geometrical problem than that of § 4, viz., where the triangle instead of being equilateral has two sides equal and enclosing a definite angle. We must now know which of the three given distances is the one drawn to the meeting point of the equal sides of the triangle; and taking this distance we make it one of the equal sides of an isosceles triangle similar to that specified, and on the third side describe a triangle with its two other sides equal to the remaining two given distances, and as before join the vertices not common to the two triangles.*

BISHOPTON, GLASGOW,
26th Dec. 1884.

Geometrical Notes.

By J. S. MACKAY, M.A.

I. A straight line KK′ meets the circumference of a circle at two real or two imaginary points K, K′, and H is the middle point of the real or imaginary chord KK′. If A, B, C, D be any four points on the circumference, and the pairs of straight lines AB,DC, AC,BD, AD,CB meet KK′ at the pairs of points E,E′, F,F′, G,G′; then if any one pair of points be equidistant from H, the two other pairs will also be equidistant.

To prove that E,E′ are equidistant from H, if F,F′ are.

First Demonstration. (*Figures* 14, 15.)

Through A draw AA′ parallel to KK′; join A′E′, A′F′, and since E′, F′ are on DC, DB, join A′D.

Then AFF′A′ may be proved to be a convex or crossed isosceles and therefore cyclic trapezium, having A′F′ = AF, and angle A′F′E′ = angle AFE.

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* Professor Chrystal pointed out that a particular case of this, viz., where the triangle is isosceles *right-angled* is dealt with in the *Annals of Mathematics*, I., p. 24, and Mr Fraser has since received from Dr Rennet, of Aberdeen, a reference to Thomas Simpson's Algebra, 2nd edition, (1755) p. 369, where a very general problem of this nature is stated and solved.

https://doi.org/10.1017/S0013091500037251 Published online by Cambridge University Press

Now angle A′AC is supplementary to angle A′DC,
and angle A′AC is equal to angle A′F′E′;
therefore angle A′F′E′ is supplementary to angle A′DC;
therefore the points A′, F′, E′, D are concyclic;
therefore angle F′A′E′ = angle F′DE′,
= angle FAE.

Hence triangles A′E′F′ and AEF are congruent, and E′F′ = EF, and consequently HE′ = HE.

The proof that G, G′ are equidistant from H, if F, F′ are, is similar to the preceding.

Second Demonstration. (*Figures* 14, 15.)

Let AC, BD, on which are F, F′, meet at L.

Consider the triangle LFF′ as cut by the transversals AB, CD, which determine the points E, E′.

We obtain $\quad LA \cdot FE \cdot F'B = FA \cdot F'E \cdot LB,$

and $\quad LC \cdot FE' \cdot F'D = FC \cdot F'E' \cdot LD;$

therefore, by multiplication and noting that $LA \cdot LC = LB \cdot LD$,

we have $FE \cdot FE' \cdot F'B \cdot F'D = FA \cdot FC \cdot F'E \cdot F'E'$.

Now $F'B \cdot F'D = F'K \cdot F'K' = FK' \cdot FK = FA \cdot FC$;

therefore $\quad FE \cdot FE' = F'E \cdot F'E'$;

therefore $\quad FE : F'E = F'E' : FE'$;

therefore $\quad FE : FF' = F'E' : FF'$.

Hence $FE = F'E'$, and consequently $HE = HE'$.

It may be proved that $FE \cdot FE' = F'E \cdot F'E'$, without introducing the segments FK, FK′, F′K, F′K′, which in figure 15 are imaginary, in the following way:—

Because $\quad FE \cdot FE' \cdot F'B \cdot F'D = FA \cdot FC \cdot F'E \cdot F'E'$,

therefore $\quad FE \cdot FE' : F'E \cdot F'E' = FA \cdot FC : F'B \cdot F'D$,

$$= \pm(r^2 - OF^2) : \pm(r^2 - OF'^2),$$

where r denotes the radius of the circle, and the upper sign is to be taken in figure 14, the lower sign in figure 15.

Now $OF = OF'$, since F, F′ are equidistant from H;

therefore $\quad FE \cdot FE' = F'E \cdot F'E'$.

Third Demonstration. (*Figure* 14.)

Join BK, BK′, CK, CK′.

Since the pairs of angles subtended at B and C by the arcs KA, AD, DK′ are equal;

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therefore the pencils B·KADK′ and C·KADK′ are superposable; therefore the anharmonic ratio of B·KADK′ is equal to the anharmonic ratio of C·KADK′,

that is, $KK' \cdot EF' : KE \cdot F'K' = KK' \cdot FE' : KF \cdot E'K'$.

Now since $F'K' = KF$, this proportion becomes

$$EF' : KE = FE' : E'K';$$

therefore $$EF' : F'K = FE' : FK'.$$

But $F'K = FK'$; therefore $EF' = FE'$.

Hence E and E′ are equidistant from F and F′, and consequently from H.

The theorem to which Mr James Taylor called attention last session (see *Proceedings of the Edinburgh Mathematical Society* for 1883-4, p. 4) is easily seen to be a particular case of the foregoing.

II. Between two sides of a triangle to inflect a straight line which shall have given ratios to the segments of the sides between it and the base.

Let ABC be the triangle, and let $p : q$ and $r : q$ be the ratios of the segments of the sides to the inflected straight line.

First Method. (*Figure* 16.)

From BA cut off $BD = p$; through D draw DE parallel to BC. Cut off $CF' = r$; with centre F′ and radius $= q$, cut DE or DE produced at the points G′; and join F′G′. Let CG′ meet AB or AB produced at G, and draw GF parallel to G′F′. GF is the line required.

For through G′ draw G′B′ parallel to GB.

Then $B'G' = BD = p$.

Now, since the quadrilaterals CB′G′F′, CBGF are similar, and either similarly or oppositely situated, C being their centre of similitude; and since $B'G' : G'F' : F'C = p : q : r$;

therefore $BG : GF : FC = p : q : r$.

Second Method. (*Figure* 17.)

Take BD such that $AC : BD = r : p$; and through D draw DE parallel to BC. With centre A and radius AG′, such that $AC : AG' = r : q$, cut DE or DE produced at the points G′; and join AG′. Let CG′ meet AB or AB produced at G, and draw GF parallel to G′A. GF is the line required.

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For through G′ draw G′B′ parallel to GB.

Then B′G′ = BD ; therefore AC : B′G′ = $r : p$.

Now since the quadrilaterals CB′G′A, CBGF are similar, and either similarly or oppositely situated, C being their centre of similitude; and since B′G′ : G′A : AC = $p : q : r$;

therefore BG : GF : FC = $p : q : r$.

Third Method. (*Figure* 18.)

Take AD′ such that AC : AD′ = $r : p$. With centre D′ and radius D′B′, such that AC : D′B′ = $r : q$, cut BC or BC produced at the points B′; and join D′B′, D′C. Through B draw BD parallel to B′D′ to meet CD′ or CD′ produced at D; through D draw DF parallel to AB to meet AC or AC produced at F; and through F draw FG parallel to BD to meet AB or AB produced at G. GF is the line required.

For BG = DF, and GF = BD.

Now since the quadrilaterals CB′D′A, CBDF are similar and either similarly or oppositely situated, C being their centre of similitude; and since AD′ : D′B : AC = $p : q : r$;

therefore FD : DB : FC = $p : q : r$;

therefore BG : GF : FC = $p : q : r$;

With these three methods, which are essentially the same, it may be interesting to compare *Proceedings of the Edinburgh Mathematical Society* for 1883-4, p. 27; *Educational Times*, vol. 37, p. 328; Vuibert's *Journal de Mathématiques Elémentaires*, 9ᵉ année, p. 45.

The following solution of the problem, in the case when p, q, r are equal, is due to Mr Robert John Dallas.

(*Figure* 19.)

Suppose the thing done.

Then since BD = ED, ∠ ADE = 2 ∠ DBE.

Similarly ∠ AED = 2 ∠ ECD.

Now ∠ ADE + ∠ AED = ∠ ABC + ∠ ACB;

therefore ∠ DBE + ∠ ECD = ½(∠ ABC + ∠ ACB);

therefore ∠ OBC + ∠ OCB = ½(∠ ABC + ∠ ACB),

= a constant angle;

therefore O lies on the circumference of a known circle.

If O could be found on this circumference, so that BD might be equal to CE, the problem would be solved; for in proving O to lie on this circumference it was assumed that BD = DE.

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Draw CF parallel to AB. Take any point G in AC, make CH = CG, and join GH cutting BC in K.
On GH describe a segment of a circle containing an angle equal to $\frac{1}{2}(\angle ABC + \angle ACB)$, and let the arc of the segment cut BC at R. Through B draw BE parallel to RG, and BF parallel to RH.

Then by similar triangles it will follow that
$EC = FC$, and $\angle EBF = \angle GRH = \frac{1}{2}(\angle ABC + \angle ACB)$.
Let BE meet at O the arc of the segment described on BC, and containing an angle equal to the supplement of $\frac{1}{2}(\angle ABC + \angle ACB)$.
Then $\angle EOC = \frac{1}{2}(\angle ABC + \angle ACB) = \angle EBF$;
therefore BDCF is a parallelogram, and $BD = FC$.
Now EC has been proved equal to FC, and O is on the circumference of the known circle;
therefore $BD = DE = EC$.

Fourth Meeting, February 13*th*, 1885.

A. J. G. Barclay, Esq., M.A., President, in the Chair.

Note on a Plane Strain.

By Professor Tait.

The object of this note is to point out, by a few remarks on a single case, how well worth the attention of younger mathematicians is the *full* study of certain problems, suggested by physics, but limited (so far as that science is concerned) by properties of matter.

In de St Venant's beautiful investigations of the flexure of prisms, there occurs a plane strain involving the displacements

$$\xi = \frac{xy}{D}, \quad \eta = \frac{y^2 - x^2}{2D}.$$

Physically, this is applicable to de St Venant's problem only when x and y are each small compared with D. But it is interesting to consider the results of extending it to all values of the coördinates. This I shall do, but very briefly.

https://doi.org/10.1017/S0013091500037251 Published online by Cambridge University Press