{k=1}^{\infty} \frac{\cos 2k\phi}{4k^2-1}, \qquad 0 \leq \phi < \pi.$$

Since

$$\left|2\sum_{k=n+1}^{\infty} \frac{\cos 2k\phi}{4k^2-1}\right| \leq \sum_{k=n+1}^{\infty} \frac{2}{4k^2-1} = \sum_{k=n+1}^{\infty}\left(\frac{1}{2k-1} - \frac{1}{2k+1}\right) = \frac{1}{2n+1},$$

we have

$$1-\frac{1}{2n+1} \leq \frac{\pi}{2}\,|\sin \phi| + 2\sum_{k=1}^{n} \frac{\cos 2k\phi}{4k^2-1} \leq 1+\frac{1}{2n+1}.$$

Taking $\phi=j\theta$, we know $0 \leq j\theta < \pi$, and hence

$$\begin{aligned}
1-\frac{1}{2n+1} &= \sum_{j=0}^{n} p_j\left(1-\frac{1}{2n+1}\right) \leq \sum_{j=0}^{n} p_j\left(\frac{\pi}{2}\,|\sin j\theta| + 2\sum_{k=1}^{n} \frac{\cos 2kj\theta}{4k^2-1}\right) \\
&\leq \frac{\pi}{2}\sum_{j=0}^{n} p_j\, |\sin j\theta| + 2\sum_{k=1}^{n} \frac{1}{4k^2-1}\left|\sum_{j=0}^{n} p_j \cos 2kj\theta\right| \\
&\leq \frac{\pi}{2}\,|P_n(x_0)| + 2\sum_{k=1}^{n} \frac{1}{4k^2-1}\,|P_n(x_k)|.
\end{aligned}$$

Now, suppose that

$$|P_n(x_k)| < \frac{1-\dfrac{1}{2n+1}}{\dfrac{\pi}{2}+1-\dfrac{1}{2n+1}} \quad \text{for} \quad k = 0, 1, \ldots, n.$$

Then

$$1-\frac{1}{2n+1} < \frac{1-\dfrac{1}{2n+1}}{\dfrac{\pi}{2}+1-\dfrac{1}{2n+1}}\left(\frac{\pi}{2} + 2\sum_{k=1}^{n} \frac{1}{4k^2-1}\right) = 1-\frac{1}{2n+1}.$$

Thus, for some value of k,

$$|P_n(x_k)| \geq \frac{1-\dfrac{1}{2n+1}}{\dfrac{\pi}{2}+1-\dfrac{1}{2n+1}} = \frac{1}{1+\dfrac{\pi}{2}+\dfrac{\pi}{4n}},$$

and hence, from (1),

$$\sum_{j=0}^{2n} |\delta_j| \geq \frac{1}{\left(1+\dfrac{\pi}{2}+\dfrac{\pi}{4n}\right)^2}.$$

We note that the right side is an increasing function of n, and for large n is greater than 0.15. Inasmuch as the polynomial of degree 6 with coefficients (0.27395196, 0.07579365, 0.09290929, 0.11469020, 0.09290929, 0.07579365, 0.27395196) yields a value of $\sum_{j=0}^{2n} |\delta_j|$ of 0.23016749, it is unlikely that a method such as that given in this theorem can yield a significantly greater lower bound.

5. **Moser's min-max problem.** Given an integer $n \geq 1$, we wish to determine the polynomial $P_n(x)=p_0+p_1x+\cdots+p_nx^n$, with $\sum_{i=0}^{n} p_i=1$, which minimizes the maximum coefficient of the polynomial $R_{2n}(x)=P_n^2(x)$. In this problem we restrict the coefficients $p_0, p_1, \ldots, p_n$ to be nonnegative.

Clements [2] observes that the coefficients in $[(1-x)^{-1/2}]^2=1+x+x^2+\cdots$ are actually equal, and conjectures that the coefficients $p_0^*, p_1^*, \ldots, p_n^*$ of the minimizing polynomial are in fact the first $(n+1)$ coefficients in the MacLaurin expansion of $(1-x)^{-1/2}$ normed so as to add to 1. That is

$$p_i^* = K(n)\binom{-\frac{1}{2}}{i}(-1)^i, \qquad i = 0, 1, \ldots, n,$$

where

$$K(n) = \left\{\sum_{i=0}^{n} \binom{-\frac{1}{2}}{i}(-1)^i\right\}^{-1} = 2(4)(6)\cdots(2n)/3(5)\cdots(2n+1).$$

(For this choice, the maximum coefficient of $R_{2n}(x)$ is $K^2(n)$.) He has proved this for the cases $n=1, 2$, and shows that this is at least a local minimum for all other values of n. His conjecture that this is a global minimum for all n is false as we shall demonstrate for the cases $n=3, 5, 6, 7, 8, 9$, and 10. The problem can be converted to a nonlinear programming problem as follows.

Let w denote the maximum coefficient in $P_n^2(x)$. We wish to minimize w subject to the constraints $A_p\mathrm{p} \leq w\mathbf{e}$, $\sum_{i=0}^{n} p_i=1$, $p_i \geq 0$, $i=0, 1, \ldots, n$ and $w \geq 0$ where $\mathbf{e}$ is the $2n+1$-dimensional column vector of ones.

This problem may be solved as a sequence of linear programming problems. At the $k+1$st stage, let $p_0^k, p_1^k, \ldots, p_n^k$ be the estimates obtained from the previous stage. The constraints for the nonlinear programming problem contain products such as p_ip_j. If we replace these terms by the linear approximation

$$p_i^kp_j^k+(p_i-p_i^k)p_j^k+(p_j-p_j^k)p_i^k,$$

then we obtain a linear programming problem. The solution of this problem by the simplex method yields new estimates p_i^{k+1}, $i=0, 1, \ldots, n$. This algorithm is not guaranteed convergence from all starting values, and may also converge to a local minimum. However, the algorithm usually converges in practice and the convergence rate is very rapid.

In Table II we list the polynomials $P_n^*(x)$ of degrees one through ten which we obtained by using the above technique repeated for each value of n for twenty different arbitrary starting polynomials $P_n(x)$. For $n=1$, 2 and 4 we reproduce the values of Clements, but for all other values of n which we tried, our method yields smaller maximum coefficients, thus disproving Clements' conjecture.

n	Coefficients of $P_n^*(x)$	Maximum Coefficient of $R^2{}_n(x)$	Clements' Maximum
1	0.333333, 0.666667	0.444444	0.444444
2	0.200000, 0.266667, 0.533333	0.284444	0.284444
3	0.248959, 0.096080, 0.394295, 0.260665	0.205558	0.208980
4	0.111111, 0.126984, 0.152381, 0.203175, 0.406349	0.165120	0.165120
5	0.162011, 0.196836, 0.292118, 0.056782, 0.079349, 0.212904	0.133397	0.136463
6	0.154412, 0.123333, 0.023393, 0.117394, 0.272579, 0.132616, 0.176272	0.113683	0.116276
7	0.117534, 0.133603, 0.161075, 0.236756, 0.040355, 0.049518, 0.069804, 0.191354	0.098694	0.101289
8	0.140374, 0.042319, 0.029668, 0.131689, 0.075393, 0.059902, 0.216415, 0.163547, 0.140693	0.087644	0.089723
9	0.176478, 0.063449, 0.044698, 0.036288, 0.031312, 0.201419, 0.137790, 0.114874, 0.101473, 0.092219	0.078309	0.080527
10	0.106261, 0.076596, 0.118225, 0.101108, 0.195597, 0.080762, 0.010309, 0.050848, 0.029258, 0.098114, 0.132922	0.071035	0.073041

TABLE II. Counter examples to Clements' conjecture.

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UNIVERSITY OF VICTORIA,
VICTORIA, BRITISH COLUMBIA