

On the smoothness of the moduli space of mathematical instanton bundles

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Abstract. In this paper we prove that the moduli spaces $MI_{\mathbb{P}^{2n+1}}(k)$ of mathematical instanton bundles on $\mathbb{P}_{\mathbb{C}}^{2n+1}$ with quantum number k are singular for $n \geq 2$ and $k \geq 3$, giving a positive answer to a conjecture made by Ancona and Ottaviani in 1993.

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Introduction

Throughout this paper \mathbf{k} will be an algebraically closed field of characteristic zero and \mathbf{P}^{2n+1} the $(2n+1)$ -dimensional projective space over the field \mathbf{k} . Let $MI_{2n+1}(k)$ be the moduli space of all mathematical instanton bundles over \mathbf{P}^{2n+1} with second Chern class $c_2 = k$. Related to the smoothness of $MI_{2n+1}(k)$ there are two important conjectures:

CONJECTURE 1. *The moduli spaces $MI_3(k)$ are smooth of dimension $8k - 3$.*

It is well known that the moduli space $MI_3(k)$ is nonsingular of dimension $8k-3$ for any $k \leq 4$ (See [B1] for $k = 1$, [H2] for $k = 2$, [ES] for $k = 3$, and [B2] or [LP] for $n = 4$); and as far as we know Conjecture 1 remains open for $k > 4$. For $n \geq 2$ the situation is quite different and we have:

CONJECTURE 2. *For all integers $n \geq 2$ and $k \geq 3$ the moduli spaces $MI_{2n+1}(k)$ are singular.*

In [AO1], Ancona and Ottaviani have recently proved: (1) the moduli spaces $MI_{2n+1}(2)$ are smooth, irreducible of dimension $4n^2 + 2n - 3$; and (2) the moduli spaces $MI_5(3)$ and $MI_5(4)$ are singular. The main goal of this paper is to show that Conjecture 2 is true.

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OUTLINE OF THE PROOF

First of all we observe that for any instanton bundle E on \mathbf{P}^{2n+1} , $H^i \mathcal{E}nd(E) = 0$ for $i \geq 3$, $H^0 \mathcal{E}nd(E) = \mathbf{k}$ (E is simple) and $h^1 \mathcal{E}nd(E) - h^2 \mathcal{E}nd(E) = -k^2 \binom{2n-1}{2} + k(8n^2) + 1 - 4n^2$ (by Hirzebruch-Riemann-Roch). Let $M^0 := MI_{2n+1}^0(k)$ be the irreducible component of $MI_{2n+1}(k)$ containing special instanton bundles. In [AO; Theorem 3.7], Ancona and Ottaviani proved that special symplectic instanton bundles are stable and the generic special instanton bundle is stable. Therefore the Zariski tangent space of $MI_{2n+1}(k)$ at special symplectic instanton bundles can be identified with $\text{Ext}^1(E, E)$.

Very recently Ottaviani and Trautmann have proved that for any special symplectic instanton bundle $E \in MI_{2n+1}^0(k)$, $h^2 \mathcal{E}nd(E) = (k-2)^2 \binom{2n-1}{2}$ [OT]. In this paper, for all integers $n \geq 2$ and $k \geq 3$, we will construct deformations E' of special symplectic instanton bundles in $MI_{2n+1}^0(k)$ satisfying (Cf. Theorem 3.1):

$$h^2 \mathcal{E}nd(E') < h^2 \mathcal{E}nd(E) = (k-2)^2 \binom{2n-1}{2}.$$

Putting altogether we get that for all integers $n \geq 2$ and $k \geq 3$, the moduli spaces $MI_{2n+1}(k)$ are singular at least in special symplectic bundles.

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Notation

Throughout this paper \mathbf{k} will be an algebraically closed field of characteristic zero and \mathbf{P}^{2n+1} the $(2n+1)$ -dimensional projective space over the field \mathbf{k} . We denote by $\mathcal{O}(d)$ the invertible sheaf of degree d on \mathbf{P}^{2n+1} . For any coherent sheaf F on \mathbf{P}^{2n+1} we use the abbreviations $F(d) := F \otimes \mathcal{O}(d)$, $H^i F := H^i(\mathbf{P}^{2n+1}, F)$ and $h^i F = \dim_{\mathbf{k}} H^i(\mathbf{P}^{2n+1}, F)$. The terms vector bundle and locally free sheaf are used synonymously. We will use the definition of stable and semistable due to Mumford-Takemoto ([OSS]).

1. Instanton bundles

We will begin this section recalling the notion of monad and the basic facts on Instanton bundles needed in the sequel.

DEFINITION 1.1. A monad over \mathbf{P}^{2n+1} is a complex of vector bundles:

$$0 \rightarrow A \xrightarrow{a} B \xrightarrow{b} C \rightarrow 0$$

with $ab = 0$, $a: A \rightarrow B$ an injective bundle-map and $b: B \rightarrow C$ surjective. The vector bundle $E := \text{Ker}(b)/\text{Im}(a)$ is called the cohomology bundle of the monad.

A monad $0 \rightarrow A \xrightarrow{a} B \xrightarrow{b} C \rightarrow 0$ has a so-called display: this is a commutative diagram with exact rows and columns:

$$\begin{array}{ccccccc}
 & & & 0 & & 0 & \\
 & & & \downarrow & & \downarrow & \\
 0 & \longrightarrow & A & \longrightarrow & K & \longrightarrow & E & \longrightarrow & 0 \\
 & & \parallel & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & A & \xrightarrow{a} & B & \longrightarrow & Q & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & b & & & & \\
 & & & & C & = & C & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
 \end{array}$$

where $K := \text{Ker}(b)$ and $Q := \text{Coker}(a)$. From the display one easily deduces that $rk(E) = rk(B) - rk(A) - rk(C)$ and $c_t(E) = c_t(B)c_t(A)^{-1}c_t(C)^{-1}$.

DEFINITION 1.2. A mathematical instanton bundle on \mathbf{P}^{2n+1} with quantum number k is a rank $2n$ vector bundle E on \mathbf{P}^{2n+1} satisfying:

- (1) E has Chern polynomial $c_t(E) = (1 - t^2)^{-k} = 1 + kt^2 + \dots$,
- (2) E has natural cohomology in the range $-2n - 1 \leq d \leq 0$, that is for any d in that range $h^i E(d) \neq 0$ for at most one i , and
- (3) the restriction of E to a general line is trivial.

Remark 1.3. In the original definition in [OS] the additional condition E is simple is imposed. However, it has been shown in [AO] that this last condition is already a consequence of (i) and (ii).

As an easy consequence of the above definition we get:

(1.4) Let E be an instanton bundle on \mathbf{P}^{2n+1} with second Chern class k . Then, E is the cohomology bundle of a monad of the following type:

$$0 \rightarrow k\mathcal{O}(-1) \xrightarrow{A} (2n + 2k)\mathcal{O} \xrightarrow{B} k\mathcal{O}(1) \rightarrow 0. \tag{*}$$

Moreover, any vector bundle E on \mathbf{P}_{2n+1} which appears as the cohomology bundle of a monad of the type (*) is a rank $2n$ vector bundle on \mathbf{P}^{2n+1} verifying the conditions (1) and (2) of definition 1.2 [AO1].

LEMMA 2.2. *Let E be an instanton bundle on \mathbf{P}^{2n+1} with second Chern class k arising from the monad:*

$$0 \rightarrow k\mathcal{O}(-1) \xrightarrow{A} (2n + 2k)\mathcal{O} \xrightarrow{B} k\mathcal{O}(1) \rightarrow 0. \tag{*}$$

Then, $H^2(\mathcal{E}nd(E)) = \text{Coker}(d_0)$ where

$$d_0: \text{Hom}(\mathcal{O}(-1)^k, \mathcal{O}^{2n+2k}) \times \text{Hom}(\mathcal{O}^{2n+2k}, \mathcal{O}(1)^k) \rightarrow \text{Hom}(\mathcal{O}(-1)^k, \mathcal{O}(1)^k)$$

is the morphism given by $d_0(a, b) = Ab + aB$.

Proof. Using Künneth Theorem (Cf. [G]; p. 100) we get that $F = E \otimes E^v$ is the cohomology bundle of the simple complex:

$$0 \rightarrow k^2\mathcal{O}(-2) \rightarrow (4k^2 + 4nk)\mathcal{O}(-1) \xrightarrow{\beta} (6k^2 + 8nk + 4n^2)\mathcal{O} \xrightarrow{\alpha} (4k^2 + 4nk)\mathcal{O}(1) \rightarrow k^2\mathcal{O}(2) \rightarrow 0$$

associated to the double complex:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \mathcal{O}(-1)^k \otimes \mathcal{O}(-1)^k & \longrightarrow & \mathcal{O}^{2n+2k} \otimes \mathcal{O}(-1)^k & \longrightarrow & \mathcal{O}(1)^k \otimes \mathcal{O}(-1)^k \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \mathcal{O}(-1)^k \otimes \mathcal{O}^{2n+2k} & \longrightarrow & \mathcal{O}^{2n+2k} \otimes \mathcal{O}^{2n+2k} & \longrightarrow & \mathcal{O}(1)^k \otimes \mathcal{O}^{2n+2k} \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \mathcal{O}(-1)^k \otimes \mathcal{O}(1)^k & \longrightarrow & \mathcal{O}^{2n+2k} \otimes \mathcal{O}(1)^k & \longrightarrow & \mathcal{O}(1)^k \otimes \mathcal{O}(1)^k \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

The above complex is exact except in the middle term and there we have $E \otimes E^v = \text{Ker}(\alpha)/\text{Im}(\beta)$. Therefore, we have the exact sequences:

$$0 \rightarrow k^2\mathcal{O}(-2) \rightarrow (4k^2 + 4nk)\mathcal{O}(-1) \xrightarrow{\beta} \text{Im}(\beta) \rightarrow 0,$$

$$0 \rightarrow \text{Im}(\beta) \rightarrow \text{Ker}(\alpha) \rightarrow E \otimes E^v \rightarrow 0,$$

$$0 \rightarrow \text{Ker}(\alpha) \rightarrow (6k^2 + 8nk + 4n^2)\mathcal{O} \rightarrow \text{Im}(\alpha) \rightarrow 0,$$

$$0 \rightarrow \text{Im}(\alpha) \rightarrow (4k^2 + 4nk)\mathcal{O}(1) \rightarrow k^2\mathcal{O}(2) \rightarrow 0,$$

and we obtain $H^2(E \otimes E^v) = H^2(\text{Ker } \alpha) = H^1(\text{Im } \alpha) = \text{Coker}(d_0)$.

Using the above notation, we have:

$$\begin{aligned} \dim \text{Ext}^2(E, E) &= \dim \text{Coker}(d_0) = \binom{2n+3}{2}k^2 - \dim \text{Im}(d_0) \\ &= \binom{2n+3}{2}k^2 - 2k(2n+2k)(2n+2) + \dim \text{Ker}(d_0). \end{aligned}$$

Assume that A (resp. B^t) is the matrix which presents a module P (resp. Q). We define $M = M(A, B^t) := M(A \otimes \text{id}, \text{id} \otimes B^t)$ as the matrix which presents the tensor product of P and Q ; and we denote by $\text{syz}_1 M$ the dimension of the \mathbf{k} -vector space of the syzygies of M of degree 1. We have $\dim \text{Ker}(d_0) = \text{syz}_1 M$ and we obtain the following useful formula:

$$\dim \text{Ext}^2(E, E) = k(2n^2k - 3nk - 5k - 8n^2 - 8n) + \text{syz}_1 M \quad (***)$$

that will be used in the sequel.

3. Main theorem

Now we are ready for proving that for all integers $k \geq 3$ and $n \geq 2$, the moduli spaces $MI_{2n+1}(k)$ are singular at least in special symplectic instanton bundles.

THEOREM 3.1. *For all integers $k \geq 3$ and $n \geq 2$, the moduli spaces $MI_{2n+1}(k)$ are singular.*

Proof. Let E_0 be a special symplectic instanton bundle on \mathbf{P}^{2n+1} with second Chern class k . By [AO; Theorem 3.7] any symplectic special instanton bundle on \mathbf{P}^{2n+1} is stable; hence using deformation theory the Zariski tangent space to the moduli space $MI_{2n+1}(k)$ at the point corresponding to E_0 is isomorphic to the vector space $\text{Ext}^1(E_0, E_0)$ and the obstructions to extending an infinitesimal deformation lie in $\text{Ext}^2(E_0, E_0)$.

From [OT; Theorem 4.1], we get:

$$\dim \text{Ext}^1(E_0, E_0) = 4k(3n-1) + (2n-5)(2n-1)$$

and

$$\dim \text{Ext}^2(E_0, E_0) = (k-2)^2 \binom{2n-1}{2}.$$

Let $M^0 \subset MI_{2n+1}(k)$ be the irreducible component containing special instanton bundles on \mathbf{P}^{2n+1} with second Chern class k . By [AO; Corollary 3.10] a generic

vector bundle $E \in M^0$ is stable. Hence for proving Theorem 3.1 it is enough to construct a vector bundle $E \in M^0$ with $\dim \text{Ext}^1(E, E) < \dim \text{Ext}^1(E_0, E_0)$. Let us denote by E_u the special instanton bundle on \mathbf{P}^{2n+1} with second Chern class k defined as the cohomology bundle of the following monad:

$$0 \rightarrow k\mathcal{O}(-1) \xrightarrow{A_u} (2n + 2k)\mathcal{O} \xrightarrow{B} k\mathcal{O}(1) \rightarrow 0, \tag{*}$$

where

$$B^t := \begin{pmatrix} x_0 & x_1 & \dots & x_n & & & & & y_0 & y_1 & \dots & y_n \\ & x_0 & x_1 & \dots & x_n & & & & y_0 & y_1 & \dots & y_n \\ & & & & x_0 & x_1 & \dots & x_n & & & & y_0 & y_1 & \dots & y_n \end{pmatrix}$$

and

$$A_u := \begin{pmatrix} & & & & y_n & \dots & y_1 & y_0 & & & & & & & -x_n & \dots & -x_1 & -x_0 \\ & & & & y_n & \dots & y_1 & y_0 & & & -x_n & \dots & -x_1 & -x_0 & & & & \\ M_n & \dots & M_1 & M_0 & & & & M & N_n & \dots & N_1 & N_0 & & & & & & N \end{pmatrix},$$

where $M_i = (1 - u)y_i + uy_{i-1}$ for $i = 1, \dots, n$, $M_0 = (1 - u)y_0$, $M = uy_n$, $N_i = (u - 1)x_i - ux_{i-1}$ for $i = 1, \dots, n$, $N_0 = (u - 1)x_0$ and $N = -ux_n$.

We have constructed a flat family $\{E_u\}_{u \in \mathbf{k}}$ of special instanton bundles on \mathbf{P}^{2n+1} with second Chern class k which is a deformation of E_0 in M^0 . Using lemma 3.2 and the above formula (***) we get:

$$\dim \text{Ext}^2(E_{u=1}, E_{u=1}) < \dim \text{Ext}^2(E_0, E_0)$$

or, equivalently,

$$\dim \text{Ext}^1(E_{u=1}, E_{u=1}) < \dim \text{Ext}^1(E_0, E_0).$$

This gives us that $MI_{2n+1}(k)$ is singular at least in the special symplectic instanton bundle E_0 .

LEMMA 3.2. For all integers $k \geq 3$ and $n \geq 2$, we take the $(k) \times (2n+2k)$ matrices:

$$B(k)^t := \begin{pmatrix} x_0 & x_1 & \dots & x_n & & & & & y_0 & y_1 & \dots & y_n \\ & x_0 & x_1 & \dots & x_n & & & & y_0 & y_1 & \dots & y_n \\ & & & & x_0 & x_1 & \dots & x_n & & & & y_0 & y_1 & \dots & y_n \end{pmatrix}$$

and

$$A_u(k) := \begin{pmatrix} & & & & y_n & \dots & y_1 & y_0 & & & & & & & -x_n & \dots & -x_1 & -x_0 \\ & & & & y_n & \dots & y_1 & y_0 & & & -x_n & \dots & -x_1 & -x_0 & & & & \\ M_n & \dots & M_1 & M_0 & & & & M & N_n & \dots & N_1 & N_0 & & & & & & N \end{pmatrix},$$

where $M_i = (1 - u)y_i + uy_{i-1}$ for $i = 1, \dots, n$, $M_0 = (1 - u)y_0$, $M = uy_n$, $N_i = (u - 1)x_i - ux_{i-1}$ for $i = 1, \dots, n$, $N_0 = (u - 1)x_0$ and $N = -ux_n$.

Assume that $A_u(k)$ (resp. $B(k)^t$) is the matrix which presents a module $P_u(k)$ (resp. $Q(k)$). We define $M_u(k) := M(A_u(k), B(k)^t)$ as the matrix which presents the tensor product of $P_u(k)$ and $Q(k)$. Then, we have:

- (1) $syzy_1(M_{u=0}(k)) = (k - 2)^2(2n - 1)(n - 1) - k(2n^2k - 3nk - 5k - 8n^2 - 8n)$;
- (2) $syzy_1(M_{u=1}(k)) < syzy_1(M_{u=0}(k))$.

Proof. (1) It follows from [OT; Theorem 4.1] and the above formula (***)

(2) We denote by $S_1(M_u(k))$ the \mathbf{k} -vector space of the syzygies of $M_u(k)$ of degree 1. We consider the \mathbf{k} -linear map

$$\Psi_k: S_1(M_{u=1}(k)) \rightarrow S_1(M_{u=0}(k - 1))$$

which sends $v = (P_1^1, \dots, P_1^{k+n}, P_1^{k+n+1}, \dots, P_1^{2k+2n}, \dots; P_k^1, \dots, P_k^{k+n}, P_k^{k+n+1}, \dots, P_k^{2k+2n}; P_{k+1}^1, \dots, P_{k+1}^{k+n}, P_{k+1}^{k+n+1}, \dots, P_{k+1}^{2k+2n}, \dots; P_{2k}^1, \dots, P_{2k}^{k+n}, P_{2k}^{k+n+1}, \dots, P_{2k}^{2k+2n}) \in S_1(M_{u=1}(k))$ to $\Psi_k(v) := (P_1^2, \dots, P_1^{k+n}, P_1^{k+n+2}, \dots, P_1^{2k+2n}, \dots; P_{k-1}^1, \dots, P_{k-1}^{k+n}, P_{k-1}^{k+n+2}, \dots, P_{k-1}^{2k+2n}, \dots; P_{k+1}^1, \dots, P_{k+1}^{k+n-1}, P_{k+1}^{k+n+1}, \dots, P_{k+1}^{2k+2n-1}, \dots; P_{2k-1}^1, \dots, P_{2k-1}^{k+n-1}, P_{2k-1}^{k+n+1}, \dots, P_{2k-1}^{2k+2n-1}) \in S_1(M_{u=0}(k - 1))$ (The indexing P_i^j has been chosen accordingly to the shape of the matrix M_u).

CLAIM. $\dim \text{Ker} \Psi_k < 12k + 20n - 10$.

Proof of the claim. For all integers $i = 1, \dots, 2k$ and $j = 1, \dots, 2k + 2n$, we consider the linear forms $P_i^j := X_{0,i}^j x_0 + \dots + X_{n,i}^j x_n + Y_{0,i}^j y_0 + \dots + Y_{n,i}^j y_n \in \mathbf{k}[x_0, \dots, x_n, y_0, \dots, y_n]$. Notice that $\text{Ker}(\Psi_k)$ is the \mathbf{k} -vector space generated by the vectors $v = (P_1^1, \dots, P_1^{2k+2n}; P_2^1, \dots, P_2^{2k+2n}, \dots; P_{2k}^1, \dots, P_{2k}^{2k+2n})$ satisfying the following conditions:

- (1) $P_i^j = 0$ for $1 \leq i \leq k - 1$ and $j \in \{1, \dots, 2k + 2n\} \setminus \{1, k + n + 1\}$
- (2) $P_i^j = 0$ for $k + 1 \leq i \leq 2k - 1$ and $j \in \{1, \dots, 2k + 2n\} \setminus \{k + n, 2k + 2n\}$
- (3) $v \in S_1(M_{u=1}(k))$

On the other hand condition (3) is equivalent to condition:

$$(3')v(M_{u=1}(k)) = (0, \dots, 0)$$

Therefore, $\text{Ker}(\Psi_k)$ is the \mathbf{k} -vector space generated by the vectors $v := (P_1^1, \dots, P_1^{2k+2n}; P_2^1, \dots, P_2^{2k+2n}, \dots; P_{2k}^1, \dots, P_{2k}^{2k+2n})$ verifying:

- (1) $P_i^j = 0$ for $1 \leq i \leq k - 1$ and $j \in \{1, \dots, 2k + 2n\} \setminus \{1, k + n + 1\}$
- (2) $P_i^j = 0$ for $k + 1 \leq i \leq 2k - 1$ and $j \in \{1, \dots, 2k + 2n\} \setminus \{k + n, 2k + 2n\}$

and the following $2k - 1$ equations:

$$\begin{aligned}
 (3a) \quad & P_i^1 y_{n-1} + P_i^{k+n+1}(-x_{n-1}) + P_{2k}^i x_0 + \dots + P_{2k}^{i+n} x_n + P_{2k}^{k+n+i} y_0 + \dots + \\
 & P_{2k}^{k+i+2n} y_n = 0 \text{ for } i = 1, \dots, k - 1; \\
 (3b) \quad & P_k^{k+1-i} y_n + \dots + P_k^{k+n+1-i} y_0 + P_k^{2k+n+1-i}(-x_n) + \dots + P_k^{2n+2k+1-i}(-x_0) + \\
 & P_{k+i}^{k+n} x_n + P_{k+i}^{2k+2n} y_n = 0 \text{ for } i = 1, \dots, k - 1; \\
 (3c) \quad & P_k^1 y_{n-1} + P_k^2 y_{n-2} + \dots + P_k^n y_0 + P_k^{n+k} y_n + P_k^{k+n+1}(-x_{n-1}) + \\
 & P_k^{k+n+2}(-x_{n-2}) + \dots + P_k^{k+2n}(-x_0) + P_k^{2n+2k}(-x_n) + P_{2k}^k x_0 + \dots + \\
 & P_{2k}^{n+k} x_n + P_{2k}^{2k+n} y_0 + \dots + P_{2k}^{2k+2n} y_n = 0.
 \end{aligned}$$

The $k - 1$ equations (3a) give rise to the following relations among the coefficients of the linear forms $P_i^j := X_{0,i}^j x_0 + \dots + X_{n,i}^j x_n + Y_{0,i}^j y_0 + \dots + Y_{n,i}^j y_n \in \mathbf{k}[x_0, \dots, x_n, y_0, \dots, y_n]$ ($i = 1, \dots, k - 1$):

$$\begin{aligned}
 & X_{\gamma,2k}^{i+\beta} + X_{\beta,2k}^{i+\gamma} = 0 \text{ for } 1 \leq \beta \leq \gamma \leq n, \beta \neq n - 1 \quad \text{and} \quad \gamma \neq n - 1; \\
 & -X_{\beta,i}^{k+n+1} + X_{n-1,2k}^{i+\beta} + X_{\beta,2k}^{i+n-1} = 0 \text{ for } 1 \leq \beta < n - 1; \\
 & -X_{n-1,i}^{k+n+1} + X_{n-1,2k}^{i+n-1} = 0; \\
 & -X_{n,i}^{k+n+1} + X_{n,2k}^{i+n-1} + X_{n-1,2k}^{i+n} = 0; \\
 & Y_{\gamma,2k}^{i+\beta} + X_{\beta,2k}^{k+n+i+\gamma} = 0 \text{ for } 1 \leq \beta, \gamma \leq n, \beta \neq n - 1 \quad \text{and} \quad \gamma \neq n - 1; \\
 & -Y_{\gamma,i}^{k+n+1} + Y_{\gamma,2k}^{i+n-1} + X_{n-1,2k}^{k+n+i+\gamma} = 0 \text{ for } 1 \leq \gamma < n \quad \text{and} \quad \gamma \neq n - 1; \\
 & X_{\beta,i}^1 + Y_{n-1,2k}^{i+\beta} + X_{\beta,2k}^{k+2n+i-1} = 0 \text{ for } 1 \leq \beta \leq n \quad \text{and} \quad \beta \neq n - 1; \\
 & X_{n-1,i}^1 - Y_{n-1,i}^{k+n+1} + Y_{n-1,2k}^{i+n-1} + X_{n-1,2k}^{k+2n+i-1} = 0; \\
 & Y_{\gamma,2k}^{k+n+i+\beta} + Y_{\beta,2k}^{k+n+i+\gamma} = 0 \text{ for } 1 \leq \beta \leq \gamma \leq n, \beta \neq n - 1 \quad \text{and} \quad \gamma \neq n - 1; \\
 & Y_{\beta,i}^1 + Y_{n-1,2k}^{k+n+i+\beta} + Y_{\beta,2k}^{k+2n+i-1} = 0 \quad \text{for } 1 \leq \beta < n - 1; \\
 & Y_{n-1,i}^1 + Y_{n-1,2k}^{k+2n+i-1} = 0; \\
 & Y_{n,i}^1 + Y_{n,2k}^{k+2n+i-1} + Y_{n-1,2k}^{k+2n+i} = 0;
 \end{aligned}$$

The $k - 1$ equations (3b) give rise to the following relations ($i = 1, \dots, k - 1$):

$$\begin{aligned}
 & X_{\gamma,k}^{2k+2n-i-\beta+1} + X_{\beta,k}^{2k+2n-i-\gamma+1} = 0 \quad \text{for } 1 \leq \beta \leq \gamma < n; \\
 & Y_{\gamma,k}^{k+n-i-\beta+1} + Y_{\beta,k}^{k+n-i-\gamma+1} = 0 \quad \text{for } 1 \leq \beta \leq \gamma < n; \\
 & X_{\beta,k}^{k+n-i-\gamma+1} - Y_{\gamma,k}^{2k+2n-i-\beta+1} = 0 \quad \text{for } 1 \leq \beta \leq \gamma < n; \\
 & -X_{\beta,k}^{2k+n+1-i} - X_{n,k}^{2k+2n-i-\beta+1} + X_{\beta,k+i}^{k+n} = 0 \quad \text{for } 1 \leq \beta < n;
 \end{aligned}$$

$$\begin{aligned}
 & -X_{n,k}^{2k+n-i+1} + X_{n,k+i}^{k+n} = 0; \\
 & Y_{\beta,k}^{k+1-i} + Y_{n,k}^{k+n-i-\beta+1} + Y_{\beta,k+i}^{2k+2n} = 0 \quad \text{for } 1 \leq \beta < n; \\
 & Y_{n,k}^{k-i+1} + Y_{n,k+i}^{2k+2n} = 0; \\
 & X_{\beta,k}^{k+1-i} - Y_{n,k}^{2k+2n-i-\beta+1} + X_{\beta,k+i}^{2k+2n} = 0 \quad \text{for } 1 \leq \beta < n; \\
 & X_{n,k}^{k+n-\beta+1-i} - Y_{\beta,k}^{2k+n-i+1} + Y_{\beta,k+i}^{k+n} = 0 \quad \text{for } 1 \leq \beta < n; \\
 & X_{n,k}^{k+1-i} - Y_{n,k}^{2k+n-i+1} + Y_{n,k+i}^{k+n} + X_{n,k+i}^{2k+2n} = 0.
 \end{aligned}$$

Finally, from the equation (3c) we obtain the following relations:

$$\begin{aligned}
 & -X_{\gamma,k}^{k+2n-\beta} - X_{\beta,k}^{k+2n-\gamma} + X_{\gamma,2k}^{k+\beta} + X_{\beta,2k}^{k+\gamma} = 0 \quad \text{for } 1 \leq \beta \leq \gamma < n; \\
 & -X_{n,k}^{k+2n-\beta} - X_{\beta,k}^{2k+2n} + X_{n,2k}^{k+\beta} + X_{\beta,2k}^{k+n} = 0 \quad \text{for } 1 \leq \beta < n; \\
 & -X_{n,k}^{2k+2n} + X_{n,2k}^{k+n} = 0; \\
 & Y_{\gamma,k}^{n-\beta} + Y_{\beta,k}^{n-\gamma} + Y_{\gamma,2k}^{2k+n+\beta} + X_{\beta,2k}^{2k+n+\gamma} = 0 \quad \text{for } 1 \leq \beta \leq \gamma < n; \\
 & Y_{n,k}^{n-\beta} + Y_{\beta,k}^{k+n} + Y_{n,2k}^{2k+n+\beta} + Y_{\beta,2k}^{2k+2n} = 0 \quad \text{for } 1 \leq \beta < n; \\
 & Y_{n,k}^{k+n} + Y_{n,2k}^{2k+2n} = 0; \\
 & X_{\beta,k}^{n-\gamma} + X_{\beta,2k}^{2k+n+\gamma} - Y_{\gamma,k}^{k+2n-\beta} + Y_{\gamma,2k}^{k+\beta} = 0 \quad \text{for } 1 \leq \beta, \gamma < n; \\
 & X_{\beta,k}^{n+k} + X_{\beta,2k}^{2k+2n} - Y_{n,k}^{k+2n-\beta} + Y_{n,2k}^{k+\beta} = 0 \quad \text{for } 1 \leq \beta < n; \\
 & X_{n,k}^{n-\gamma} + X_{n,2k}^{2k+n+\gamma} - Y_{\gamma,k}^{2k+2n} + Y_{\gamma,2k}^{k+n} = 0 \quad \text{for } 1 \leq \gamma < n; \\
 & X_{n,k}^{k+n} + X_{n,2k}^{2k+2n} - Y_{n,k}^{2k+2n} + Y_{n,2k}^{k+n} = 0.
 \end{aligned}$$

Putting altogether, we get, after an intricate computation, that $\dim \text{Ker } \Psi_k < 12k + 20n - 10$, which proves our claim.

Using the \mathbf{k} -linear map Ψ_k we obtain:

$$\begin{aligned}
 \text{syz}_1(M_{u=1}(k)) &= \dim S_1(M_{u=1}(k)) = \dim(\text{Ker } \Psi_k) + \dim \text{Im}(\Psi_k) \\
 &\leq \dim(\text{Ker } \Psi_k) + \text{syz}_1(M_{u=0}(k-1)) < \text{(Claim)} \\
 &< 12k + 20n - 10 + \text{syz}_1(M_{u=0}(k-1)) = \text{(Lemma 3.2(1))} \\
 &= \binom{2n-1}{2} (k-2)^2 - k(2n^2k - 3nk - 5k - 8n^2 - 8n) \\
 &= \text{syz}_1(M_{u=0}(k))
 \end{aligned}$$

which gives what we want.

Remark 3.2.1. For our purpose it is enough to see that $\text{syz}_1(M_{u=1}(k)) < \text{syz}_1(M_{u=0}(k))$ and we do not need to know the precise value of $\text{syz}_1(M_{u=1}(k))$. After computing the first cases using the computer program ‘Macaulay’ [BS], we guess:

$$\text{syz}_1(M_{u=1}(k)) = \text{syz}_1(M_{u=0}(k)) - (4n - 5)(k - 2).$$

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