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Hausdorff Prime Matrices

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Abstract. In this paper we give the form of every multiplicative Hausdorff prime matrix, thus answering a long-standing open question.

Define $G = \{z : \mathbb{R}ez > 0\}$, $H(G)$ the set of analytic functions defined on G, and $f \in H(G)$.

In 1917 Hurwitz and Silverman ([8]) raised the question of which matrices commute with C, the Cesàro matrix of order one. They found the answer to that question to be the set of all Hausdorff matrices. In 1921 Hausdorff [5] investigated these matrices, which now bear his name, in connection with the solution of the moment problem over $[0, 1]$.

A Hausdorff matrix is a lower triangular matrix with entries $h_{nk} = \binom{n}{k}\Delta^{n-k}\mu_k$, where $\{\mu_n\}$ is any real or complex sequence, and Δ is the forward difference operator defined by $\Delta\mu_k = \mu_k - \mu_{k+1}$, $\Delta^{n+1}\mu_k = \Delta(\Delta^n\mu_k)$.

A matrix is conservative if and only if it is a selfmap of c, the space of convergent sequences. Hausdorff proved that a Hausdorff matrix is conservative if and only if $\int_0^1 |d\chi(t)| < \infty$, where $\chi \in BV[0, 1]$, and the integral is a Riemann–Stieltjes one. Moreover, the integral is the norm of the matrix.

Every Hausdorff matrix has row sums μ_0. If it is conservative, then every column limit is zero, except possibly the first one, and that column limit exists. Let $\mathcal{H}$ denote the set of multiplicative Hausdorff matrices. (A conservative matrix is said to be multiplicative if every column limit is zero.) With each $H_\mu \in \mathcal{H}$ there exists a uniquely defined mass function $\chi(t)$ and a corresponding moment function $\mu(z) = \int_0^1 t^z d\chi(t)$ that is analytic for $\mathbb{R}ez > 0$ and continuous over $\mathbb{R}ez \geq 0$. Conversely, each moment function or mass function determines a unique Hausdorff matrix. Let V and M denote, respectively, the algebras of mass functions and moment functions associated with members of $\mathcal{H}$. Then the three algebras $\mathcal{H}$, M, and V can be made isomorphic and isometric. (See, *e.g.*, [6, p. 615].)

Hurwitz and Silverman ([8]) showed that each Hausdorff matrix H has the decomposition $H = \delta\mu\delta$, where μ is the diagonal matrix with diagonal entries μ_n, and δ is a lower triangular matrix with entries $\delta_{nk} = (-1)^k\binom{n}{k}$.

Using this decomposition it is easy to establish the well-known result that $\mathcal{H}$ forms an integral domain. Thus the concepts of divisibility, factor, multiple, unit, associate, and prime can be defined on $\mathcal{H}$, and these concepts carry over to M and V as well.

The convergence domain of a matrix A, written c_A, is the set of sequences $\{x_n\}$ that A maps into a convergent sequence. A Hausdorff matrix H_μ is called a unit if

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$c_{H_\mu} = c$, and H_μ is called a prime if $c_{H_\mu} \neq c$, but every H_λ for which $c_{H_\mu} \supsetneqq c_{H_\lambda}$ implies that $c_{H_\lambda} = c$.

Theorem 1 *Let H be a Hausdorff matrix in $\mathcal{H}$. Then H is prime if and only if*

$$c_H = c \oplus x \tag{1}$$

for some unbounded sequence x.

Proof If (1) is satisfied, then it is obvious from the definition of a prime that H is prime.

Suppose now that H is prime. If H sums a bounded divergent sequence x, then, from [2, Corollary 2.5.8], H must also sum an unbounded divergent sequence, and therefore has too large a convergence domain to be prime. If H sums more than one unbounded divergent sequence, then again H cannot be prime. Consequently, H sums one unbounded and divergent sequence x, and the convergence domain of H is of the form (1). ■

In 1933 Hille and Tamarkin ([7]) proved that every Hausdorff matrix with moment function

$$f(z) = \frac{z-a}{z+1}, \quad \mathbb{R}e\, a > 0$$

is prime and raised the question of whether each prime is of this form. We answer their seventy-five year old open question by means of the following theorem.

Theorem 2 *Let H_f be a multiplicative Hausdorff matrix. Then H_f is prime if and only if*

$$f(z) = \left(\frac{z-a}{z+1}\right) g(z), \quad Re(a) > 0,$$

where g is a unit.

Proof The sufficiency is obvious from [7], since multiplication of Hausdorff matrices is commutative.

Suppose that H_f is prime. Since H_f is multiplicative, f has the representation

$$f(z) = \int_0^1 t^z d\chi(t),$$

where $\chi(t) \in BV[0,1]$, $\chi(0+) = \chi(0) = 0$, and $\chi(t) = [\chi(t+0) + \chi(t-0)]/2$ for each $0 < t < 1$.

It then follows that $f \in H(G)$, and f is continuous and bounded on $\overline{G}$ in $\mathbb{C}$.

Let $\sigma(H_\mu)$ denote the spectrum of H_μ. Sharma [9] has shown that $\sigma(H_f) \supset \overline{f(G)}$. Either $0 \in \sigma(H_f)$ or $0 \notin \sigma(H_f)$. If $0 \notin \sigma(H_f)$, then H_f is invertible, hence a unit, and hence not prime.

Grahame Bennett has shown that $\mu_n \to 0$ implies that H_μ is not prime. Therefore, we need consider only those Hausdorff matrices for which $\mu_n \nrightarrow 0$ and $0 \in \sigma(H_f)$. Since $\mu_n \nrightarrow 0, 0 \in \overline{f(G)}$ implies that either there exists a $z_0 \in G$ with $f(z_0) = 0$, or there exists a sequence $\{w_n\} \subset f(G)$ with $\lim w_n = 0$. But, in the latter case, for each

n there exists a $z_n \in G$ such that $w_n = f(z_n)$. Also $\{z_n\}$ is bounded, since $\mu_n \nrightarrow 0$. Since f is continuous on $\overline{G}$, $0 = \lim_n f(z_n) = f(\lim z_n)$, and $\lim z_n \in \overline{G}$, since it is closed. Thus $\mu_n \nrightarrow 0$ and $0 \in \sigma(H_f)$ imply that there exists a $z_0 \in \overline{G}$ with $f(z_0) = 0$.

There are two possibilities; either $\mathbb{R}ez = 0$ or $\mathbb{R}ez > 0$.

Case IA $\mathbb{R}ez = 0$ and $f(z) \neq 0$ for $z \in G$.

Since G is simply connected, by [3, Theorem 2.2(h), p. 202], there exists a $g \in H(G)$ such that $f(z) = [g(z)]^2$. Since f is bounded and continuous in $\overline{G}$, so is g. From [4], $c_{H_f} \supseteq c_{H_g}$.

We now need to show that $c_{H_g} \neq c$. From [1], $g(z_0) = 0$, since $z_0 \in \overline{G}$ implies that H_q sums the sequence

$$s_n = \frac{\Gamma(n+1)}{\Gamma(n+1-z_0)}$$

to zero.

If $z_0 \neq 0$, then $\{s_n\}$ is a bounded divergent sequence, and $c_{H_g} \neq c$. If $z_0 = 0$, then $\mu_0 = g(0) = 0$ and H_g is conull. It is well known that every conull matrix sums a bounded divergent sequence. Therefore, in all cases, $c_{H_g} \neq c$ and f is not prime.

Case IB Suppose that f also has a zero in G. Call it z_1. Then we may write

$$f(z) = (z - z_1)^k g_1(z), \quad \text{where} \quad g_1 \in H(G), g(z_1) \neq 0.$$

Moreover, since f is also bounded in $\overline{G}$, it must be the case that $g_1(z) = O(|z|^k)$ in $\overline{G}$. Therefore we may write

$$f(z) = \Big(\frac{z - z_1}{z+1}\Big)^k g(z), \quad \text{where} \quad g(z) = (z+1)^k g_1(z),$$

and where k is finite and $k \geq 1$. Therefore, $c_{H_f} \supseteq c_{H_g} \neq c$ since $g(z_0) = 0$, from Case IA, and f is not prime.

Case II $\mathbb{R}ez > 0$.

As in Case IA we may write

$$f(z) = \Big(\frac{z - z_0}{z+1}\Big)^k,$$

where $g \in H(G)$ and g is bounded and continuous in $\overline{G}$.

Clearly H_f cannot be prime if $k > 1$, since then $c_{H_f} \supseteq c_{H_k}$, where $k(z) = \frac{z-z_0}{z+1}$. Using the same argument, if g has any zeros in $\overline{G}$, then, since $c_{H_f} \supseteq c_{H_g}$, f cannot be prime.

But, if g does not vanish in $\overline{G}$, it is a unit. Therefore H_f prime implies that f has the desired representation. ■

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