Canad. J. Math. Vol. **52** (2), 2000 pp. 306–331

Characters of Depth-Zero, Supercuspidal Representations of the Rank-2 Symplectic Group

Clifton Cunningham

Abstract. This paper expresses the character of certain depth-zero supercuspidal representations of the rank-2 symplectic group as the Fourier transform of a finite linear combination of regular elliptic orbital integrals—an expression which is ideally suited for the study of the stability of those characters. Building on work of F. Murnaghan, our proof involves Lusztig's Generalised Springer Correspondence in a fundamental way, and also makes use of some results on elliptic orbital integrals proved elsewhere by the author using Moy-Prasad filtrations of p-adic Lie algebras. Two applications of the main result are considered toward the end of the paper.

Introduction

This paper describes a simple linear relationship between two families of distributions: characters of admissible representations of a p-adic group G and the Fourier transform of orbital integrals on its Lie algebra $\mathfrak{g}$. When π is an admissible representation of G, it defines a trace-class operator π on the space $C_c^\infty(G)$ of locally constant, compactly supported function on G; in this case, the trace $\operatorname{Tr}\pi$ of π is defined, and if we let Θ_π be the locally integrable function representing $\operatorname{Tr}\pi$, then Θ_π is locally constant on the set of regular elements in G, called the character of π. On the other hand, each X in $\mathfrak{g}$ defines a linear functional on the space $C_c^\infty(\mathfrak{g})$ of locally constant, compactly supported functions on $\mathfrak{g}$; if we write $I_G(X,\phi)$ for the orbital integral of ϕ at X, and $\widehat{\mu}_X$ for the locally integrable function representing the Fourier transform of the distribution $I_G(X,\cdot)$, then $\widehat{\mu}_X$ is locally constant on the set of regular elements in $\mathfrak{g}$.

Received by the editors September 25, 1997.
AMS subject classification: Primary: 22E50; secondary: 22E35.

Although received wisdom states that these two families of distributions are closely related, explicit information relating characters and orbital integrals is rare. The local character expansion is a striking example of a such a relationship; it shows that the space of distributions spanned by germs of characters of admissible representations is finite-dimensional by expressing each such character (composed with the exponential map) as a linear combination of the Fourier transform of nilpotent orbital integrals. However, the coefficients appearing in this expansion are notoriously difficult to calculate for reasons that are not likely to be resolved soon. And even in the few cases when it is known how to calculate these coefficients in principle, it is still difficult to answer questions about stability of the character from this expression.

Here, we produce a relationship similar to the local character expansion, but different in some rather important respects. Specifically, when G is the rank-2 p-adic symplectic group, we find a finite set of regular elliptic orbits in $\mathfrak{g}$ with representatives $\mathcal{E}_0$ such that that, for certain depth-zero supercuspidal representations π of G,

$$\Theta_\pi(\exp Y) = \sum_{X \in \mathcal{E}_0} e_X(\pi)\widehat{\mu}_X(Y),$$

for all Y in a large subset $\mathcal{V}$ of the topologically nilpotent elements in $\mathfrak{g}$, defined in 5.5. The coefficients in this expansion, which we dub the *elliptic character expansion* and present as Proposition 6.4, are easily calculated *integers*. Our proof of the elliptic character expansion is constructive: we produce the finite set $\mathcal{E}_0$ and then calculate the integers $e_X(\pi)$ for each X in $\mathcal{E}_0$, for certain supercuspidal representations π. In some sense, the orbits represented by $\mathcal{E}_0$ are naturally associated to depth-zero representations.

The fact that the coefficients $e_X(\pi)$ are easily calculated distinguishes the elliptic character expansion from the local character expansion. More importantly, the fact that we are dealing with regular elliptic orbits means that J.-L. Waldspurger's recent results on stability on the Fourier transform in [Wa.2] can be used to study the stability of the character. This line of thought is briefly explored in Section 7, where we show that certain sums of characters are stable on the set of topologically unipotent elements in G.

More often than not, the elliptic character expansion requires only one non-zero coefficient, in which case we say that the representation is Kirillov, borrowing nomenclature from F. Murnaghan who studied such representations extensively in [Mu]. In Section 3 of this paper we study Kirillov representations using Springer's Hypothesis [K] together with Waldspurger's connection between certain anisotropic tori defined over a finite field and unramified tori defined over a p-adic field [Wa.1]. Surprisingly, even though Section 3 actually carries over to any classical group, none of these results help with the elliptic character expansion for non-Kirillov representations. Sections 4 and 5 introduce ideas which, roughly speaking, take the role played by Waldspurger [Wa.1] in Section 3: Section 4 is inspired by Spaltenstein's treatment [Sp] of Kazhdan and Lusztig's notion of induction of nilpotent orbits using affine Springer fibres [KL]; Section 5 introduces a truncation functor which is inspired by the so-called open sets also found in [KL]. In Section 6, Lusztig's Generalised Springer Correspondence [L.2] is added to the brew—its importance cannot be understated for the present work, neither theoretically nor computationally.

As remarked above, most depth-zero supercuspidal representations are Kirillov. However, as germs, most of these representations have identical characters. In fact, the characters of depth-zero Kirillov representations account for a minority of germs of characters of depth-zero representations. For example, from the 16 germs of characters Θ_π produced as π ranges over the set of all depth-zero supercuspidal representations of the rank-2 symplectic group, only 5 come from Kirillov representations. This phenomenon is not special to low-rank groups—if we consider higher rank symplectic groups, the characters of Kirillov representations account for a minority of the germs of characters of all depth-zero supercuspidal representations. For this reason, we believe that any comprehensive theory of the characters of supercuspidal representations must include non-Kirillov representations in a significant way. It is for these representations that the elliptic character expansion has been designed.

Acknowledgements This paper is based on thesis work completed under the supervision of James Arthur to whom I am hugely indebted—and very happy to thank. It will also be clear that this work is built on results of Fiona Murnaghan, who graciously shared her ideas and time with me. I would also like to thank Joe Repka, who has been a source of inspiration and good advice for many years. This paper is dedicated to the memory of Peter Botta.

0 Notation and Preliminaries

Let F be a p-adic field with ring of integers $\mathfrak{O}$, prime ideal $\mathfrak{p}$ and residue field $\mathbb{F}_q$. Pick a generator ϖ for $\mathfrak{p}$ in $\mathfrak{O}$ and a non-square unit ϵ in $\mathfrak{O}$.

Henceforth, $\mathbf{G}$ will denote Sp_4 as an algebraic group over F and $\mathbf{g}$ will denote its Lie algebra. We will represent $\mathbf{G}$ as elements g of GL_4 such that $g^t Jg = J$, where J has anti-diagonal entries $1, -1, 1, -1$ and 0 elsewhere. Following standard convention, we write G for the group of F-rational points of $\mathbf{G}$, and $\mathfrak{g}$ for the F-rational points of $\mathbf{g}$.

The reader is referred to [MP] for a definition of the subsets $G_{x,r}$ and G_{x,r^+} of G, where x is any point in the building for G and r is a real number. We write M_x for $G_{x,0}/G_{x,0^+}$ and $\rho_x\colon G_x \to M_x$ for the quotient map. We also recall the definition of the lattices $\mathfrak{g}_{x,r}$ and $\mathfrak{g}_{x,r^+}$ in $\mathfrak{g}$. We write $\mathfrak{m}_{x,r}$ for $\mathfrak{g}_{x,r}/\mathfrak{g}_{x,r^+}$ and $\rho_{x,r}\colon \mathfrak{g}_{x,r} \to \mathfrak{m}_{x,r}$ for the quotient map. There is a natural action of M_x on $\mathfrak{m}_{x,r}$. For any function $\varphi\colon \mathfrak{m}_{x,r} \to \mathbb{C}$, let $\varphi_{x,r}$ denote the function on $\mathfrak{g}$ defined as $\varphi \circ \rho_{x,r}$ on $\mathfrak{g}_{x,r}$ and equal to 0 elsewhere. We call $\varphi_{x,r}$ the function produced by inflation from φ.

Let F^{unr} be an unramified closure of F. All the notions of the preceding paragraph make sense for $\mathbf{G}(F^{unr})$. In particular, if x is any point in the building for $\mathbf{G}(F^{unr})$, then the quotient group $\mathbf{M}_x := \mathbf{G}(F^{unr})_{x,0}/\mathbf{G}(F^{unr})_{x,0^+}$ is an algebraic group defined over the algebraic closure of $\mathbb{F}_q$; moreover, $M_x = \mathbf{M}_x(\mathbb{F}_q)$. Likewise, $\mathbf{m}_{x,r} := \mathbf{g}(F^{unr})_{x,r}/\mathbf{g}(F^{unr})_{x,r^+}$ is defined over the algebraic closure of $\mathbb{F}_q$; moreover, $\mathfrak{m}_{x,r} = \mathbf{m}_{x,r}(\mathbb{F}_q)$. The group $\mathbf{M}_x$ acts naturally on $\mathbf{m}_{x,r}$.

To every point x in the building for G there is a function $d_x\colon \mathfrak{g}_{\mathrm{reg}} \to \mathbb{R}$ defined as follows: if $X \in \mathfrak{g}$ is regular, then $d_x(X)$ is the largest real number such that $X \in \mathfrak{g}_{x,d_x(X)}$. We refer to this as the depth of X with respect to x.

Let $\{a_1, a_2\}$ be a basis for the root system for $\mathbf{G}$ (with respect to the torus of diagonal elements in $\mathbf{G}$) such that the Borel subgroup of $\mathbf{G}$ determined by this basis is upper triangular. Let α_1 be the affine root $(a_1, 0)$, let α_2 be the affine root $(a_2, 0)$ and let $\alpha_0 = (-d, 1)$, where d is the dominant root. The Iwahori subgroup $\mathfrak{I}$ of G determined by the basis $\{\alpha_0, \alpha_1, \alpha_2\}$ for the affine root system for G (with respect to the torus of diagonal elements in G), is called the standard Iwahori subgroup. Vertices in the chamber of the building for G corresponding to $\mathfrak{I}$ are called standard vertices and will be denoted x_0, x_1 and x_2 so as to correspond to the roots above and also to the vertices in the Dynkin diagram for G (this imposes a condition on a_1 and a_2). Thus, x_0 and x_2 are special vertices and G_{x_0} and G_{x_2} are hyperspecial maximal parahoric subgroups of G.

When T is a tamely ramified torus in G, the building for T may be identified with a subset of the building for G; in this case, we refer to that subset as "the building for T in G". The reader is referred to [AD] for details.

Let Ψ_F be an additive character of F which is trivial on $\mathfrak{p}$ but not trivial on $\mathfrak{O}$. Let $\Psi_{\mathbb{F}_q}$ be an additive character of $\mathbb{F}_q$ such that $\Psi_F(a) = \Psi_{\mathbb{F}_q}(\bar{a})$ for all $a \in \mathfrak{O}$, where $\bar{a}$ is the image of a under the linear map $\mathfrak{O} \to \mathbb{F}_q$ with kernel $\mathfrak{p}$. Let $\langle \cdot, \cdot \rangle : \mathfrak{g} \times \mathfrak{g} \to F$ be a Killing form on $\mathfrak{g}$ and define $\Psi_{\mathfrak{g}}(\cdot, \cdot) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ by $\Psi_{\mathfrak{g}}(X, Y) := \Psi_F(\langle X, Y \rangle)$. The Fourier transform on the space $C_c^\infty(\mathfrak{g})$ of locally constant, compactly supported functions is defined by $\widehat{\phi}$ of $\phi \in C_c^\infty(\mathfrak{g})$ where

$$\widehat{\phi}(X) := \int_{\mathfrak{g}} \phi(Y) \Psi_{\mathfrak{g}}(X, Y)\, dY.$$

For each regular elliptic X in $\mathfrak{g}$ equip the centraliser $G_X(F)$ of X in G with a normalised Haar measure and write

$$I_G(X, \phi) := \int_{G_X \backslash G} \phi\big(\mathrm{Ad}(g)^{-1} X\big)\, d_X(g),$$

for the orbital integral of $\phi \in C_c^\infty(\mathfrak{g})$ at X, where $d_X(g)$ is the quotient measure. (By 'normalised' we mean unit total measure.) As in the introduction, write $\widehat{\mu}_X$ for the the locally integrable function on $\mathfrak{g}$ representing the Fourier transform of the linear functional $I_G(X, \cdot)$, as defined by $\phi \mapsto I_G(X, \widehat{\phi})$.

In this paper, we suppose that p is large. Among other consequences of this assumption, it follows that the exponential map is a diffeomorphism from the set $\mathfrak{g}^{tn}$ of topologically nilpotent elements in $\mathfrak{g}$ onto the set G^{tu} of topologically unipotent elements of G. Let $C_c^\infty(G^{tu})$ be the set of functions in $C_c^\infty(G)$ supported by topologically unipotent elements in G. Likewise, let $C_c^\infty(\mathfrak{g}^{tn})$ be the set of functions in $C_c^\infty(\mathfrak{g})$ supported by topologically nilpotent elements in $\mathfrak{g}$.

The only non-standard definition in this section in the following: for any two functions ϕ_1 and ϕ_2 in $C_c^\infty(\mathfrak{g})$, let

$$D_G(\phi_1, \phi_2) := \int_G \int_{\mathfrak{g}} \phi_1(Y) \phi_2\big(\mathrm{Ad}(g) Y\big)\, dY\, dg.$$

This definition is purely formal since the integral is often divergent.

1 Depth-Zero Representations and Their Characters

Definition 1.1 A representation π of G is *depth-zero* if there is a point x in the building for G such that such that the space of vectors fixed by $G_{x,0^+}$ is non-trivial.

Definition 1.2 When σ is a representation of M_x, define

$$\pi_x(\sigma) := \mathrm{cInd}_{G_x}^G(\sigma \circ \rho_x).$$

Proposition 1.3 *An irreducible supercuspidal representation of G is depth-zero if and only if it is equivalent to $\pi_x(\sigma)$, where x is a vertex in the building for G and where σ is an irreducible cuspidal representation of M_x.*

Proof For a proof in a notation close to ours, the reader is referred to Section 6.8 of [MP], from which 1.3 follows.

The simple inducing data for a depth-zero supercuspidal representation guaranteed by Proposition 1.3 makes this class of representations easy to study—they are all equivalent to the representations $\pi_x(\sigma)$, where σ is an irreducible cuspidal representation of M_x, and x is a standard vertex, so $x = x_0, x_1$ or x_2. In fact, this paper is concerned with the character $\Theta_{\pi_x(\sigma)}$ on a large subset of the topologically unipotent elements in G, which makes for a further simplification, as the rest of this section shows.

Proposition 1.4 *Let σ be an irreducible cuspidal representation of M_x. Then, for every $f \in C_c^\infty(G)$,*

$$\mathrm{Tr}\,\pi_x(\sigma)(f) = \int_G \int_{G_x} (\mathrm{Tr}\,\sigma \circ \rho_x)(y) f(gyg^{-1})\, dy\, dg,$$

where the measure of G_x with respect to dy and dg is 1.

Proof This is the standard Frobenius form for the character of an induced representation, so, with apologies, the proof is omitted.

The next result shows that if f is supported by topologically unipotent elements, then the behavior of $\mathrm{Tr}\,\sigma$ off the unipotent set in M_x is irrelevant to the distribution. Besides simplifying the inducing data required to describe depth-zero supercuspidal representations even further, this also moves the whole problem to the Lie algebra $\mathfrak{g}$.

Proposition 1.5 *Let σ be an irreducible cuspidal representation of M_x and define $\varphi\colon \mathfrak{m}_{x,0} \to \mathbb{C}$ by $\varphi(\mathfrak{Y}) := (\mathrm{Tr}\,\sigma \circ \exp)(\mathfrak{Y})$ on $\mathfrak{m}_{x,0}^{nilp}$ and* 0 *elsewhere. Then,*

$$\mathrm{Tr}\,\pi_x(\sigma)(f) = D_G(\varphi_{x,0}, \phi),$$

where f is any element of $C_c^\infty(G^{tu})$ and $\phi\colon \mathfrak{g} \to \mathbb{C}$ is defined by $\phi(Y) := (f \circ \exp)(Y)$ on $\mathfrak{g}^{tn}$ and 0 *elsewhere.*

Remark The measures on G and $\mathfrak{g}$ appearing in the definition of D_G are specified in the proof.

Proof In light of Proposition 1.4, it is enough to show that

$$D_G(\varphi_{x,0}, \phi) = \int_G \int_{G_x} (\mathrm{Tr}\,\sigma \circ \rho_x)(y) f(gyg^{-1})\, dy\, dg, \tag{1.5.1}$$

with the measures as in 1.4. Fix g in G and y in G_x and consider the integrand of the right-hand side of (1.5.1). Since the support of f is contained in G^{tu}, we may assume without loss of generality that gyg^{-1} is topologically unipotent, or equivalently, that y is topologically unipotent. Recall that the exponential map defines a diffeomorphism from $\mathfrak{g}^{tn}$ to G^{tu} (since p is large). From the definition of G_x and $\mathfrak{g}_{x,0}$ we see that this restricts to a diffeomorphism from $\mathfrak{g}^{tn} \cap \mathfrak{g}_{x,0}$ to $G^{tu} \cap G_x$. Thus,

$$\int_{G_x} (\operatorname{Tr} \sigma \circ \rho_x)(y) f(gyg^{-1})\, dy = \int_{G^{tu}\cap G_x} (\operatorname{Tr} \sigma \circ \rho_x)(y) f(gyg^{-1})\, dy$$
$$= \int_{\mathfrak{g}^{tn}\cap \mathfrak{g}_{x,0}} (\operatorname{Tr} \sigma \circ \rho_x)(\exp Y) f(g \exp Y g^{-1})\, dY.$$

(Note the condition on dY.) We make use of the following elementary statements which follow from the definitions above: $\rho_x(\exp Y) = \exp \rho_{x,0}(Y)$; $f(g \exp Y g^{-1}) = \phi\big(\operatorname{Ad}(g)Y\big)$ for all $Y \in \mathfrak{g}^{tn}$; and $\operatorname{Tr} \sigma(\exp \mathcal{Y}) = \varphi(\mathcal{Y})$ for all $\mathcal{Y} \in \mathfrak{m}_{x,0}^{nilp}$. Thus,

$$\int_{\mathfrak{g}^{tn}\cap \mathfrak{g}_{x,0}} (\operatorname{Tr} \sigma \circ \rho_x)(\exp Y) f(g \exp Y g^{-1})\, dY = \int_{\mathfrak{g}} \varphi_{x,0}(Y)\phi\big(\operatorname{Ad}(g)Y\big)\, dY,$$

and therefore

$$\int_G \int_{G_x} (\operatorname{Tr} \sigma \circ \rho_x)(y) f(gyg^{-1})\, dy\, dg = \int_G \int_{\mathfrak{g}} \varphi_{x,0}(Y)\phi\big(\operatorname{Ad}(g)Y\big)\, dY.$$

Since the integral on the right-hand side is $D_G(\varphi_{x,0}, \phi)$, this proves the proposition. ∎

Definition 1.6 Let $C_0(M_x)$ denote the set of characters of irreducible cuspidal representations of M_x and let $C_0(M_x^{unip})$ denote the image of $C_0(M_x)$ under the map induced by restriction from M_x to M_x^{unip}.

According to Proposition 1.5, the character of any depth-zero, supercuspidal representation π evaluated $f \in C_c^\infty(G^{tu})$ is of the form $D_G(\varphi_{x,0}, \phi)$, with $x = x_0$, x_1 or x_2, and where $\varphi \circ \log$ is an element of $C^0(M_x^{unip})$, and $\phi := f \circ \exp$ is an element of $C_c^\infty(\mathfrak{g}^{tn})$. Using Tables 8.1 and 8.2 we find there are 16 characters of irreducible, depth-zero, supercuspidal representations on G^{tu}: 5 characters are induced from G_{x_0}, 6 characters are induced from G_{x_1} and 5 characters are induced from G_{x_2}.

2 The Fourier Transform of Elliptic Orbital Integrals

The material in this section, together with the proofs omitted here, will be found in [Cu]. Throughout this section, x denotes an arbitrary point in the building for G and r denotes any real number.

We introduce a Fourier transform sending functions on $\mathfrak{m}_{x,r}$ to functions on $\mathfrak{m}_{x,-r}$ and then relate that notion to the usual Fourier transform on $\mathfrak{g}$ by way of inflation of functions, as defined in Section 0.

Definition 2.1 Let $C(\mathfrak{m}_{x,r})$ be the space of M_x-invariant functions on $\mathfrak{m}_{x,r}$. (That is, $\varphi\colon \mathfrak{m}_{x,r} \to \mathbb{C}$ is an element of $C(\mathfrak{m}_{x,r})$ if $\varphi(m \cdot \mathcal{X}) = \varphi(\mathcal{X})$, for all $m \in M_x$ and $\mathcal{X} \in \mathfrak{m}_{x,r}$.) Let $\Psi_{x,r}$ be a pairing of $\mathfrak{m}_{x,r}$ with $\mathfrak{m}_{x,-r}$ defined by $\Psi_{x,r}(\mathcal{X}, \mathcal{Y}) := \Psi_{\mathfrak{g}}(X, Y)$, where X is any element of $\rho_{x,r}^{-1}(\mathcal{X})$ and Y is any element of $\rho_{x,-r}^{-1}(\mathcal{Y})$. We define a map from $C(\mathfrak{m}_{x,r})$ to $C(\mathfrak{m}_{x,-r})$ by $\varphi \mapsto \hat{\varphi}$, where

$$\hat{\varphi}(\mathcal{Y}) := \sum_{\mathcal{Z} \in \mathfrak{m}_{x,r}} \Psi_{x,r}(\mathcal{Z}, \mathcal{Y})\varphi(\mathcal{Z}).$$

In fact, some work is required to see that Definition 2.1 makes sense. That lacuna is filled by the following proposition.

Proposition 2.2 *Fix $\mathcal{X} \in \mathfrak{m}_{x,r}$ and $\mathcal{Y} \in \mathfrak{m}_{x,-r}$. Then*

$$\Psi_{\mathfrak{g}}(X_1, Y_1) = \Psi_{\mathfrak{g}}(X_2, Y_2),$$

for all $X_1, X_2 \in \rho_{x,r}^{-1}(\mathcal{X})$ and for all $Y_1, Y_2 \in \rho_{x,-r}^{-1}(\mathcal{Y})$.

Proof See [Cu]. The work necessary to prove this simple result may also be found in [AD].

The following result, closely related to 2.2, shows that the Fourier transform (properly interpreted) commutes with inflation, up to a multiple.

Proposition 2.3 *For all $\varphi \in C(\mathfrak{m}_{x,r})$, $\widehat{\varphi_{x,r}} = \mathrm{vol}(\mathfrak{g}_{x,r^+})\, \hat{\varphi}_{x,-r}$.*

Proof This follows easily from 2.1 and 2.2, as presented in [Cu]. (Here, vol refers to the Haar measure on $\mathcal{Y}$.)

Definition 2.4 For any $\mathcal{X} \in \mathfrak{m}_{x,r}(X)$, let $\mathcal{O}_{M_x}(\mathcal{X})$ be the orbit of $\mathcal{X}$ under the action of M_x on $\mathfrak{m}_{x,r}$. Define $\psi_{\mathcal{X}} \in C(\mathfrak{m}_{x,r})$ by

$$\psi_{\mathcal{X}}(\mathcal{Y}) = \begin{cases} |\mathcal{O}_{M_x}(\mathcal{X})|^{-1}, & \text{if } \mathcal{Y} \in \mathcal{O}_{M_x}(\mathcal{X}); \\ 0, & \text{otherwise.} \end{cases}$$

Proposition 2.5 *Let $\mathbf{T}$ be a maximal torus in $\mathbf{G}$, defined over F. Suppose that $X \in \mathrm{Lie}\, T$ is regular and that x is a point in the building for T in G. Set $r = d_x(X)$ and $\mathcal{X} = \rho_{x,r}(X)$. Then, for all Y in the G-orbit of $\mathfrak{g}_{x,-r}$,*

$$(\widehat{\psi_{\mathcal{X}}})_{x,-r}(Y) = \int_{G_x} \Psi_{\mathfrak{g}}\big(\mathrm{Ad}(h)X, Y\big)\, dh.$$

Here, the measure on G_x has been normalised.

Proof This is the main result of [Cu], where it is proved in the context of any connected reductive group **G**.

Remark It is important to notice that the left-hand side depends only on $\mathcal{X} \in \mathfrak{m}_{x,r}$ and therefore that the value taken by the right-hand side would remain unaltered if X were replaced by any regular, elliptic X' in $\rho_{x,r}^{-1}(\mathcal{X})$. That fact provides a very useful characterisation of the "mesh size" of the distribution $\widehat{\mu}_X$ on the G-orbit of $\mathfrak{g}_{x,r}$, as described in [Cu], by way of the following result.

Proposition 2.6 *Let* $\mathbf{T}$ *be an elliptic maximal torus in* $\mathbf{G}$*, defined over F. Let* X*,* x*,* r *and* $\mathcal{X}$ *be as in Proposition 2.5. If* $\phi \in C_c^\infty(\mathfrak{g})$ *is supported by the G-orbit of* $\mathfrak{g}_{x,r}$*, then*

$$I_G(X, \widehat{\phi}) = D_G\big((\widehat{\psi_{\mathcal{X}}})_{x,-r}, \phi\big).$$

Proof This follows easily from 2.5 and is proved in [Cu]. Recall from Section 0 that the measure of the centraliser of X in G appearing in the definition of $I_G(X, \phi)$ has been normalised.

3 The Characters of Uniform Depth-Zero Representations

This section presents F. Murnaghan's formula, as found in [Mu], giving a Kirillov theory for certain depth-zero representations of G. Throughout this section, x denotes any standard vertex of the building for G.

Definition 3.1 A representation of M_x is *Deligne-Lusztig* if it is equivalent to $R^{\mathbf{M}_x}_{\mathbf{T},\theta}$, for some maximal torus $\mathbf{T}$ defined over $\mathbb{F}_q$, and for some character θ of $\mathbf{T}(\mathbb{F}_q)$. (Necessarily, $\mathbf{T}$ must be anisotropic over $\mathbb{F}_q$ and θ must be in general position.)

Proposition 3.2 *Suppose that* $R^{\mathbf{M}_x}_{\mathbf{T}_w,\theta}$ *is an irreducible, cuspidal representation and let* $\pi = \pi_x(R^{\mathbf{M}_x}_{\mathbf{T}_w,\theta})$*. There is a regular elliptic* X_π *such that, for all topologically nilpotent* Y *in* $\mathfrak{g}$*,*

$$\Theta_\pi(\exp Y) = q^{-\operatorname{rank} \mathbf{M}_x} |M_x|\, |\mathbf{T}_w(\mathbb{F}_q)|^{-1} \widehat{\mu}_{X_\pi}(Y).$$

Remark As an equality of germs, this is a special case of the main result in [Mu], proved using Springer's Hypothesis and L. Morris' lattice chains. We briefly re-cast her work here using 2.5, 2.6 and [Wa.1], in part to fit the proof into the framework required for Section 6, but also to show that the equality holds for all topologically nilpotent Y in $\mathfrak{g}$.

Proof Using [Wa.1], find the conjugacy class of elliptic unramified maximal tori in G associated to $\mathbf{T}_w$. Straightforward calculations show that the building for any such torus is conjugate to $\{x\}$; choose one such torus so that its building in G is $\{x\}$ and let $\mathfrak{h}$ be its Lie algebra. Let $\mathcal{X}_w$ be any strongly regular element from Lie $\mathbf{T}_w(\mathbb{F}_q)$ and let X_π be any regular element in the $\rho_{x,0}^{-1}(\mathcal{X}_w) \cap \mathfrak{h}$. (From [Wa.1] we see that this intersection is non-empty. Notice also that $d_x(X_\pi) = 0$ and $\rho_{x,0}(X_\pi) = \mathcal{X}_w$.) By 2.6,

$$I_G(X_\pi, \widehat{\phi}) = D_G\big((\widehat{\psi_{\mathcal{X}_w}})_{x,0}, \phi\big). \tag{3.2.1}$$

for all $\phi \in C_c^\infty\big(\mathcal{O}_G(\mathfrak{g}_x)\big)$. By Springer's Hypothesis [K],

$$|M_x|\,|\mathbf{T}_w(\mathbb{F}_q)|^{-1}\widehat{\psi_{\mathcal{X}_w}}(\mathcal{Y}) = q^{\operatorname{rank}(M_x)} Q^{\mathbf{M}_x}_{\mathbf{T}_w}(\exp \mathcal{Y}),$$

for all nilpotent $\mathcal{Y}$ in $\mathfrak{m}_{x,0}$. Thus,

$$q^{-\operatorname{rank}(M_x)}|M_x|\,|\mathbf{T}_w(\mathbb{F}_q)|^{-1} D_G\big((\widehat{\psi_{\mathcal{X}_w}})_{x,0}, \phi\big) = D_G\big((Q^{\mathbf{M}_x}_{\mathbf{T}_w} \circ \exp)_{x,0}, \phi\big). \tag{3.2.2}$$

Now suppose that ϕ is supported by the G-orbit of $\mathfrak{g}_{x,0}$. Notice that this set contains $\mathfrak{g}^{tn}$. Let f be defined by $f \circ \exp = \phi$. (Recall that p is large, so these conditions uniquely define f.) By 1.4,

$$D_G\big((Q^{\mathbf{M}_x}_{\mathbf{T}} \circ \exp)_{x,0}, \phi\big) = \operatorname{Tr} \pi(f), \tag{3.2.3}$$

where the measure on G appearing in the definition of D_G has been normalised as in 1.4. Putting equations (3.2.1), (3.2.2) and (3.2.3) together gives

$$\operatorname{Tr} \pi(f) = q^{-\operatorname{rank}(M_x)}|M_x|\,|\mathbf{T}_w(\mathbb{F}_q)|^{-1} I_G(X_\pi, \widehat{\phi}). \tag{3.2.4}$$

Since ϕ is supported by $\mathfrak{g}^{tn}$, equation (3.2.4) may be re-written as

$$\Theta_\pi(\exp Y) = q^{-\operatorname{rank}(M_x)}|M_x|\,|\mathbf{T}_w(\mathbb{F}_q)|^{-1}\widehat{\mu}_{X_\pi}(Y),$$

for all $Y \in \mathfrak{g}^{tn}$. ■

Definition 3.3 Suppose that $\mathcal{X} \in \mathfrak{m}_{x,0}$ is strongly regular, semi-simple, and that its centraliser in $\mathbf{M}_x$ is an anisotropic torus. Let $\mathfrak{o}$ denote the $\mathbf{M}_x$-orbit of $\mathcal{X}$ in $\mathfrak{m}_{x,0}$. We say that Lie T is *induced* from $\mathfrak{o}$ if T is in the conjugacy class of elliptic unramified maximal tori in G associated to the centraliser of $\mathcal{X}$ in $\mathbf{M}_x$ by [Wa.1], and if the building for T is in the standard chamber. We also say that an adjoint orbit $\mathcal{O}$ in $\mathfrak{g}$ is *induced* from an adjoint orbit ϑ in $\mathfrak{o}(\mathbb{F}_q)$ if the intersection of $\mathcal{O}$ with $\rho_x^{-1}(\vartheta) \cap \operatorname{Lie} T$ is not empty, where T is induced from $\mathfrak{o}$. We choose a representative for each orbit induced from a strongly regular, semi-simple, anisotropic orbit defined over $\mathbb{F}_q$, and gather these to form the set $\mathcal{E}^{ss}_{x,0}$.

Remark The elements in $\mathcal{E}^{ss}_{x,0}$ are not uniquely defined by this definition, although the orbits they represent are. The reader is referred to Tables 8.3 and 8.4 for an explicit description of one realisation of $\mathcal{E}^{ss}_{x,0}$.

Definition 3.4 Let $C(M_x^{unip})$ denote the space of M_x-invariant functions with unipotent support. A function in $C(M_x^{unip})$ is *uniform* if it is a linear combination of Green's polynomials $Q^{\mathbf{M}_x}_{\mathbf{T}_w}$, where $\mathbf{T}_w$ indicates the split torus twisted by the element w from the Weyl group for $\mathbf{M}_x$. The vector space of uniform functions is denoted $C(M_x^{unip})_{unif}$.

Remark A basis for $C(M_x^{unip})_{unif}$ is the set of all $Q^{\mathbf{M}_x}_{\mathbf{T}_w}$, where w runs over a set of representatives for the conjugacy classes in the Weyl group for $\mathbf{M}_x$. We will use that basis in 3.6.

Proposition 3.5 *If σ is an irreducible cuspidal representation of M_x and if the restriction of* $\operatorname{Tr}\sigma$ *to M_x^{unip} is uniform, then,*

$$\Theta_{\pi_x(\sigma)}(\exp Y) = \sum_{X_w \in \mathcal{E}^{ss}_{x,0}} e_{X_w}\big(\pi_x(\sigma)\big)\widehat{\mu}_{X_\varpi}(Y),$$

for all for all topologically nilpotent Y in $\mathfrak{g}$, where

$$e_{X_w}\big(\pi_x(\sigma)\big) = q^{-\operatorname{rank} M_x} c_w |M_x|\, |\mathbf{T}_w(\mathbb{F}_q)|^{-1},$$

where c_w is given by (3.5.1).

Proof Define $\varphi\colon \mathfrak{m}_{x,0} \to \mathbb{C}$ by $\varphi(\mathcal{Y}) = \operatorname{Tr}\sigma(\exp\mathcal{Y})$ for all nilpotent $\mathcal{Y}$ in $\mathfrak{m}_{x,0}$ and 0 elsewhere. Since the restriction of Tr σ to M_x^{unip} is uniform, there are uniquely defined complex numbers c_w such that

$$\varphi(\mathcal{Y}) = \sum_{w\in(W)} c_w Q^{\mathbf{M}_x}_{\mathbf{T}_w}(\exp\mathcal{Y}), \tag{3.5.1}$$

for all nilpotent $\mathcal{Y}$ in $\mathfrak{m}_{x,0}$, where the summation is over a set of representatives for conjugacy classes in W. Since σ is cuspidal, c_w is zero unless w is elliptic in W, in which case $\mathbf{T}_w$ is anisotropic. From Springer's Hypothesis [K] recall that for any anisotropic maximal torus $\mathbf{T}_w$ there is a strongly regular $\mathcal{X}_w$ in Lie $\mathbf{T}_w(\mathbb{F}_q)$ such that

$$Q^{\mathbf{M}_x}_{\mathbf{T}_w}(\exp\mathcal{Y}) = q^{-\operatorname{rank} M_x}|M_x|\, |\mathbf{T}_w(\mathbb{F}_q)|^{-1}\widehat{\psi_{\mathcal{X}_w}}(\mathcal{Y}), \tag{3.5.2}$$

for all nilpotent $\mathcal{Y}$ in $\mathfrak{m}_{x,0}$. As in the proof of Proposition 3.2 we find that

$$D_G\big((\widehat{\psi_{\mathcal{X}_w}})_{x,0},\phi\big) = I_G(X_w,\widehat{\phi}), \tag{3.5.3}$$

where the orbit of X_w in $\mathfrak{g}$ is induced from the orbit of $\mathcal{X}_w$ in $\mathfrak{m}_{x,0}$. By equation (3.5.1),

$$\varphi(\mathcal{Y}) = q^{-\operatorname{rank}\mathbf{M}_x}|M_x| \sum_{w\in(W)} c_w |\mathbf{T}_w(\mathbb{F}_q)|^{-1}\widehat{\psi_{\mathcal{X}_w}}(\mathcal{Y}). \tag{3.5.4}$$

Without worrying about convergence for the moment, it follows from (3.5.4) that

$$D_G(\varphi_{x,0},\phi) = q^{-\operatorname{rank}\mathbf{M}_x}|M_x| \sum_{w\in(W)} c_w |\mathbf{T}_w(\mathbb{F}_q)|^{-1} D_G\big((\widehat{\psi_{\mathcal{X}_w}})_{x,0},\phi\big), \tag{3.5.5}$$

where ϕ is any element of $C_c^\infty(\mathfrak{g}^{tn})$. By equation (3.5.3),

$$D_G(\varphi_x,\phi) = q^{-\operatorname{rank}\mathbf{M}_x}|M_x| \sum_{w\in(W)} c_w |\mathbf{T}_w(\mathbb{F}_q)|^{-1} I_G(X_w,\widehat{\phi}), \tag{3.5.6}$$

which shows that any concerns concerning convergence of the terms in (3.5.5) were unfounded. By Proposition 1.5 and equation (3.5.6), it follows that

$$\text{(3.5.7)} \qquad \operatorname{Tr} \pi_x(\sigma)(f) = q^{-\operatorname{rank} \mathbf{M}_x} |M_x| \sum_{w \in (W)} c_w |\mathbf{T}_w(\mathbb{F}_q)|^{-1} I_G(X_w, \widehat{\phi}),$$

where $f \in C_c^\infty(G^{tu})$ is defined by $f(\exp Y) = \phi(Y)$ for all $Y \in \mathfrak{g}^{tn}$ and 0 elsewhere. Since $\mathcal{E}_{x,0}^{\text{ss}}$ is precisely the set of all X_w as w runs over a set of representatives for the conjugacy classes of elliptic elements in the Weyl group for $\mathbf{M}_x$, equation (3.5.7) may be re-written as

$$\operatorname{Tr} \pi_x(\sigma)(f) = \sum_{X_w \in \mathcal{E}_{x,0}^{\text{ss}}} q^{-\operatorname{rank} \mathbf{M}_x} c_w |M_x|\, |\mathbf{T}_w(\mathbb{F}_q)|^{-1} I_G(X_w, \widehat{\phi}),$$

which proves the proposition. ■

Remark The coefficients $e_{X_w}\big(\pi_x(\sigma)\big)$ are listed in the first three rows of Tables 8.9 and 8.10, when x is a standard special vertex.

4 Nilpotent Orbits and Induced Cartan Subalgebras

In Section 3 we saw that certain regular elliptic orbits in $\mathfrak{g}$ are naturally associated to strongly regular semi-simple anisotropic orbits in $\mathfrak{m}_{x,0}$, using ideas from [Wa.1]. In this section, we see that certain elliptic orbits in $\mathfrak{g}$ are naturally associated to nilpotent orbits in $\mathfrak{m}_{x,0}$, using [KL] and [Sp]. Throughout this section, x denotes a standard vertex in the building for G.

All the claims made by the fourth paragraph of Section 0 hold when F^{unr} is replaced by any non-Archimedean field. For example, if we were to replace F^{unr} with formal power series $\mathbb{C}((\varpi))$ in a variable ϖ with coefficients in $\mathbb{C}$, then $\mathbf{M}_x$ (respectively, $\mathfrak{m}_{x,0}$) would be an algebraic group (respectively, a Lie algebra) defined over $\mathbb{C}$. In this context, Kazhdan and Lusztig [KL] used affine Springer fibres above topologically nilpotent elements in $\mathfrak{m}_{x,0}$ to define, for each vertex in the building for $\mathbf{G}\big(\mathbb{C}((\varpi))\big)$, a map from nilpotent orbits in $\mathfrak{m}_{x,0}$ to conjugacy classes of Cartan subalgebras in $\mathbf{G}\big(\mathbb{C}((\varpi))\big)$. This section is motivated by Spaltenstein's combinatorial description of that map, as it applies to symplectic groups defined over any complete discrete valuation field with algebraically closed residue field, such as F^{unr}.

Definition 4.1 Let $\mathfrak{o}$ be a nilpotent orbit in $\mathfrak{m}_{x,0}$. We say that an elliptic Cartan $\mathfrak{h}$ is *induced* from $\mathfrak{o}$ if $\mathfrak{h}(F^{unr})$ is the Lie algebra of a maximal torus associated to the centraliser of $\mathcal{X}$ in $\mathbf{M}_x$ by [Sp], and if the building of that torus is a subset of the standard chamber in the building for G. We also say that an orbit $\mathcal{O}$ in $\mathfrak{g}$ is *induced* from an adjoint orbit ϑ in $\mathfrak{o}(\mathbb{F}_q)$ if the intersection of $\mathcal{O}$ with $\rho_{x,0}^{-1}(\vartheta) \cap \mathfrak{h}$ is not empty, where $\mathfrak{h}$ is induced from $\mathfrak{o}$. We choose a representative for each orbit induced from a nilpotent orbit in $\mathfrak{m}_{x,0}$ and gather these to form the set $\mathcal{E}_{x,0}^{nilp}$. The set of elements in $\mathcal{E}_{x,0}^{nilp}$ which are induced from nilpotent orbits not appearing in the (classical) Springer correspondence is denoted $\mathcal{E}_{x,0}^{nilp\,\prime}$.

Remark Although the elements in $\mathcal{E}_{x,0}^{nilp}$ are not uniquely defined by this definition, although the orbits they represent are. The reader is referred to Tables 8.5 and 8.6 for an explicit description of one realisation of $\mathcal{E}_{x,0}^{nilp\,\prime}$.

Definition 4.2 For any subset U of $\mathfrak{m}_{x,0}^{nilp}$, define

$$\mathcal{E}_{x,0}(U) := \mathcal{E}_{x,0}^{nilp} \cap \rho_{x,0}^{-1}(U).$$

Definition 4.3 For each nilpotent orbit $\mathfrak{o}$ in $\mathbf{m}_{x,0}$, the building for the centraliser of X in G is one and the same point for every X in $\mathcal{E}_{x,0}\big(\mathfrak{o}(\mathbb{F}_q)\big)$. Since this point is naturally associated to $\mathfrak{o}$ and x, we denote this point by $y_{\mathfrak{o},x}$. Likewise, for each nilpotent orbit $\mathfrak{o}$ in $\mathbf{m}_{x,0}$, the number $d_{y_{\mathfrak{o},x}}(X)$ is the same for every X in $\mathcal{E}_{x,0}\big(\mathfrak{o}(\mathbb{F}_q)\big)$. Since this number is naturally associated to $\mathfrak{o}$ and x, we call it $s_{\mathfrak{o},x}$.

Remark For each standard vertex x, we have defined a map sending a nilpotent orbit $\mathfrak{o}$ to the point $y_{\mathfrak{o},x}$, $s_{\mathfrak{o},s}$ in the building for G crossed with $\mathbb{R}$; Tables 8.7 and 8.8 record the values of that map when x is special. Moreover, this map does not depend on the choices we made in the definition of $\mathcal{E}_{x,0}^{nilp}$. In fact this map admits a simple combinatorial description, for any connected reductive group, which can be used to define $\mathcal{E}_{x,0}^{nilp}$ without ambiguity, and also to characterise the "open subsets" of [KL], without reference to affine Springer fibres or to the Newton polygons of [Sp]. That perspective will be explored elsewhere.

5 Some Technical Results

This section carries most of the weight of this paper. With apologies to the reader, we use musical notation to define what is essentially a truncation functor. For each function ϕ in $C_c^\infty(\mathfrak{g}^{tn})$ we define a new function $\phi^\natural$ in $C_c^\infty(\mathfrak{g}^{tn})$. The main result is Proposition 5.8, which compares the Fourier transform of $\phi^\natural$ with the Fourier transform of ϕ, when ϕ is produced by inflation from the characteristic function of a local system on $\mathbf{m}_{x,0}$. In this section, x is a special standard vertex in the building for G.

Definition 5.1 To simplify notation somewhat, when $\mathfrak{o}$ is a nilpotent orbit it $\mathbf{m}_{x,0}$ and when X is induced from $\mathfrak{o}$, we write $y(X)$ for $y_{\mathfrak{o},x}$, $s(X)$ for $s_{\mathfrak{o},x}$ and $\bar{X}$ for $\rho_{y(X),s(X)}(X)$. For each $\phi \in C_c^\infty(\mathfrak{g}^{tn})$, define $\phi^\flat \in C_c^\infty(\mathfrak{g}^{tn})$ by

$$\phi^\flat(Y) = \sum_{X \in \mathcal{E}(\mathfrak{m}_{x,0}^{nilp})} \phi(X)\,\mathrm{vol}(\mathfrak{g}_{y(X),s(X)^+})^{-1}(\psi_{\bar{X}})_{y(X),s(X)}(Y).$$

Here, vol refers to the Haar measure on $\mathfrak{Y}$.

Remark The fact that each $X \in \mathcal{E}_{x,0}^{nilp}$ is induced from exactly one nilpotent orbit in $\mathbf{m}_{x,0}$ follows, with a small amount of work, from the definition of $\mathcal{E}_{x,0}^{nilp}$. Also, recall the definition of $\psi_{\bar{X}} \in C(\mathfrak{m}_{x,0}^{nilp})$ from 2.4.

Definition 5.3 Let $\mathcal{N}(\mathbf{m}_{x,0}^{nilp})$ denote the set of $\mathbf{M}_x$-equivariant, irreducible l-adic étale local systems on $\mathbf{m}_{x,0}^{nilp}$, where l is a fixed prime not dividing p. Also, let $\mathcal{N}(\mathbf{m}_{x,0}^{nilp})_{unif}$ denote the set of $\mathbf{M}_x$-equivariant, irreducible local systems on $\mathbf{m}_{x,0}^{nilp}$ which appear in the (classical)

Springer correspondence. Finally, let $\mathcal{N}(\mathfrak{m}_{x,0}^{nilp})'$ denote the complement of $\mathcal{N}(\mathfrak{m}_{x,0}^{nilp})_{unif}$ in $\mathcal{N}(\mathfrak{m}_{x,0}^{nilp})$.

Remark In the case at hand, there are exactly two such local systems $\mathcal{L}_{\text{reg}}$ and $\mathcal{L}_{\text{min}}$ in in $\mathcal{N}(\mathfrak{m}_{x,0}^{nilp})'$: $\mathcal{L}_{\max}$ is the non-constant local system supported by the nilpotent orbit $\mathfrak{o}_{\max}$ corresponding to the symplectic partition (4), and $\mathcal{L}_{\min}$ is the non-constant local system supported by the nilpotent orbit $\mathfrak{o}_{\min}$ corresponding to the symplectic partition $(2, 1^2)$.

Definition 5.4 For each $\mathcal{L}$ in $\mathcal{N}(\mathfrak{m}_{x,0}^{nilp})$, let $[\mathcal{L}]$ denote the characteristic function of $\mathcal{L}$ with respect to an $\mathbb{F}_q$-rational structure chosen according to Section 25 of [L], composed with a fixed embedding $\overline{\mathbb{Q}_l} \hookrightarrow \mathbb{C}$.

Definition 5.5 Let $\mathcal{V}$ be the set of all topologically nilpotent regular semi-simple Y in $\mathfrak{g}$ such that the affine Springer fibre above Y contains a line of type α_0, as defined in [KL].

Proposition 5.6 *If φ is the characteristic function of a local system in $\mathcal{N}(\mathfrak{m}_{x,0}^{nilp})'$, then*

$$\operatorname{supp}(\widehat{\varphi_{x,0}^{\flat}}) \cap \mathcal{V} \subseteq \mathfrak{g}_{x,0}.$$

Proof We will suppose that $Y \in \mathcal{V}$ is not an element of $\mathfrak{g}_{x,0}$ and show that $\widehat{\varphi_{x,0}^{\flat}}(Y) \neq 0$ leads to a contradiction. We suppose that x is the special vertex x_0—the argument given below adapts easily to the case of any other special standard vertex.

Let φ be the characteristic function of $\mathcal{L}$, and let $\mathfrak{o}$ be the support of $\mathcal{L}$. Set $y = y_{\mathfrak{o},x}$ and $s = s_{\mathfrak{o},s}$. From Definition 5.5 and values in Table 8.7, it can be seen that $\mathcal{V}$ is a subset of $\mathcal{O}_G(\mathfrak{g}_{y,-s})$, in each case considered below. Then, from Proposition 2.3 and Definitions 4.3 and 5.1, it follows that

$$\widehat{\varphi_{x,0}^{\flat}}(Y) = \sum_{X \in \mathcal{E}_{x,0}(\mathfrak{o}(\mathbb{F}_q))} \varphi_{x,0}(X)(\widehat{\psi_{\bar{X}}})_{y,-s}(Y), \tag{5.6.1}$$

where $\bar{X} = \rho_{y,s}(X)$. Since we have assumed that $\widehat{\varphi_{x,0}^{\flat}}(Y) \neq 0$, from (5.6.1) it follows that Y must be an element of $\mathfrak{g}_{y,-s}$. Let $\mathcal{Y} = \rho_{y,-s}(Y)$; then

$$\widehat{\psi_{\bar{X}}}(\mathcal{Y}) = |M_y|^{-1} \sum_{m \in M_y} \Psi_{y,s}(m \cdot \bar{X}, \mathcal{Y}). \tag{5.6.2}$$

Below, we describe this sum for each local system in $\mathcal{N}(\mathfrak{m}_{x,0}^{nilp})'$, and then use that description to evaluate $\widehat{\varphi_{x,0}^{\flat}}(Y)$.

Suppose that φ is the characteristic function of $\mathcal{L}_{\max}$. The support of φ is the support of $\mathcal{L}_{\max}$, which is $\mathfrak{o}(\mathbb{F}_q)$, where $\mathfrak{o}$ is the nilpotent orbit in $\mathfrak{m}_{x,0}$ corresponding to the symplectic partition (4). (Note that $\mathfrak{o}(\mathbb{F}_q)$ is a union of two nilpotent orbits in $\mathrm{m}_{x,0}$.) Let $y = y_{\mathfrak{o},x}$ and let $s = s_{\mathfrak{o},x}$. From Table 8.5 we find eight elements in $\mathcal{E}_{x,0}\big(\mathfrak{o}(\mathbb{F}_q)\big)$:

$$X_{(4)}^{i,j} = \begin{pmatrix} 0 & x & 0 & 0 \\ 0 & 0 & x\epsilon^i & 0 \\ 0 & 0 & 0 & x \\ x\varpi\epsilon^j & 0 & 0 & 0 \end{pmatrix} + X', \tag{5.6.3}$$

where $i = 0, 1$ and $j = 0, 1, 2$ or 3. Without loss of generality, set $x = 1$. Note that in the case at hand, $\varphi_x(X^{i,j}_{(4)}) = \operatorname{sign}(\epsilon^i) = (-1)^i$. Recall y from Table 8.7; the filtration of $\mathfrak{g}$ associated to y is the period-4 lattice chain

$$\cdots \supset \mathfrak{g}_{y,0} \supset \mathfrak{g}_{y,\frac{1}{4}} \supset \mathfrak{g}_{y,\frac{1}{2}} \supset \mathfrak{g}_{y,\frac{3}{4}} \supset \mathfrak{g}_{y,1} = \varpi\mathfrak{g}_{y,0} \supset \cdots$$

where

$$\mathfrak{g}_{y,0} = \left\{ \begin{pmatrix} \mathfrak{O} & \mathfrak{O} & \mathfrak{O} & \mathfrak{O} \\ \mathfrak{p} & \mathfrak{O} & \mathfrak{O} & \mathfrak{O} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{O} & \mathfrak{O} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{O} \end{pmatrix} \right\}, \quad \mathfrak{g}_{y,\frac{1}{4}} = \left\{ \begin{pmatrix} \mathfrak{p} & \mathfrak{O} & \mathfrak{O} & \mathfrak{O} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{O} & \mathfrak{O} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{O} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{pmatrix} \right\},$$

$$\mathfrak{g}_{y,\frac{1}{2}} = \left\{ \begin{pmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{O} & \mathfrak{O} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{O} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{pmatrix} \right\}, \quad \mathfrak{g}_{y,\frac{3}{4}} = \left\{ \begin{pmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{O} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} \end{pmatrix} \right\}.$$

(From this and (5.6.3) we see that $s = \frac{1}{4}$; *cf.* Table 8.7.) Write

$$(5.6.4) \qquad M_y = \left\{ \begin{pmatrix} t_1 & 0 & 0 & 0 \\ 0 & t_2 & 0 & 0 \\ 0 & 0 & t_2^{-1} & 0 \\ 0 & 0 & 0 & t_1^{-1} \end{pmatrix} \right\} \quad \text{and} \quad \mathcal{Y} = \begin{pmatrix} 0 & 0 & 0 & \varpi^{-1}u \\ v_1 & 0 & 0 & 0 \\ 0 & v_2 & 0 & 0 \\ 0 & 0 & v_1 & 0 \end{pmatrix},$$

where t_1, t_2, v_1, v_2 and u are elements of $\mathbb{F}_q$. From the fact that $Y \in \mathcal{V}$ and $Y \neq \mathfrak{g}_{x,0}$ it follows that $u \neq 0$ and $v_1 v_2 = 0$. If $X = X^{i,j}_{(4)}$ is any element in $\mathcal{E}_{x,0}\big(\mathbf{o}(\mathbb{F}_q)\big)$ (*cf.* (5.6.3)), then

$$(5.6.5) \qquad \widehat{\psi_{\bar{X}}}(\mathcal{Y}) = (q-1)^{-2} \sum_{t_1, t_2 \in \mathbb{F}_q{}^\times} \Psi_{\mathbb{F}_q}(2v_1 t_1 t_2^{-1} + \epsilon^i v_2 t_2^2 + \epsilon^j u t_1^2).$$

To evaluate $\widehat{\varphi^\flat_{x,0}}(Y)$, consider the following cases separately: i) $v_1 \neq 0$ and $v_2 = 0$; ii) $v_1 = 0$ and $v_2 \neq 0$.

i) Suppose $v_1 \neq 0$ and $v_2 = 0$. Then, combining (5.6.1) and (5.6.1) gives

$$\begin{aligned} \widehat{\varphi^\flat_{x,0}}(Y) &= \sum_{X \in \mathcal{E}_{x,0}(\mathbf{o}(\mathbb{F}_q))} \varphi_{x,0}(X) \widehat{\psi_{\bar{X}}}(\mathcal{Y}) \\ &= (q-1)^{-2} \sum_{i,j} (-1)^i \sum_{t_1, t_2 \in \mathbb{F}_q{}^\times} \Psi_{\mathbb{F}_q}(2v_1 t_1 t_2^{-1} + \epsilon^j u t_1^2) \\ &= (q-1)^{-2} \sum_{i=0,1} (-1)^i \sum_{j, t_1, t_2} \Psi_{\mathbb{F}_q}(2v_1 t_1 t_2^{-1} + \epsilon^j u t_1^2) \\ &= 0. \end{aligned}$$

This is the desired contradiction for this case.

ii) Suppose $v_1 = 0$ and $v_2 \neq 0$. From the description of the lattice chain above, it is clear that Y must lie in

$$\mathcal{M} := \begin{Bmatrix} \mathfrak{O} & \mathfrak{O} & \mathfrak{O} & \varpi^{-1}\mathfrak{O}^* \\ \mathfrak{p} & \mathfrak{O} & \mathfrak{O} & \mathfrak{O} \\ \mathfrak{p} & \mathfrak{O}^* & \mathfrak{O} & \mathfrak{O} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{O} \end{Bmatrix} \cap \mathfrak{g},$$

which is itself contained in the maximal parahoric $\mathfrak{g}_{x_1,0}$. Since Y is topologically nilpotent, its image under $\rho_{x_1,0}$ must be nilpotent. Since $\rho_{x_1,0}(Y)$ is regular nilpotent, there must be some $k \in G_{x_1}$ such that $\mathrm{Ad}(k)Y$ is actually an element of

$$\mathcal{M}' := \begin{Bmatrix} \mathfrak{p} & \mathfrak{O} & \mathfrak{O} & \varpi^{-1}\mathfrak{O}^* \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{O} \\ \mathfrak{p} & \mathfrak{O}^* & \mathfrak{p} & \mathfrak{O} \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{Bmatrix} \cap \mathfrak{g}.$$

Now we see that there is a $w \in W$ such that $\mathrm{Ad}(n_w)$ sends $\mathcal{M}'$ to

$$\mathcal{M}'' := \begin{Bmatrix} \mathfrak{p} & \mathfrak{O} & \mathfrak{O} & \mathfrak{O} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{O}^* & \mathfrak{O} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{O} \\ \varpi\mathfrak{O}^* & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{Bmatrix} \cap \mathfrak{g},$$

which is a subset of the standard Iwahori subalgebra of $\mathfrak{g}$. Let Y'' denote the image of Y under $\mathrm{Ad}(n_w k)$. Use the line bundles from Section 4 of [KL] to see that the affine Springer fibre above Y'' does not contain the projective line $\mathbb{P}^1_{\alpha_0}$, in the notation of [KL]. Thus, the affine Springer fibre above Y does not contain this line, which contradict the assumption that $Y \in \mathcal{V}$ (*cf.* 5.5). This is the desired contradiction in this case.

Next, suppose that φ is the characteristic function of $\mathcal{L}_{\min}$. The support of φ is the support of $\mathcal{L}_{\min}$, which is $\mathfrak{o}(\mathbb{F}_q)$, where $\mathfrak{o}$ is the nilpotent orbit in $\mathfrak{m}_{x,0}$ corresponding to the symplectic partition $(2, 1^2)$. (Note that $\mathfrak{o}(\mathbb{F}_q)$ is a union of two nilpotent orbits in $\mathfrak{m}_{x,0}$.) From Table 8.5 we find four elements in $\mathcal{E}_{x,0}\big(\mathfrak{o}(\mathbb{F}_q)\big)$:

(5.6.6)
$$X^{i,j}_{(2,1^2)} = \begin{pmatrix} 0 & 0 & 0 & x\epsilon^i \\ 0 & 0 & y & 0 \\ 0 & y\epsilon & 0 & 0 \\ x\varpi\epsilon^j & 0 & 0 & 0 \end{pmatrix} + X',$$

where $i = 0, 1$ and $j = 0, 1$ and $y \in \varpi\mathfrak{O}^\times$ is arbitrary. Without loss of generality, set $x = 1$. Note that in the case at hand, $\varphi_x(X^{i,j}_{(2,1^2)}) = \mathrm{sign}(\epsilon^i) = (-1)^i$. Recall y from Table 8.7; the filtration of $\mathfrak{g}$ associated to y is the period-4 lattice chain

$$\cdots \supset \mathfrak{g}_{y,0} \supset \mathfrak{g}_{y,\frac{1}{4}} \supset \mathfrak{g}_{y,\frac{1}{2}} \supset \mathfrak{g}_{y,\frac{3}{4}} \supset \mathfrak{g}_{y,1} = \varpi\mathfrak{g}_{y,0} \supset \cdots$$

where

$$\mathfrak{g}_{y,0} = \left\{ \begin{pmatrix} \mathfrak{O} & \mathfrak{O} & \mathfrak{O} & \mathfrak{O} \\ \mathfrak{p} & \mathfrak{O} & \mathfrak{O} & \mathfrak{O} \\ \mathfrak{p} & \mathfrak{O} & \mathfrak{O} & \mathfrak{O} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{O} \end{pmatrix} \right\}, \quad \mathfrak{g}_{y,\frac{1}{4}} = \left\{ \begin{pmatrix} \mathfrak{p} & \mathfrak{O} & \mathfrak{O} & \mathfrak{O} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{O} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{O} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{pmatrix} \right\},$$

$$\mathfrak{g}_{y,\frac{1}{2}} = \left\{ \begin{pmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{O} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{pmatrix} \right\}, \quad \mathfrak{g}_{y,\frac{3}{4}} = \left\{ \begin{pmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{pmatrix} \right\}.$$

(From this and (5.6.6) it is clear that $s_{\mathfrak{o},x} = \frac{1}{2}$; *cf.* Table 8.7.) Write

$$M_y = \left\{ \begin{pmatrix} t & 0 & 0 & 0 \\ 0 & * & * & 0 \\ 0 & * & * & 0 \\ 0 & 0 & 0 & t^{-1} \end{pmatrix} \right\} \quad \text{and} \quad \mathcal{Y} = \begin{pmatrix} 0 & 0 & 0 & \varpi^{-1}u \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ v & 0 & 0 & 0 \end{pmatrix}, \tag{5.6.7}$$

where t, u and v are elements of $\mathbb{F}_q$. (The inner, starred matrix is an element of $\mathrm{SL}_2(\mathbb{F}_q)$.) From the fact that $Y \in \mathcal{V}$ and Y is not an element of $\mathfrak{g}_{x,0}$, it follows that $u \neq 0$ and $v = 0$. If $X = X^{i,j}_{(2,1^2)}$ is any element in $\mathcal{E}_{x,0}\big(\mathfrak{o}(\mathbb{F}_q)\big)$ (*cf.* (5.6.6)), then

$$\widehat{\psi_{\bar{X}}}(\mathcal{Y}) = (q-1)^{-1} \sum_{t \in \mathbb{F}_q{}^\times} (-1)^i \Psi_{\mathbb{F}_q}(\epsilon^i v t^2 + \epsilon^j u t^{-2}). \tag{5.6.8}$$

Combining (5.6.1) and (5.6.8) we have

$$\begin{aligned} \widehat{\varphi^\flat_{x,0}}(Y) &= \sum_{X \in \mathcal{E}_{x,0}(\mathfrak{o}(\mathbb{F}_q))} \varphi_{x,0}(X) \widehat{\psi_{\bar{X}}}(\mathcal{Y}) \\ &= (q-1)^{-1} \sum_{i,j=0,1} (-1)^i \sum_{t \in \mathbb{F}_q{}^\times} \Psi_{\mathbb{F}_q}(\epsilon^j u t^{-2}) \\ &= (q-1)^{-1} \sum_{i=0,1} (-1)^i \sum_{j=0,1} \sum_{t \in \mathbb{F}_q{}^\times} \Psi_{\mathbb{F}_q}(\epsilon^j u t^{-2}) \\ &= 0. \end{aligned}$$

This is the desired contradiction for this case. ∎

Definition 5.7 For each $\phi \in C_c^\infty(\mathfrak{g}^{tn})$, define $\phi^\natural \in C_c^\infty(\mathfrak{g}^{tn})$ by

$$\phi^\natural = \int_{G_x} \phi^\flat\big(\mathrm{Ad}(k)Y\big)\, dk.$$

(Recall the definition of $\phi^\flat$ from 5.1.)

Proposition 5.8 *If φ is the characteristic function of a local system in $\mathcal{N}(\mathbf{m}^{nilp}_{x,0})'$, then, for all $Y \in \mathcal{V}$,*

$$\hat{\varphi}_{x,0}(Y) = |\operatorname{supp}\varphi|\, |\mathcal{E}_{x,0}(\operatorname{supp}\varphi)|^{-1}\, \widehat{\varphi^\natural_{x,0}}(Y).$$

Proof Begin with an elementary observation: for any $\phi \in C_c^\infty(\mathfrak{g}^{tn})$ and any $Z \in \mathfrak{g}$,

$$\widehat{\phi^\natural}(Z) = \int_{G_y} \widehat{\phi^\flat}\big(\mathrm{Ad}(k)Z\big)\, dk. \tag{5.8.1}$$

Now suppose that $\phi = \varphi_x$ and also that $Y \in \mathcal{V}$. Let φ be the characteristic function of $\mathcal{L}$, and let $\mathbf{o}$ be the support of $\mathcal{L}$, so the support of φ is $\mathbf{o}(\mathbb{F}_q)$. Set $y = y_{\mathbf{o},x}$ and $s = s_{\mathbf{o},x}$ as in the proof of Proposition 5.6. Observe that $\varphi_{x,0}(X) = 0$, unless $X \in \mathcal{E}_{x,0}\big(\mathbf{o}(\mathbb{F}_q)\big)$. By (5.9.1) and (5.6.1),

$$\widehat{\varphi^\natural_{x,0}}(Y) = \sum_{X \in \mathcal{E}_{x,0}(\mathbf{o}(\mathbb{F}_q))} \varphi_{x,0}(X) \int_{G_x} (\widehat{\psi_{\tilde{X}}})_{y,-s}\big(\mathrm{Ad}(k)Y\big). \tag{5.8.2}$$

Using Proposition 2.5,

$$\widehat{\varphi^\natural_{x,0}}(Y) = \sum_{X \in \mathcal{E}_{x,0}(\mathbf{o}(\mathbb{F}_q))} \varphi_{x,0}(X) \int_{G_x}\int_{G_y} \Psi_{\mathfrak{g}}\big(\mathrm{Ad}(h)X, \mathrm{Ad}(k)Y\big)\, dh\, dk. \tag{5.8.3}$$

From Table 8.6 we find that G_y is a subgroup of G_x. Thus, (5.8.3) becomes

$$\widehat{\varphi^\natural_{x,0}}(Y) = \sum_{X \in \mathcal{E}_{x,0}(\mathbf{o}(\mathbb{F}_q))} \varphi_{x,0}(X) \int_{G_x} \Psi_{\mathfrak{g}}\big(\mathrm{Ad}(k)X, Y\big)\, dk. \tag{5.8.4}$$

(Recall that the measure on G_y was normalised.) From 5.6, recall that the intersection of $\mathcal{V}$ with the support of the Fourier transform of $\varphi^\flat_{x,0}$ is contained in $\mathfrak{g}_{x,0}$. Since G_x stabilises $\mathfrak{g}_{x,0}$, it follows from the definition of $\varphi^\natural_{x,0}$ that the intersection of $\mathcal{V}$ with the support of the Fourier transform of $\varphi^\natural_{x,0}$ is contained in $\mathfrak{g}_{x,0}$. To prove the proposition it is sufficient, therefore, to study the Fourier transform of $\varphi^\natural_{x,0}$ on $\mathfrak{g}_{x,0}$. Henceforth, $Y \in \mathfrak{g}_{x,0}$. Thus,

$$\int_{G_x} \Psi_{\mathfrak{g}}\big(\mathrm{Ad}(k)X, Y\big)\, dk = |M_x|^{-1} \sum_{m \in M_x} \Psi_{\mathfrak{m}_{x,0}}\big(m \cdot \rho_{x,0}(X), \mathcal{Y}\big), \tag{5.8.5}$$

where $\mathcal{Y} := \rho_{x,0}(Y)$. Note that the right-hand side of equation (5.8.5) is $\widehat{\psi_{\rho_{x,0}(X)}}(\mathcal{Y})$. Now, the map fibres of the map $\rho_{x,0}\colon \mathcal{E}_{x,0}\big(\mathbf{o}(\mathbb{F}_q)\big) \to \mathbf{o}(\mathbb{F}_q)$ have cardinality $|\mathcal{E}_{x,0}\big(\mathbf{o}(\mathbb{F}_q)\big)|/2$ (simply because $\mathbf{o}(\mathbb{F}_q)$ contains two adjoint orbits of equal size), so re-write $\varphi_{x,0}(X)$ as $\varphi\big(\rho_{x,0}(X)\big)$ and combine (5.8.4) and (5.8.5) to see

$$\widehat{\varphi^\natural_{x,0}}(Y) = |\mathcal{E}_{x,0}\big(\mathbf{o}(\mathbb{F}_q)\big)| \frac{1}{2} \sum_{\mathcal{X}} \varphi(\mathcal{X}) \widehat{\psi_{\mathcal{X}}}(\mathcal{Y}), \tag{5.8.6}$$

where the sum runs over a set of representatives for the adjoint orbits in $\mathbf{o}(\mathbb{F}_q)$. On the other hand, again using the fact that each adjoint orbit in $\mathbf{o}(\mathbb{F}_q)$ has cardinality $|\mathbf{o}(\mathbb{F}_q)|2^{-1}$, we have

$$\sum_{\mathcal{X}} \varphi(\mathcal{X}) \psi_{\mathcal{X}}(\mathcal{Y}) = 2\,|\mathbf{o}(\mathbb{F}_q)|^{-1} \varphi(\mathcal{Y}), \tag{5.8.7}$$

where the sum runs over a set of representatives for the adjoint orbits in $\mathfrak{o}(\mathbb{F}_q)$. Take the Fourier transform of (5.8.7), and from (5.8.6) we have

$$\widehat{\varphi^{\natural}_{x,0}}(Y) = |\mathcal{E}_{x,0}\big(\mathfrak{o}(\mathbb{F}_q)\big)|\,|\mathfrak{o}(\mathbb{F}_q)|^{-1}\,\widehat{\varphi}(\mathcal{Y}).$$

Thus,

$$\hat{\varphi}_{x,0}(Y) = |\mathfrak{o}(\mathbb{F}_q)|\,|\mathcal{E}_{x,0}\big(\mathfrak{o}(\mathbb{F}_q)\big)|^{-1}\,\widehat{\varphi^{\natural}_{x,0}}(Y).$$

This proves the proposition. ■

6 The Elliptic Character Expansion

In this section, x denotes a standard special vertex in the building for G.

Definition 6.1 Let $C(\mathfrak{m}^{nilp}_{x,0})_{unif}$ denote the logarithm of $C(M^{unip}_x)_{unif}$, as in Definition 3.4. Let $C(\mathfrak{m}^{nilp}_{x,0})^{\perp}_{unif}$ denote the space of invariants orthogonal to $C(\mathfrak{m}^{nilp}_{x,0})_{unif}$ under the usual inner product.

Remark From the Generalised Springer Correspondence [L.1] we find that $C(\mathfrak{m}^{nilp}_{x,0})_{unif}$ is the complex span of the characteristic functions of $\mathcal{L}$, as $\mathcal{L}$ runs over the elements of $\mathcal{N}(\mathbf{m}^{nilp}_{x,0})_{unif}$; likewise, $C(\mathfrak{m}^{nilp}_{x,0})^{\perp}_{unif}$ is the span of the functions $[\mathcal{L}]$, as $\mathcal{L}$ runs over the local systems in $\mathcal{N}(\mathbf{m}^{nilp}_{x,0})'$.

Proposition 6.2 *If $\varphi \in C(\mathfrak{m}^{nilp}_{x,0})^{\perp}_{unif}$ and $\phi \in C^{\infty}_c(\mathcal{V})$, then*

$$D_G(\varphi_{x,0}, \phi) = \sum_{X \in \mathcal{E}^{nilp\,\prime}_{x,0}} e_X(\varphi_{x,0}) I_G(X, \widehat{\phi}),$$

where

$$e_X(\varphi_{x,0}) = \sum_{\mathcal{L} \in \mathcal{N}(\mathbf{m}^{nilp}_{x,0})'} c_{\mathcal{L}}\,|\operatorname{supp}[\mathcal{L}]|\,|\mathcal{E}_{x,0}(\operatorname{supp}[\mathcal{L}])|^{-1}[\mathcal{L}]_{x,0}(X),$$

where $c_{\mathcal{L}}$ is given by (6.2.1).

Proof The Generalised Springer Correspondence [L.1] shows that when the Fourier transform (as defined in 2.1) is followed by restriction to $\mathfrak{m}^{nilp}_{x,0}$, the result is an automorphism of $C(\mathfrak{m}^{nilp}_{x,0})^{\perp}_{unif}$. In fact, the Generalised Springer Correspondence does more—it shows how to actually calculate the Fourier transform of any element in $C(\mathfrak{m}^{nilp}_{x,0})$, using Chapter V of [L.2]. We use that fact in the calculations behind Tables 8.9 and 8.10. It follows that there are uniquely defined complex numbers $c_{\mathcal{L}}$ such that

$$\varphi(\mathcal{Y}) = \sum_{\mathcal{L} \in \mathcal{N}(\mathbf{m}^{nilp}_{x,0})'} c_{\mathcal{L}}\,\widehat{[\mathcal{L}]}(\mathcal{Y}), \tag{6.2.1}$$

for all nilpotent $\mathcal{Y}$ in $\mathfrak{m}_{x,0}$. Without worrying about convergence for a moment, from (6.2.1) we have

$$D_G(\varphi_{x,0},\phi) = \sum_{\mathcal{L}\in\mathcal{N}(\mathbf{m}^{nilp}_{x,0})'} c_{\mathcal{L}} D_G(\widehat{[\mathcal{L}]}_{x,0},\phi). \tag{6.2.2}$$

By Proposition 5.8,

$$\widehat{[\mathcal{L}]}_{x,0}(Y) = |\operatorname{supp}[\mathcal{L}]|\,|\mathcal{E}_{x,0}(\operatorname{supp}[\mathcal{L}])|^{-1}\,\widehat{[\mathcal{L}]^{\natural}_{x,0}}(Y), \tag{6.2.3}$$

for all Y in $\mathcal{V}$. Since the support of ϕ is contained in $\mathcal{V}$, from (6.2.2) and (6.2.3) it follows that

$$D_G(\widehat{[\mathcal{L}]}_{x,0},\phi) = |\operatorname{supp}[\mathcal{L}]|\,|\mathcal{E}_{x,0}(\operatorname{supp}[\mathcal{L}])|^{-1} D_G(\widehat{[\mathcal{L}]^{\natural}_{x,0}},\phi). \tag{6.2.4}$$

Let the support of $\mathcal{L}$ be $\mathfrak{o}$ and let $y = y_{\mathfrak{o},x}$ and $s = s_{\mathfrak{o},x}$ and $\bar{X} = \rho_{y,s}(X)$. Then, by Definitions 5.1 and 5.7, together with Proposition 2.3,

$$\widehat{[\mathcal{L}]^{\natural}_{x,0}}(Y) = \sum_{X\in\mathcal{E}^{nilp}_{x,0}} [\mathcal{L}]_{x,0}(X) \int_{G_x} (\widehat{\psi_{\bar{X}}})_{y,-s}\big(\operatorname{Ad}(k)Y\big)\,dk, \tag{6.2.5}$$

for all Y in $\mathcal{V}$. (We used a similar argument in (5.6.1).) Thus,

(6.2.6)

$$D_G(\widehat{[\mathcal{L}]^{\natural}_{x,0}},\phi) = \sum_{X\in\mathcal{E}^{nilp}_{x,0}} [\mathcal{L}]_{x,0}(X) \int_G\int_{\mathfrak{g}}\int_{G_x} (\widehat{\psi_{\bar{X}}})_{y,-s}\big(\operatorname{Ad}(k)Y\big)\,dk\,\phi\big(\operatorname{Ad}(g)Y\big)\,dY\,dg.$$

Since ϕ has compact support,

$$\begin{aligned}&\int_G\int_{\mathfrak{g}}\int_{G_x} (\widehat{\psi_{\bar{X}}})_{y,-s}\big(\operatorname{Ad}(k)Y\big)\,dk\,\phi\big(\operatorname{Ad}(g)Y\big)\,dY\,dg\\ &\qquad = \int_G\int_{G_x}\int_{\mathfrak{g}} (\widehat{\psi_{\bar{X}}})_{y,-s}\big(\operatorname{Ad}(k)Y\big)\phi\big(\operatorname{Ad}(g)Y\big)\,dY\,dk\,dg.\end{aligned} \tag{6.2.7}$$

By a change of variables, the right-hand side of (6.2.7) is

$$\int_G\int_{\mathfrak{g}} (\widehat{\psi_{\bar{X}}})_{y,-s}\big(\operatorname{Ad}(k)Y\big)\phi\big(\operatorname{Ad}(g)Y\big)\,dY\,dg =: D_G\big((\widehat{\psi_{\bar{X}}})_{y,-s},\phi\big). \tag{6.2.8}$$

(Recall that the measure on G_x was normalised.) By Proposition 6.2, this equals $I_G(X,\widehat{\phi})$, so from (6.2.6) we have

$$D_G(\widehat{[\mathcal{L}]^{\natural}_{x,0}},\phi) = \sum_{X\in\mathcal{E}^{nilp}_{x,0}} [\mathcal{L}]_{x,0}(X) I_G(X,\widehat{\phi}). \tag{6.2.9}$$

Combining this with (6.2.4) yields

$$D_G(\widehat{[\mathcal{L}]}_{x,0},\phi) = |\operatorname{supp}[\mathcal{L}]|\,|\mathcal{E}_{x,0}(\operatorname{supp}[\mathcal{L}])|^{-1} \sum_{X\in\mathcal{E}^{nilp}_{x,0}} [\mathcal{L}]_{x,0}(X) I_G(X,\widehat{\phi}). \tag{6.2.10}$$

So, by (6.2.2),

$$D_G(\varphi_{x,0},\phi) = \sum_{\mathcal{L}\in\mathcal{N}(\mathbf{m}^{nilp}_{x,0})'} c_{\mathcal{L}}\,|\operatorname{supp}[\mathcal{L}]|\,|\mathcal{E}_{x,0}(\operatorname{supp}[\mathcal{L}])|^{-1} \sum_{X\in\mathcal{E}^{nilp}_{x,0}} [\mathcal{L}]_{x,0}(X) I_G(X,\widehat{\phi}). \tag{6.2.11}$$

This resolves any doubts we may have had about the convergence of the integrals in (6.2.2). Re-arranging (6.2.11), we have

$$D_G(\varphi_{x,0},\phi) = \sum_{X\in\mathcal{E}^{nilp}_{x,0}} \Big\{ \sum_{\mathcal{L}\in\mathcal{N}(\mathbf{m}^{nilp}_{x,0})'} c_{\mathcal{L}}\,|\operatorname{supp}[\mathcal{L}]|\,|\mathcal{E}_{x,0}(\operatorname{supp}[\mathcal{L}])|^{-1}[\mathcal{L}]_{x,0}(X)\Big\} I_G(X,\widehat{\phi}),$$

which proves the proposition. ∎

Definition 6.3 For any standard vertex x, let $\mathcal{E}_{x,0}$ denote the union of $\mathcal{E}^{ss}_{x,0}$ and $\mathcal{E}^{nilp\,\prime}_{x,0}$. Also, let $\mathcal{E}_0$ be the unions of $\mathcal{E}_{x,0}$, as x runs over the standard vertices.

Proposition 6.4 *Let $\pi_x(\sigma)$ be a depth-zero supercuspidal representation of G. There are integers $e_X\big(\pi_x(\sigma)\big)$ such that, for all Y in $\mathcal{V}$,*

$$\Theta_{\pi_x(\sigma)}(\exp Y) = \sum_{X\in\mathcal{E}_{x,0}} e_X\big(\pi_x(\sigma)\big)\widehat{\mu}_X(Y).$$

Proof Let φ be any element of $C^\infty_c(\mathcal{V})$ and define $f\in C^\infty_c(G^{tu})$ by $f\circ\exp=\phi$. We will prove the proposition by finding the integers $e_X\big(\pi_x(\sigma)\big)$ such that

$$\operatorname{Tr}\pi_x(\sigma)(f) = \sum_{X\in\mathcal{E}_{x,0}} e_X\big(\pi_x(\sigma)\big) I_G(X,\widehat{\phi}). \tag{6.4.1}$$

Define $\varphi\in C(\mathfrak{m}^{nilp}_{x,0})$ by $\varphi(\mathcal{Y})=\operatorname{Tr}\sigma(\exp\mathcal{Y})$, for all nilpotent $\mathcal{Y}$ in $\mathfrak{m}_{x,0}$. Then, by Proposition 1.5,

$$\operatorname{Tr}\pi_x(\sigma)(f) = D_G(\varphi_{x,0},\phi), \tag{6.4.2}$$

where the measure on G is normalised so that G_x has measure 1. Let φ^0 be the projection of φ from $C(\mathfrak{m}^{nilp}_{x,0})$ to $C(\mathfrak{m}^{nilp}_{x,0})_{unif}$ and let φ' be the projection of φ from $C(\mathfrak{m}^{nilp}_{x,0})$ to $C(\mathfrak{m}^{nilp}_{x,0})^{\perp}_{unif}$. Again, without worrying about convergence for a moment, we have

$$D_G(\varphi_{x,0},\phi) = D_G(\varphi^0_{x,0},\phi) + D_G(\varphi'_{x,0},\phi). \tag{6.4.3}$$

As in the proof of Proposition 3.5,

$$D_G(\varphi^0_{x,0},\phi) = \sum_{X_w\in \mathcal{E}^{\mathrm{ss}}_{x,0}} e_{X_w}(\varphi^0) I_G(X,\widehat{\phi}), \tag{6.4.4}$$

where

$$e_{X_w}(\varphi^0_{x,0}) = q^{-\operatorname{rank} M_x} c_w |M_x|\, |\mathbf{T}_w(\mathbb{F}_q)|^{-1}, \tag{6.4.5}$$

where the complex numbers c_w are given by (3.5.1), *mut. mut.* Likewise, by Proposition 6.2,

$$D_G(\varphi'_{x,0},\phi) = \sum_{X\in \mathcal{E}^{nilp\,\prime}_{x,0}} e_X(\varphi'_{x,0}) I_G(X,\widehat{\phi}), \tag{6.4.6}$$

where

$$e_X(\varphi'_{x,0}) = \sum_{\mathcal{L}\in\mathcal{N}(\mathbf{m}^{nilp}_{x,0})'} c_{\mathcal{L}}\, |\operatorname{supp}[\mathcal{L}]|\, |\mathcal{E}_{x,0}(\operatorname{supp}[\mathcal{L}])|^{-1} [\mathcal{L}]_{x,0}(X), \tag{6.4.7}$$

where the complex numbers $c_{\mathcal{L}}$ are given by (6.2.1), *mut. mut.* Gather equations (6.4.3) through (6.4.7) to see that

$$D_G(\varphi_{x,0},\phi) = \sum_{X\in \mathcal{E}_{x,0}} e_X\big(\pi_x(\sigma)\big) I_G(X,\widehat{\phi}), \tag{6.4.8}$$

where

$$e_X\big(\pi_x(\sigma)\big) = \begin{cases} e_X(\varphi^0_{x,0}), & \text{if } X = X_w \in \mathcal{E}^{\mathrm{ss}}_{x,0};\\ e_X(\varphi'_{x,0}), & \text{if } X \in \mathcal{E}^{nilp\,\prime}_{x,0}. \end{cases}$$

Together with (6.4.2), this proves (6.4.1) and therefore proves the proposition. The fact that each $e_X(\varphi_x)$ is an integer comes from a direct calculation of these numbers, as found in Tables 8.9 and 8.10. ■

7 Some Applications of the Elliptic Character Expansion

This section briefly explores two applications of the elliptic character expansion. In this section x is a special vertex of the building for G.

Proposition 7.1 *For any characters θ_0 and θ_2 of $\mathbf{T}_c(\mathbb{F}_q)$ in general position, the character of*

$$\pi_{x_0}(R^{\mathbf{M}_{x_0}}_{\mathbf{T}_c,\theta_0}) + \pi_{x_2}(R^{\mathbf{M}_{x_2}}_{\mathbf{T}_c,\theta_2})$$

is stable on the set of topologically unipotent elements in G.

Proof Let X_0 denote X_c from Table 8.3 and let X_2 denote X_c from Table 8.4. The orbits represented by X_0 and X_2 form a stable conjugacy class in $\mathfrak{g}$, thus

$$I_G^{(st)}(X_0,\cdot) := I_G(X_0,\cdot) + I_G(X_2,\cdot)$$

is a stable distribution on $C_c^\infty(\mathfrak{g})$. By [Wa.2], the Fourier transform of this distribution is also stable; that is,

$$\widehat{\mu}_{X_0}^{(st)}(\,\cdot\,) := \widehat{\mu}_{X_0}(\,\cdot\,) + \widehat{\mu}_{X_2}(\,\cdot\,)$$

is a stable distribution on $C_c^\infty(\mathfrak{g})$. From Tables 8.9 and 8.10 respectively,

$$\Theta_{\pi_{x_0}(R^{\mathbf{M}_{x_0}}_{\mathbf{T}_c,\theta_0})}(\exp Y) = q^2(q-1)^2(q+1)^2\widehat{\mu}_{X_0}(Y),$$

$$\Theta_{\pi_{x_2}(R^{\mathbf{M}_{x_2}}_{\mathbf{T}_c,\theta_2})}(\exp Y) = q^2(q-1)^2(q+1)^2\widehat{\mu}_{X_2}(Y),$$

for all topologically nilpotent Y in $\mathfrak{g}$. It follows that

$$\Theta_{\pi_{x_0}(R^{\mathbf{M}_{x_0}}_{\mathbf{T}_c,\theta_0})} + \Theta_{\pi_{x_2}(R^{\mathbf{M}_{x_2}}_{\mathbf{T}_c,\theta_2})}$$

is a stable distribution on the set of all topologically unipotent elements in G. ■

Proposition 7.2 *Let $\pi_x(\sigma)$ be a depth-zero supercuspidal representation. The coefficient of the local character expansion corresponding to the nilpotent orbit $\mathcal{O}$ in $\mathfrak{g}$ is*

$$c_{\mathcal{O}}\big(\pi_x(\sigma)\big) = \sum_{X\in\mathcal{E}_{x,0}} e_X\big(\pi_x(\sigma)\big)\Gamma_{\mathcal{O}}(X).$$

Proof Let $\pi = \pi_x(\sigma)$ and let

(7.2.1) $$\Theta_\pi(\exp Y) = \sum_{X\in\mathcal{E}_{x,0}} e_X(\pi)\widehat{\mu}_X(Y)$$

be the elliptic character expansion for π, where Y is any element of $\mathcal{V}$. The Shalika germ expansion (augmented by Waldspurger to $\mathfrak{g}^{tn}$) gives

(7.2.2) $$\widehat{\mu}_X(Y) = \sum_{\mathcal{O}} \Gamma_{\mathcal{O}}(X)\widehat{\mu}_{\mathcal{O}}(Y),$$

where the summation is over the nilpotent orbits in $\mathfrak{g}$ and where $\widehat{\mu}_{\mathcal{O}}$ is the Fourier transform of the nilpotent orbital integral at $\mathcal{O}$. Combining (7.2.1) and (7.2.2) we have

(7.2.3) $$\Theta_\pi(\exp Y) = \sum_{\mathcal{O}}\Big\{\sum_{X\in\mathcal{E}_{x,0}} e_X(\pi)\Gamma_{\mathcal{O}}(X)\Big\}\widehat{\mu}_{\mathcal{O}}(Y).$$

On the other hand, by the local character expansion (also augmented by Waldspurger to $\mathfrak{g}^{tn}$),

(7.2.4) $$\Theta_\pi(\exp Y) = \sum_{\mathcal{O}} c_{\mathcal{O}}\big(\pi_x(\sigma)\big)\widehat{\mu}_{\mathcal{O}}(Y).$$

Since the $\widehat{\mu}_{\mathcal{O}}$ are linearly independent distributions on $\mathcal{V}$ (even as germs), a comparison of (7.2.3) and (7.2.4) proves the proposition. ■

Remark It is interesting to note that the sum in 7.2 involves both ramified and unramified orbits in $\mathfrak{g}$.

8 Tables

Table 8.1 The elements of $C_0(M_x^{unip})$, where x is any special vertex.

$Q_{\mathbf{T}_c}^{\mathbf{M}_x}$	This Green's polynomial is the restriction of the Deligne-Lusztig representation $R_{\mathbf{T}_c,\theta}^{\mathbf{M}_x}$ to the set of unipotent elements in M_x, where θ is any character in general position of the twist $\mathbf{T}_c(\mathbb{F}_q)$ of the split torus by the Coxeter element c in the Weyl group for $\mathbf{M}_x$.
$Q_{\mathbf{T}_{c^2}}^{\mathbf{M}_x}$	This is the restriction of $R_{\mathbf{T}_{c^2},\theta}^{\mathbf{M}_x}$ to the set of unipotent elements in M_x, where θ is any character of $\mathbf{T}_{c^2}(\mathbb{F}_q)$ in general position, and where the subscript c^2 refers to the square of the Coxeter element in the Weyl group for $\mathbf{M}_x$.
$\frac{1}{4}(Q_{\mathbf{T}_c}^{\mathbf{M}_x} - Q_{\mathbf{T}_{c^2}}^{\mathbf{M}_x})$	This is the restriction of the cuspidal unipotent representation θ_{10} to the set of unipotent elements in $\mathbf{M}_x$.
$\chi_{\mathbf{M}_x}^+$ and $\chi_{\mathbf{M}_x}^-$	These are the restrictions to the set of unipotent elements in M_x of the two irreducible cuspidal representation which appear the Lusztig series for $R_{\mathbf{T}_{c^2},\theta}^{\mathbf{M}_x}$, for certain non-trivial characters θ of $\mathbf{T}_{c^2}(\mathbb{F}_q)$ not in general position.

Table 8.2 The elements of $C_0(M_x^{unip})$, where x is any non-special vertex. Here we use an isomorphism $\mathbf{M}_x \cong \mathrm{Sp}_2 \times \mathrm{Sp}_2$.

$Q_{\mathbf{T}_c}^{\mathrm{Sp}_2} \otimes Q_{\mathbf{T}_c}^{\mathrm{Sp}_2}$	This is the restriction of $R_{\mathbf{T}_c,\theta_1}^{\mathrm{Sp}_2} \otimes R_{\mathbf{T}_c,\theta_2}^{\mathrm{Sp}_2}$ to the set of unipotent elements in M_x, where θ_1 and θ_2 are characters of $\mathbf{T}_c(\mathbb{F}_q)$ in general position, and where the subscript c refers to the Coxeter element in the Weyl group for Sp_2.
$Q_{\mathbf{T}_c}^{\mathrm{Sp}_2} \otimes \chi_{\mathrm{Sp}_2}^{\pm}$	The functions $\chi_{\mathrm{Sp}_2}^{\pm}$ are the restriction of the irreducible cuspidal representations which appear in the Lusztig series for $R_{\mathbf{T}_c,\theta}^{\mathrm{Sp}_2}$, where θ is a non-trivial character
$\chi_{\mathrm{Sp}_2}^{\pm} \otimes \chi_{\mathrm{Sp}_2}^{\pm}$	of $\mathbf{T}_c(\mathbb{F}_q)$ which is not in general position.

Table 8.3 The elements in $\mathcal{E}_{x_0,0}^{\mathrm{ss}}$ are chosen so as to have the following form. The subscripts c and c^2 are explained in Table 8.1.

$$X_c = \begin{pmatrix} 0 & x & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & x \\ x\epsilon & 0 & 0 & 0 \end{pmatrix} + X' \qquad \begin{matrix} x \in \mathfrak{O}^* \\ X' \in \mathfrak{g}_{x_0,1} \end{matrix}$$

$$X_{c^2} = \begin{pmatrix} 0 & 0 & 0 & x \\ 0 & 0 & y & 0 \\ 0 & y\epsilon & 0 & 0 \\ x\epsilon & 0 & 0 & 0 \end{pmatrix} + X' \qquad \begin{matrix} x, y \in \mathfrak{O}^* \\ X' \in \mathfrak{g}_{x_0,1} \end{matrix}$$

Table 8.4 The elements in $\mathcal{E}^{ss}_{x_2,0}$ are chosen so as to have the following form. The subscripts c and c^2 are explained in Table 8.1.

$$X_c = \begin{pmatrix} 0 & x & 0 & 0 \\ 0 & 0 & x\varpi^{-1} & 0 \\ 0 & 0 & 0 & x \\ x\epsilon\varpi & 0 & 0 & 0 \end{pmatrix} + X' \qquad \begin{matrix} x \in \mathfrak{O}^* \\ X' \in \mathfrak{g}_{x_2,1} \end{matrix}$$

$$X_{c^2} = \begin{pmatrix} 0 & 0 & 0 & x\varpi^{-1} \\ 0 & 0 & y\varpi^{-1} & 0 \\ 0 & y\epsilon\varpi & 0 & 0 \\ x\epsilon\varpi & 0 & 0 & 0 \end{pmatrix} + X' \qquad \begin{matrix} x, y \in \mathfrak{O}^* \\ X' \in \mathfrak{g}_{x_2,1} \end{matrix}$$

Table 8.5 The elements of $\mathcal{E}^{nilp\,'}_{x_0,0}$ are chosen so as to have the following form.

$$X^{i,j}_{(4)} = \begin{pmatrix} 0 & x & 0 & 0 \\ 0 & 0 & x\epsilon^i & 0 \\ 0 & 0 & 0 & x \\ x\epsilon^j\varpi & 0 & 0 & 0 \end{pmatrix} + X', \qquad \begin{matrix} x \in \mathfrak{O}^*, \\ i = 1, 0 \\ j = 0, 1, 2, 3 \\ X' \in \mathfrak{g}_{x_0,2} \end{matrix}$$

$$X^{i,j}_{(2,1^2)} = \begin{pmatrix} 0 & 0 & 0 & x\epsilon^i \\ 0 & 0 & y & 0 \\ 0 & y\epsilon & 0 & 0 \\ x\epsilon^j\varpi & 0 & 0 & 0 \end{pmatrix} + X', \qquad \begin{matrix} x \in \mathfrak{O}^*; y \in \mathfrak{p} \\ i, j = 0, 1 \\ X' \in \mathfrak{g}_{x_0,2} \end{matrix}$$

Table 8.6 The elements of $\mathcal{E}^{nilp\,'}_{x_2,0}$ are chosen so as to have the following form.

$$X^{i,j}_{(4)} = \begin{pmatrix} 0 & x & 0 & 0 \\ 0 & 0 & x\epsilon^i\varpi^{-1} & 0 \\ 0 & 0 & 0 & x \\ x\epsilon^j\varpi^2 & 0 & 0 & 0 \end{pmatrix} + X', \qquad \begin{matrix} x \in \mathfrak{O}^*, \\ i = 1, 0 \\ j = 0, 1, 2, 3 \\ X' \in \mathfrak{g}_{x_2,2} \end{matrix}$$

$$X^{i,j}_{(2,1^2)} = \begin{pmatrix} 0 & 0 & 0 & x\epsilon^i\varpi^{-1} \\ 0 & 0 & y\varpi^{-1} & 0 \\ 0 & y\epsilon\varpi & 0 & 0 \\ x\epsilon^j\varpi^2 & 0 & 0 & 0 \end{pmatrix} + X', \qquad \begin{matrix} x \in \mathfrak{O}^*; y \in \mathfrak{p} \\ i, j = 0, 1 \\ X' \in \mathfrak{g}_{x_2,2} \end{matrix}$$

Table 8.7 Values of $y_{\mathfrak{o},x}$ and $s_{\mathfrak{o},x}$ corresponding to the nilpotent orbits $\mathfrak{o}$ in $\mathfrak{m}_x$, when $x = x_0$. In these expressions, we make the apartment into a vector space by choosing x_0 as the origin 0.

$$\begin{array}{lll} \mathfrak{o} \mapsto (4) & y_{\mathfrak{o},x} = \frac{1}{2}x_1 + \frac{1}{4}x_2 & s_{\mathfrak{o},x} = \frac{1}{4} \\ \mathfrak{o} \mapsto (2^2) & y_{\mathfrak{o},x} = \frac{1}{2}x_2 & s_{\mathfrak{o},x} = \frac{1}{2} \\ \mathfrak{o} \mapsto (2,1^2) & y_{\mathfrak{o},x} = \frac{1}{2}x_1 & s_{\mathfrak{o},x} = \frac{1}{2} \\ \mathfrak{o} \mapsto (1^4) & y_{\mathfrak{o},x} = x_0 = 0 & s_{\mathfrak{o},x} = 0 \end{array}$$

Table 8.8 Values of $y_{\mathfrak{o},x}$ and $s_{\mathfrak{o},x}$ corresponding to the nilpotent orbits $\mathfrak{o}$ in $\mathfrak{m}_x$, when $x = x_2$. In these expressions, we make the apartment into a vector space by choosing x_0 as the origin 0.

$$\begin{array}{lll} \mathfrak{o} \mapsto (4) & y_{\mathfrak{o},x} = \frac{1}{2}x_1 + \frac{1}{4}x_2 & s_{\mathfrak{o},x} = \frac{1}{4} \\ \mathfrak{o} \mapsto (2^2) & y_{\mathfrak{o},x} = \frac{1}{2}x_2 & s_{\mathfrak{o},x} = \frac{1}{2} \\ \mathfrak{o} \mapsto (2,1^2) & y_{\mathfrak{o},x} = \frac{1}{2}x_1 & s_{\mathfrak{o},x} = \frac{1}{2} \\ \mathfrak{o} \mapsto (1^4) & y_{\mathfrak{o},x} = x_2 & s_{\mathfrak{o},x} = 0 \end{array}$$

Table 8.9 Coefficients $e_X(\pi_x(\sigma))$ of the elliptic character expansion when the restriction of $\operatorname{Tr}\sigma$ to M_x^{unip} is an element of $C_0(M_x^{unip})$, when $x = x_0$. (Refer to Table 8.1 for $C_0(M_x^{unip})$.) We write χ_{10} for the restriction of θ_{10} to M_x^{unip}. The first row lists the elements of $\mathcal{E}_{x,0}$ as labeled in Tables 8.3 and 8.5. All other rows list the integers $e_X(\pi_x(\sigma))$, where the restriction of $\operatorname{Tr}\sigma$ to M_x^{unip} is given at the left-hand end of that row, and X is found at the top of the corresponding column.

	X_c	X_{c^2}	$X^{i,j}_{(4)}$	$X^{i,j}_{(2,1^2)}$
$Q^{\mathbf{M}_x}_{\mathbf{T}_c}$	$\frac{q^2(q-1)^2(q+1)^2}{1}$	0	0	0
$Q^{\mathbf{M}_x}_{\mathbf{T}_{c^2}}$	0	$\frac{q^2(q-1)^2(q^2+1)}{1}$	0	0
χ_{10}	$\frac{q^2(q-1)^2(q+1)^2}{4}$	$\frac{-q^2(q-1)^2(q^2+1)}{4}$	0	0
$\chi^+_{\mathbf{M}_x}$	$\frac{q^2(q-1)^2(q^2+1)}{2}$	0	$\frac{(-1)^i q(q-1)(q^4-1)}{16}$	$\frac{(-1)^{i+1} q^2(q^2-1)(q^4-1)}{8}$
$\chi^-_{\mathbf{M}_x}$	$\frac{q^2(q-1)^2(q^2+1)}{2}$	0	$\frac{(-1)^{i+1} q(q-1)(q^4-1)}{16}$	$\frac{(-1)^i q^2(q^2-1)(q^4-1)}{8}$

Table 8.10 Coefficients $e_X(\pi_x(\sigma))$ of the elliptic character expansion when the restriction of $\operatorname{Tr}\sigma$ to M_x^{unip} is an element of $C_0(M_x^{unip})$, when $x = x_2$. (Refer to Table 8.1 for $C_0(M_x^{unip})$.) We write χ_{10} for the restriction of θ_{10} to M_x^{unip}. The first row lists the elements

of $\mathcal{E}_{x,0}$ as labeled in Tables 8.4 and 8.6. All other rows list the integers $e_X(\pi_x(\sigma))$, where the restriction of Tr σ to M_x^{unip} is given at the left-hand end of that row, and X is found at the top of the corresponding column.

	X_c	X_{c^2}	$X^{i,j}_{(4)}$	$X^{i,j}_{(2,1^2)}$
$Q^{\mathbf{M}_x}_{\mathbf{T}_c}$	$\frac{q^2(q-1)^2(q+1)^2}{1}$	0	0	0
$Q^{\mathbf{M}_x}_{\mathbf{T}_{c^2}}$	0	$\frac{q^2(q-1)^2(q^2+1)}{1}$	0	0
χ_{10}	$\frac{q^2(q-1)^2(q+1)^2}{4}$	$\frac{-q^2(q-1)^2(q^2+1)}{4}$	0	0
$\chi^+_{\mathbf{M}_x}$	$\frac{q^2(q-1)^2(q^2+1)}{2}$	0	$\frac{(-1)^i q(q-1)(q^4-1)}{16}$	$\frac{(-1)^{i+1} q^2(q^2-1)(q^4-1)}{8}$
$\chi^-_{\mathbf{M}_x}$	$\frac{q^2(q-1)^2(q^2+1)}{2}$	0	$\frac{(-1)^{i+1} q(q-1)(q^4-1)}{16}$	$\frac{(-1)^i q^2(q^2-1)(q^4-1)}{8}$

References

[AD] J. Adler and S. Debacker, *Moy-Prasad filtrations and harmonic analysis.* Draft, 1997.

[Cu] C. Cunningham, *Some results on elliptic orbital integrals.* In preparation.

[K] D. Kazhdan, *Proof of Springer's hypothesis.* Israel J. Math. **28**(1977), 272–286.

[KL] D. Kazhdan and G. Lusztig, *Fixed point varieties on affine flag manifolds.* Israel J. Math. (2) **62**(1988), 129–168.

[L.1] G. Lusztig, *Intersection cohomology complexes on a reductive group.* Invent. Math. **75**(1984), 205–272.

[L.2] ———, *Character sheaves I.* Adv. Math. **56**(1985), 193–297; *Character sheaves II.* Adv. Math. **57**(1985), 226–265; *Character sheaves III.* Adv. Math. **57**(1985), 266–315; *Character sheaves IV.* Adv. Math. **59**(1986), 1–63; *Character sheaves V.* Adv. Math. **61**(1986), 103–155.

[Mu] F. Murnaghan, *Characters of supercuspidal representations of classical groups.* Ann. Sci. École Norm Sup. (4), (1) **29**(1996), 45–105.

[MP] A. Moy and G. Prasad, *Jacquet functors and unrefined minimal K-types.* Comment. Math. Helv. (1) **71**(1996), 98–121.

[Sp] N. Spaltenstein, *Polynomials over local fields, nilpotent orbits and conjugacy classes in Weyl groups.* Astérisque **168**(1988), 191–217.

[Wa.1] J.-L. Waldspurger, *Quelques questions sur les intégrales orbitales unipotentes et les algèbres de Hecke.* Bull. Soc. Math. France **124**(1996), 1–34.

[Wa.2] ———, *Le lemme fondamental implique le transfert.* Compositio Math. **105**(1997), 153–236.

Department of Mathematics
University of Massachusetts
Amherst MA 01003-4515
USA