A REFINEMENT OF THEOREMS OF KIRCHBERGER AND CARATHÉODORY

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1. Introduction

I can indicate the type of refinement mentioned in the title by referring to Kirchberger's theorem [4]. Its picturesque form in the plane is: if sheep and goats are grazing in a field and for every four animals there exists a line separating the sheep from the goats then there exists such a line for all the animals. The refinement is that the words 'every four animals' may be replaced by 'every four animals including an arbitrarily chosen animal'; this reduces the 'Kirchberger number' from four to, effectively, three.

The starting point will be Helly's theorem [3] in d-dimensional Euclidean space E_d. Then follow refinements of the following chain of theorems: Helly's theorem in the surface S_d of a sphere in E_{d+1}, Kirchberger's theorem in S_d and in E_d, Carathéodory's theorem in E_d. For convenience of exposition the original name is retained, in coded form, for the refined version of a theorem. In each case the original form may be restored by deleting the reference to an arbitrarily chosen point.

In what follows convexity in S_d will mean strong convexity; that is, a set in S_d is called convex if it contains no pair of diametrically opposite points, and together with any two of its points contains the whole of the minor arc of the great circle containing them.

2. Helly's Theorems in S_d

First let us state Helly's theorem [3] in E_d:–

THEOREM HE$_d$. *If every $d+1$ members of a finite family of convex sets in E_d have a common point then all menbers of the family have a common point.*

We can readily deduce from this a refinement of Helly's theorem in S_d:

THEOREM HS$_d$. *Let F be a finite family of convex sets in S_d; if every $d+2$*

of the sets, including an arbritarily chosen set, have a common point then all sets of F have a common point.

To prove this let C be the arbitrarily chosen set and consider the family H of intersections of members of F with C. H will have the property that every $d+1$ of its members have a common point. Now project C, from the centre of S_d, onto a hyperplane in E_{d+1}. We see, by theorem HE_d that all members of the projections of H have a common point; hence all members of H have a common point. It follows that all members of F have a common point.

An immediate corollary to theorem HS_d is obtained by specifying F to be a finite family of open hemi-spheres. It is a simple step then to the dual of theorem HS_d; in replacing an open hemisphere by its antipodal pole, and vice versa, properties of incidence are preserved and theorem HS_d becomes

THEOREM DHS_d: *Let F be a finite family of points in S_d, if every $d+2$ points, including an arbitrarily chosen one, lie in some open hemisphere, then all points of F lie in an open hemisphere.*

3. Kirchberger's Theorem

A simple proof of Kirchberger's theorem was given by Baker [1]; the same method can be used to derive the refinement:–

THEOREM KS_d. *Let S and G be disjoint finite sets of points in S_d; and let p be an arbitrarily chosen point in the union A of S and G. If for each set K satisfying*

$$|K| = d+2, \qquad p \in K, \qquad K \subset A$$

there exists a hyperplane through the centre stricly separating $K \cap S$ from $K \cap G$, then there exists a hyperplane through the centre stricly separating S from G.

To prove this we reflect each point of G through the centre into its antipodal point. Call the set of such points G'. Then $S \cup G'$ satisfy the conditions placed on F in theorem DHS_d, hence the members of $S \cup G'$ lie in some open hemisphere. A reflection of G' back onto G yields the theorem.

An immediate consequence of the preceding theorem is:–

THEOREM KE_d. *Let S and G be disjoint finite sets of points in E_d; and let p be any point in the union A of S and G. If for each set K satisfying*

$$|K| = d+2, \qquad p \in K, \qquad K \subset A$$

there exists a hyperplane strictly separating $K \cap S$ from $K \cap G$, then there exists a hyperplane strictly separating S from G.

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The proof is effected by embedding E_d in E_{d+1} and projecting E_d onto an open hemisphere of S_d by projecting through the centre of S_d. Theorem KE_d then follows from theorem KS_d.

4. Carathéodory's Theorem

To obtain the refinement of Carathéodory's theorem [2] we express theorem KE_d in contrapositive form and specialise it by taking $|S| = 1, p \in G$. In E_d the statement that a point can not be strictly separated by a hyperplane from a set of points is equivalent to the statement that the point is in the convex hull of the set. Hence we have:–

THEOREM CE_d. *If in E_d a point is in the convex hull of a finite set of points G, where $|G| > d+1$, then it is in the convex hull of some $d+1$ points of G including an arbitrarily chosen point of G.*

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References

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