8
Field transformations

The previous chapters take a pragmatic, almost engineering, approach to the solution of field theories. The recipes of chapter 5 are invaluable in generating solutions to field equations in many systems, but the reason for their effectiveness remains hidden. This chapter embarks upon a train of thought, which lies at the heart of the theory of dynamical systems, which explain the fundamental reasons why field theories look the way they do, how physical quantities are related to the fields in the action, and how one can construct theories which give correct answers regardless of the perspective of the observer. Before addressing these issues directly, it is necessary to understand some core notions about symmetry on a more abstract level.

8.1 Group theory

To pursue a deeper understanding of dynamics, one needs to know the language of transformations: *group theory*. Group theory is about families of transformations with special symmetry. The need to *parametrize* symmetry groups leads to the idea of *algebras*, so it will also be necessary to study these.

Transformations are central to the study of dynamical systems because all changes of variable, coordinates or measuring scales can be thought of as transformations. The way one parametrizes fields and spacetime is a matter of convenience, but one should always be able to transform any results into a new perspective whenever it might be convenient. Even the dynamical development of a system can be thought of as a series of transformations which alter the system's state progressively over time. The purpose of studying groups is to understand the implications posed by constraints on a system: the field equations and any underlying symmetries – but also the rules by which the system unfolds on the background spacetime. In pursuit of this goal, we shall find universal themes which enable us to understand many structures from a few core principles.

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8.1.1 Definition of a group

A group is a set of objects, usually numbers or matrices, which satisfies the following conditions.

(1) There is a rule of composition for the objects. When two objects in a group are combined using this rule, the resulting object also belongs to the group. Thus, a group is *closed* under the action of the composition rule. If a and b are two matrices, then $a \cdot b \neq b \cdot a$ is not necessarily true. If $a \cdot b = b \cdot a$, the group is said to be Abelian, otherwise it is non-Abelian.

(2) The combination rule is associative, i.e. $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.

(3) The identity element belongs to the set, i.e. an object which satisfies $a \cdot I = a$.

(4) Every element a in the set has a right-inverse a^{-1}, such that $a^{-1} \cdot a = I$.

A group may contain one or more *sub-groups*. These are sub-sets of the whole group which also satisfy all of the group axioms. Sub-groups always overlap with one another because they must all contain the identity element. Every group has two trivial or improper sub-groups, namely the identity element and the whole group itself. The *dimension* of a group d_G is defined to be the number of independent degrees of freedom in the group, or the number of generators required to represent it. This is most easily understood by looking at the examples in the next section. The *order* of a group O_G is the number of distinct elements in the group. In a continuous group the order is always infinite.

If the ordering of elements in the group with respect to the combination rule matters, i.e. the group elements do not commute with one another, the group is said to be non-Abelian. In that case, there always exists an Abelian sub-group which commutes with *every* element of the group, called the *centre*. *Schur's lemma* tells us that any element of a group which commutes with every other must be a multiple of the identity element. The centre of a group is usually a discrete group, Z_N, with a finite number, N, of elements called the *rank* of the group.

8.1.2 Group transformations

In field theory, groups are used to describe the relationships between components in a multi-component field, and also the behaviour of the field under spacetime transformations. One must be careful to distinguish between two vector spaces in the discussions which follow. It is also important to be very clear about what is being transformed in order to avoid confusion over the names.

- *Representation space*. This is the space on which the group transformations act, or the space in which the objects to be transformed live. In field theory, when transformations relate to internal symmetries, the components of field multiplets $(\phi_1, \phi_2, \ldots, \phi_{d_R})$ are the coordinates on representation space. When transformations relate to changes of spacetime frame, then spacetime coordinates are the representation space.
- *Group space*. This is an abstract space of dimension d_G. The dimension of this space is the number of independent transformations which the group is composed of. The coordinates $(\theta_1, \theta_2, \ldots, \theta_{d_G})$ in this space are measured with respect to a set of basis matrices called the *generators* of the group.

Since fields live on spacetime, the full representation space of a field consists of spacetime (μ, ν indices) combined with any hidden degrees of freedom: spin, charge, colour and any other hidden labels or indices (all denoted with indices $A, B, a, b, \alpha, \beta$) which particles might have. In practice, some groups (e.g. the Lorentz group) act only on spacetime, others (e.g. $SU(3)$) act only on hidden indices. In this chapter, we shall consider group theory on a mainly abstract level, so this distinction need not be of concern.

A field, $\phi(x)$, might be a spacetime-scalar (i.e. have no spacetime indices), but also be vector on representation space (have a single group index).

$$\phi(x)_A = \begin{pmatrix} \phi_1(x) \\ \phi_2(x) \\ \vdots \\ \phi_{d_R}(x) \end{pmatrix}. \tag{8.1}$$

The transformation rules for fields with spacetime (coordinate) indices are therefore

$$\begin{aligned} \phi &\to \phi' \\ A_\mu &\to U_\mu{}^\nu\, A_\nu \\ g_{\mu\nu} &\to U_\mu{}^\rho U_\nu{}^\lambda\, g_{\rho\lambda}, \end{aligned} \tag{8.2}$$

and for multiplet transformations they are

$$\begin{aligned} \phi^A &\to U_{AB}\, \phi^B \\ A_\mu^a &\to U_{ab}\, A_\mu^b \\ g_{\mu\nu}^A &\to U_{AB}\, g_{\mu\nu}^B. \end{aligned} \tag{8.3}$$

All of the above have the generic form of a vector $\mathbf{v}$ with Euclidean components $v^A = v_A$ transforming by matrix multiplication:

$$\mathbf{v} \to U\mathbf{v}, \tag{8.4}$$

or

$$v^{A'} = U^{A}_{\ B}\, v^{B}. \tag{8.5}$$

The label $A = 1, \ldots, d_R$, where d_R is the dimension of the representation. Thus, the transformation matrix U is a $d_R \times d_R$ matrix and $\mathbf{v}$ is a d_R-component column vector. The group space is Euclidean, so raised and lowered A, B indices are identical here.

Note that multiplet indices (those which do not label spacetime coordinates) for general group representations G_R are labelled with upper case Latin characters $A, B = 1, \ldots, d_R$ throughout this book. Lower case Latin letters $a, b = 1, \ldots, d_G$ are used to distinguish the components of the adjoint representation G_{adj}.

In general, the difference between a representation of a group and the group itself is this: while a group might have certain unique abstract properties which define it, the realization of those properties in terms of numbers, matrices or functions might not be unique, and it is the explicit representation which is important in practical applications. In the case of Lie groups, there is often a variety of possible locally isomorphic groups which satisfy the property (called the Lie *algebra*) that defines the group.

8.1.3 Use of variables which transform like group vectors

The property of transforming a dynamical field by simple matrix multiplication is very desirable in quantum theory where symmetries are involved at all levels. It is a direct representation of the Markov property of physical law. In chapter 14, it becomes clear that invariances are made extremely explicit and are algebraically simplest if transformation laws take the multiplicative form in eqn. (8.5).

An argument against dynamical variables which transform according to group elements is that they cannot be observables, because they are non-unique. Observables can only be described by invariant quantities. A vector is, by definition, not invariant under transformations; however, the scalar product of vectors is invariant.

In classical particle mechanics, the dynamical variables $q(t)$ and $p(t)$ do not transform by simple multiplication of elements of the Galilean symmetry. Instead, there is a set of eqns. (14.34) which describes how the variables change under the influence of group generators. Some would say that such a formulation is most desirable, since the dynamical variables are directly observable, but the price for this is a more complicated set of equations for the symmetries.

As we shall see in chapter 14, the quantum theory is built upon the idea that the dynamical variables should transform like linear combinations of vectors on some group space. Observables are extracted from these vectors with the help of operators, which are designed to pick out actual data as eigenvalues of the operators.

8.2 Cosets and the factor group

8.2.1 Cosets

Most groups can be decomposed into non-overlapping sub-sets called *cosets*. Cosets belong to a given group and one if its sub-groups. Consider then a group G of order O_G, which has a sub-group H of order O_H. A coset is defined by acting with group elements on the elements of the sub-group. In a non-Abelian group one therefore distinguishes between *left* and *right* cosets, depending on whether the group elements pre- or post-multiply the elements of the sub-group. The left coset of a given group element is thus defined by

$$GH \equiv \left\{GH_1, GH_2, \ldots, GH_{d_H}\right\} \tag{8.6}$$

and the right coset is defined by

$$HG = \left\{H_1G, H_2G, \ldots, H_{d_H}G\right\}. \tag{8.7}$$

The cosets have order O_H and one may form a coset from every element of G which is not in the sub-group itself (since the coset formed by a member of the coset itself is simply that coset, by virtue of the group axioms). This means that cosets do not overlap.

Since cosets do not overlap, one can deduce that there are $\mathrm{O}_G - \mathrm{O}_H$ distinct cosets of the sub-group. It is possible to go on forming cosets until all these elements are exhausted. The full group can be written as a sum of a sub-group and all of its cosets.

$$G = H + G_1H + G_2H + \cdots + G_pH, \tag{8.8}$$

where p is some integer. The value of p can be determined by counting the orders of the elements in this equation:

$$\mathrm{O}_G = \mathrm{O}_H + \mathrm{O}_H + \mathrm{O}_H + \cdots + \mathrm{O}_H = (p+1)\,\mathrm{O}_H. \tag{8.9}$$

Thus,

$$\mathrm{O}_G = (p+1)\,\mathrm{O}_H. \tag{8.10}$$

Notice that the number of elements in the sub-group must be a factor of the number of elements in the whole group. This is necessarily true since all cosets are of order O_H.

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8.2.2 Conjugacy and invariant sub-groups

If g_1 is an element of a group G, and g_2 is another element, then g_c defined by

$$g_c = g_2 \, g \, g_2^{-1} \tag{8.11}$$

is said to be an element of the group G which is conjugate to g_1. One can form conjugates from every other element in the group. Every element is conjugate to itself since

$$g = I \, g \, I^{-1}. \tag{8.12}$$

Similarly, all elements in an Abelian group are conjugate only to themselves. Conjugacy is a mutual relationship. If g_1 is conjugate to g_2, then g_2 is conjugate to g_1, since

$$\begin{aligned} g_1 &= g \, g_2 \, g^{-1} \\ g_2 &= g^{-1} \, g_1 \, (g^{-1})^{-1}. \end{aligned} \tag{8.13}$$

If g_1 is conjugate to g_2 and g_2 is conjugate to g_3, then g_1 and g_3 are also conjugate. This implies that conjugacy is an equivalence relation.

Conjugate elements of a group are similar in the sense of similarity transformations, e.g. matrices which differ only by a change of basis:

$$A' = \Lambda \, M \, \Lambda^{-1}. \tag{8.14}$$

The *conjugacy class* of a group element g is the set of all elements conjugate to g:

$$\left\{ I \, g \, I^{-1}, \; g_1 \, g \, g_1^{-1}, \; g_2 \, g \, g_2^{-1}, \ldots \right\}. \tag{8.15}$$

A sub-group H of G is said to be an *invariant sub-group* if every element of the sub-group is conjugate to another element in the sub-group:

$$H_c = G \, H \, G^{-1} = H. \tag{8.16}$$

This means that the sub-group is invariant with respect to the action of the group, or that the only action of the group is to permute elements of the sub-group. It follows trivially from eqn. (8.16) that

$$GH = HG, \tag{8.17}$$

thus the left and right cosets of an invariant sub-group are identical. This means that all of the elements within H commute with G. H is said to belong to the *centre* of the group.

8.2.3 Schur's lemma and the centre of a group

Schur's lemma states that any group element which commutes with every other element of the group must be a multiple of the identity element. This result proves to be important in several contexts in group theory.

8.2.4 The factor group G/H

The *factor group*, also called the group of cosets is formed from an invariant sub-group H of a group G. Since each coset formed from H is distinct, one can show that the set of cosets of H with G forms a group which is denoted G/H. This follows from the Abelian property of invariant sub-groups. If we combine cosets by the group rule, then

$$Hg_1 \cdot Hg_2 = H\,H\,g_1\,g_2 = H(g_1 \cdot g_2,) \tag{8.18}$$

since $H \cdot H = H$. The group axioms are satisfied.

(1) The combination rule is the usual combination rule for the group.

(2) The associative law is valid for coset combination:

$$(Hg_1 \cdot Hg_2) \cdot Hg_3 = H(g_1 \cdot g_2) \cdot Hg_3 = H((g_1 \cdot g_2) \cdot g_3). \tag{8.19}$$

(3) The identity of G/H is $H \cdot I$.

(4) The inverse of Hg is Hg^{-1}.

The number of independent elements in this group (the order of the group) is, from eqn. (8.10), $p + 1$ or $\mathrm{O}_G/\mathrm{O}_H$. Initially, it might appear confusing from eqn. (8.7) that the number of elements in the sub-group is in fact multiplied by the number of elements in the group, giving a total number of elements in the factor group of $\mathrm{O}_G \times \mathrm{O}_H$. This is wrong, however, because one must be careful not to count cosets which are *similar* more than once; indeed, this is the point behind the requirement of an invariant sub-group. Cosets which are merely permutations of one another are considered to be equivalent.

8.2.5 Example of a factor group: $SU(2)/Z_2$

Many group algebras generate groups which are the same except for their maximal Abelian sub-group, called the *centre*. This virtual equivalence is determined by factoring out the centre, leaving only the factor group which has a trivial centre (the identity); thus, factor groups are important in issues of spontaneous symmetry breaking in physics, where one is often interested in the precise group symmetry rather than algebras. As an example of a factor group, consider $SU(2)$. The group elements of $SU(2)$ can be parametrized in terms of $d_G = 3$ parameters, as shown in eqn. (8.131). There is a redundancy in these parameters. For example, one can generate the identity element from each of the matrices $g_1(\theta_1)$, $g_2(\theta_2)$, $g_3(\theta_3)$ by choosing θ_A to be zero.

A non-trivial Abelian sub-group in these generators must come from the diagonal matrix $g_3(\theta_3)$. Indeed, one can show quite easily that g_3 commutes with any of the generators for any $\theta_A \neq 0$, if and only if $\exp(\mathrm{i}\frac{1}{2}\theta_3) = \exp(-\mathrm{i}\frac{1}{2}\theta_3) = \pm 1$. Thus, there are two possible values of θ_3, arising from one of the generators; these lead to an Abelian sub-group, and the group elements they correspond to are:

$$H = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\}, \tag{8.20}$$

which form a 2×2 representation of the discrete group Z_2. This sub-group is invariant, because it is Abelian, and we may therefore form the right cosets of H for every other element of the group:

$$\begin{aligned} H \cdot H &= \{ \ \mathbf{1} \ , \ \ -\mathbf{1} \ \} \\ H \cdot g_1(\theta_1) &= \{ g_1(\theta_1) \ , \ -g_1(\theta_1) \} \\ H \cdot g_1(\theta_1') &= \{ g_1(\theta_1') \ , \ -g_1(\theta_1') \} \\ H \cdot g_1(\theta_1'') &= \{ g_1(\theta_1'') \ , \ -g_1(\theta_1'') \} \\ &\vdots \\ H \cdot g_2(\theta_2) &= \{ g_2(\theta_2) \ , \ -g_2(\theta_2) \} \\ H \cdot g_2(\theta_2') &= \{ g_2(\theta_2') \ , \ -g_2(\theta_2') \} \\ &\vdots \\ H \cdot g_3(\theta_3) &= \{ g_3(\theta_3) \ , \ -g_2(\theta_3) \} \\ &\vdots \end{aligned} \tag{8.21}$$

The last line is assumed to exclude the members of g_3, which generate H, and the elements of g_1 and g_2, which give rise to the identity in Z_2, are also excluded from this list. That is because we are listing distinct group elements rather than the combinations, which are produced by a parametrization of the group.

The two columns on the right hand side of this list are two equivalent copies of the factor group $SU(2)/Z_2$. They are simply mirror images of one another which can be transformed into one another by the action of an element of Z_2. Notice that the full group is divided into two invariant pieces, each of which has half the total number of elements from the full group. The fact that these coset groups are possible is connected with *multiple coverings*. In fact, it turns out that this property is responsible for the double-valued nature of electron spin, or, equivalently, the link between the real rotation group $SO(3)$ ($d_G = 3$) and the complexified rotation group, $SU(2)$ ($d_G = 3$).

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8.3 Group representations

A *representation* of a group is a mapping between elements of the group and elements of the general linear group of either real matrices, $GL(n, R)$, or complex matrices, $GL(n, C)$. Put another way, it is a correspondence between the abstract group and matrices such that each group element can be represented in matrix form, and the rule of combination is replaced by matrix multiplication.

8.3.1 Definition of a representation G_R

If each element g of a group G can be assigned a non-singular $d_R \times d_R$ matrix $U_R(g)$, such that matrix multiplication preserves the group combination rule $g_{12} = g_1 \cdot g_2$,

$$U_R(g_{12}) = U_R(g_1 \cdot g_2) = U_R(g_1)\, U_R(g_2), \tag{8.22}$$

then the set of matrices is said to provide a d_R dimensional representation of the group G. The representation is denoted collectively G_R and is composed of matrices U_R. In most cases we shall call group representations U to avoid excessive notation.

8.3.2 Infinitesimal group generators

If one imagines a continuous group geometrically, as a vector space in which every point is a new element of the group, then, using a set of basis vectors, it is possible to describe every element in this space in terms of coefficients to these basis vectors. Matrices too can be the basis of a vector space, which is why matrix representations are possible. The basis matrices which span the vector space of a group are called its generators.

If one identifies the identity element of the group with the origin of this geometrical space, the number of linearly independent vectors required to reach every element in a group, starting from the identity, is the dimension of the space, and is also called the dimension of the group d_G. Note that the number of independent generators, d_G, is unrelated to their size d_R as matrices.

Thus, given that every element of the group lies in this vector space, an arbitrary element can be described by a vector whose components (relative to the generator matrices) uniquely identify that element. For example, consider the group $SU(2)$, which has dimension $d_G = 3$. In the fundamental representation, it has three generators (the Pauli matrices) with $d_R = 2$:

$$T_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\; T_2 = \frac{1}{2}\begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix},\; T_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{8.23}$$

A general point in group space may thus be labelled by a d_G dimensional vector $(\theta_1, \theta_2, \theta_3)$:

$$\Theta = \theta_1\, T_1 + \theta_2\, T_2 + \theta_3\, T_3. \tag{8.24}$$

A general element of the group is then found by exponentiating this generalized generator:

$$U_R = \exp(\mathrm{i}\Theta). \tag{8.25}$$

U_R is then a two-dimensional matrix representation of the group formed from two-dimensional generators. Alternatively, one may exponentiate each generator separately, as in eqn. (8.131) and combine them by matrix multiplication to obtain the same result. This follows from the property that multiplication of exponentials leads to the addition of the arguments.

For continuous groups generally, we can formalize this by writing a Taylor expansion of a group element $U(\theta)$ about the identity $I \equiv U(\mathbf{0})$,

$$U(\theta_A) = \sum_{A=1}^{d_G} \theta_A \left(\frac{\partial U}{\partial \theta_A}\right)\Big|_{\theta_A=0} + \cdots, \tag{8.26}$$

where d_G is the dimension of the group. We can write this

$$\begin{aligned} U(\theta) &= U(0) + \sum_{A=1}^{d_G} \theta_A T_A + \frac{1}{2!}\theta_A\theta_B T_A T_B + \cdots + \mathrm{O}(\theta^3) \\ &= I + \sum_{A=1}^{d_G} \theta_A T_A + \frac{1}{2!}\theta_A\theta_B T_A T_B + \cdots + \mathrm{O}(\theta^3), \end{aligned} \tag{8.27}$$

where

$$T_A = \left(\frac{\partial U}{\partial \theta_A}\right)\Big|_{\theta_A=0}. \tag{8.28}$$

T_A is a matrix generator for the group.

8.3.3 Proper group elements

All infinitesimal group elements can be parametrized in terms of linear combinations of generators T_A; thus, it is normal for group transformations to be discussed in terms of infinitesimal transformations. In terms of the geometrical analogy, infinitesimal group elements are those which are very close to the identity. They are defined by taking only terms to first order in θ in the sum in eqn. (8.27). The coefficients θ_A are assumed to be infinitesimally small, so that all higher powers are negligible. This is expressed by writing

$$U(\delta\theta) = U(0) + \delta\theta_A T_A, \tag{8.29}$$

with an implicit summation over A. With infinitesimal transformations, one does not get very far from the origin; however, the rule of group composition

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may be used to build (almost) arbitrary elements of the group by repeated application of infinitesimal elements. This is analogous to adding up many infinitesimal vectors to arrive at any point in a vector space.

We can check the consistency of repeatedly adding up N group elements by writing $\delta\theta_A = \theta_A/N$, combining $U(\theta) = U(\delta\theta)^N$ and letting $N \to \infty$. In this limit, we recover the exact result:

$$U(\theta) = \lim_{N\to\infty} \left(I + \mathrm{i}\frac{\delta\theta_A}{N} T_A \right) = \mathrm{e}^{\mathrm{i}\theta_A T_A}, \tag{8.30}$$

which is consistent with the series in eqn. (8.27). Notice that the finite group element is the exponential of the infinitesimal combination of the generators. It is often stated that we obtain a group by *exponentiation* of the generators.

It will prove significant to pay attention to another form of this exponentiation in passing. Eqn. (8.30) may also be written

$$U(\theta) = \exp\left(\mathrm{i} \int_0^\theta T_A \mathrm{d}\theta'_A \right). \tag{8.31}$$

From this we note that

$$\frac{\partial U(\theta)}{\partial \theta_A} = \mathrm{i}T_A\, U(\theta), \tag{8.32}$$

and hence

$$\frac{\mathrm{d}U}{U} = \mathrm{i}T_A \mathrm{d}\theta \equiv \Gamma. \tag{8.33}$$

This quantity, which we shall often label Γ in future, is an infinitesimal linear combination of the generators of the group. Because of the exponential form, it can also be written as a differential change in the group element $U(\theta)$ divided by the value of $U(\theta)$ at that point. This quantity has a special significance in geometry and field theory, and turns up repeatedly in the guise of gauge fields and 'connections'.

Not all elements of a group can necessarily be generated by combining infinitesimal elements of the group. In general, it is only a sub-group known as the *proper group* which can be generated in this way. Some transformations, such as *reflections* in the origin or coordinate reversals with respect to a group parameter are, by nature, discrete and discontinuous. A reflection is an all-or-nothing transformation; it cannot be broken down into smaller pieces. Groups which contain these so-called *large* transformations are expressible as a direct product of a connected, continuous group and a discrete group.

8.3.4 Conjugate representations

Given a set of infinitesimal generators, T^A, one can generate infinitely many more by similarity transformations:

$$T^A \rightarrow \Lambda\, T^A\, \Lambda^{-1}. \tag{8.34}$$

This has the effect of generating an equivalent representation. Any two representations which are related by such a similarity transformation are said to be *conjugate* to one another, or to lie in the same *conjugacy class*. Conjugate representations all have the same dimension d_R.

8.3.5 Congruent representations

Representations of different dimension d_R also fall into classes. Generators which exponentiate to a given group may be classified by *congruency class*. All group generators with different d_R exponentiate to groups which are congruent, modulo their centres, i.e. those which are the same up to some multiple covering. Put another way, the groups formed by exponentiation of generators of different d_R are identical only if one factors out their centres.

A given matrix representation of a group is not necessarily a one-to-one mapping from algebra to group, but might cover all of the elements of a group one, twice, or any integer number of times and still satisfy all of the group properties. Such representations are said to be multiple coverings.

A representation U_R and another representation $U_{R'}$ lie in different *congruence classes* if they cover the elements of the group a different number of times. Congruence is a property of discrete tiling systems and is related to the ability to lay one pattern on top of another such that they match. It is the properties of the generators which are responsible for congruence [124].

8.4 Reducible and irreducible representations

There is an infinite number of ways to represent the properties of a given group on a representation space. A representation space is usually based on some physical criteria; for instance, to represent the symmetry of three quarks, one uses a three-dimensional representation of $SU(3)$, although the group itself is eight-dimensional. It is important to realize that, if one chooses a large enough representation space, the space itself might have more symmetry than the group which one is using to describe a particular transformation. Of the infinity of possible representations, some can be broken down into simpler structures which represent truly invariant properties of the *representation space*.

8.4.1 Invariant sub-spaces

Suppose we have a representation of a group in terms of matrices and vectors; take as an example the two-dimensional rotation group $SO(2)$, with the representation

$$U = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \tag{8.35}$$

so that the rotation of a vector by an angle θ is accomplished by matrix multiplication:

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_3 \end{pmatrix}. \tag{8.36}$$

It is always possible to find higher-dimensional representations of the same group by simply embedding such a group in a larger space. If we add an extra dimension x_3, then the same rotation is accomplished, since x_1 and x_2 are altered in exactly the same way:

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} \cos\theta_3 & \sin\theta_3 & 0 \\ -\sin\theta_3 & \sin\theta_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}. \tag{8.37}$$

This makes sense: it is easy to make a two-dimensional rotation in a three-dimensional space, and the same generalization carries through for any number of extra dimensions. The matrix representation of the transformation has zeros and a diagonal 1, indicating that nothing at all happens to the x_3 coordinate. It is irrelevant or ignorable:

$$U = \begin{pmatrix} \cos\theta_3 & \sin\theta_3 & 0 \\ -\sin\theta_3 & \sin\theta_3 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{8.38}$$

A six-dimensional representation would look like this:

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \\ x_5' \\ x_6' \end{pmatrix} = \begin{pmatrix} \cos\theta_3 & \sin\theta_3 & 0 & 0 & 0 & 0 \\ -\sin\theta_3 & \sin\theta_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix}. \tag{8.39}$$

The matrix has a block-diagonal form. These higher-dimensional representations are said to be *reducible*, since they contain *invariant sub-spaces*, or

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coordinates which remain unaltered by the group. In the six-dimensional case above, the 6×6 matrix factorizes into a direct sum of block-diagonal pieces: a 2×2 piece, which is the actual $SO(2)$ part, and a trivial four-dimensional group composed of only the identity I_4. The direct sum is written

$$SO(2)_6 = SO(2)_2 \oplus I_4. \tag{8.40}$$

When a matrix has the form of eqn. (8.39), or is related to such a form by a similarity transformation

$$\Lambda^{-1}\, U\, \Lambda, \tag{8.41}$$

it is said to be a completely reducible representation of the group. In block-diagonal form, each block is said to be an *irreducible representation* of the group. The smallest representation with all of the properties of the group intact is called the *fundamental representation*. A representation composed of $d_G \times d_G$ matrices, where d_G is the dimension of the group, is called the *adjoint representation*. In the case of $SO(3)$, the fundamental and adjoint representations coincide; usually they do not.

Whilst the above observation might seem rather obvious, it is perhaps less obvious if we turn the argument around. Suppose we start with a 6×6 matrix parametrized in terms of some group variables, θ_A, and we want to know which group it is a representation of. The first guess might be that it is an irreducible representation of $O(6)$, but if we can find a linear transformation Λ which changes that matrix into a block-diagonal form with smaller blocks, and zeros off the diagonal, then it becomes clear that it is really a reducible representation, composed of several sub-spaces, each of which is invariant under a smaller group.

8.4.2 Reducibility

The existence of an invariant sub-space S in the representation space R implies that the matrix representation G_R is reducible. Suppose we have a representation space with a sub-space which is unaffected by the action of the group. By choosing coordinates we can write a group transformation g as

$$\begin{pmatrix} X'_R \\ X'_S \end{pmatrix} = \begin{pmatrix} A(g) & B(g) \\ 0 & C(g) \end{pmatrix} \begin{pmatrix} X_R \\ X_S \end{pmatrix}, \tag{8.42}$$

which shows that the coordinates X_S belonging to the sub-space are independent of the remaining coordinates X_R. Thus no matter how X_R are transformed, X_S will be independent of this. The converse is not necessarily true, but often is. Our representation,

$$U_R(g) = \begin{pmatrix} A(g) & B(g) \\ 0 & C(g) \end{pmatrix}, \tag{8.43}$$

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satisfies the group composition law; thus,

$$U_R(g_1)\,U_R(g_2) = \begin{pmatrix} A(g_1) & B(g_1) \\ 0 & C(g_1) \end{pmatrix}\begin{pmatrix} A(g_2) & B(g_2) \\ 0 & C(g_2) \end{pmatrix}$$
$$= \begin{pmatrix} A(g_1)\,A(g_2) & A(g_1)B(g_2) + B(g_1)C(g_2) \\ 0 & C(g_1)C(g_2) \end{pmatrix} \tag{8.44}$$

Comparing this with the form which a true group representation would have:

$$\begin{pmatrix} A(g_1 \cdot g_2) & B(g_1 \cdot g_2) \\ 0 & C(g_1 \cdot g_2) \end{pmatrix}, \tag{8.45}$$

one sees that A and C also form representations of the group, of smaller size. B does not, however, and its value is constrained by the condition $B(g_1 \cdot g_2) = A(g_1)B(g_2) + B(g_1)C(g_2)$. A representation of this form is said to be partially reducible.

If $B = 0$ in the above, then the two sub-spaces decouple: both are invariant under transformations which affect the other. The representation is then said to be completely reducible and takes the block-diagonal form mentioned in the previous section.

8.5 Lie groups and Lie algebras

Groups whose elements do not commute are called *non-Abelian*. The commutativity or non-commutativity of the group elements $U(\theta)$ follows from the commutation properties of the generators T_a, as may be seen by writing the exponentiation operation as a power series. In a non-Abelian group the commutation relations between generators may be written in this form:

$$[T_a, T_b] = C_{ab}. \tag{8.46}$$

A special class of groups which is interesting in physics is the *Lie groups*, which satisfy the special algebra,

$$[T_a, T_b] = -\mathrm{i} f_{ab}{}^c T_c. \tag{8.47}$$

$f_{ab}{}^c$ is a set of *structure constants*, and all the labels a, b, c run over the group indices from $1, \ldots, d_G$. Eqn. (8.47) is called a *Lie algebra*. It implies that the matrices which generate a Lie group are not arbitrary; they are constrained to satisfy the algebra relation. The matrices satisfy the algebraic Jacobi identity

$$[T^a, [T^b, T^c]] + [T^b, [T^c, T^a]] + [T^c, [T^a, T^b]] = 0. \tag{8.48}$$

Many of the issues connected to Lie algebras are analogous to those of the groups they generate. We study them precisely because they provide a deeper level of understanding of groups. One also refers to representations, equivalence classes, conjugacy for algebras.

8.5.1 Normalization of the generators

The structure of the $d_R \times d_R$ dimensional matrices and the constants f_{abc} which make up the algebra relation are determined by the algebra relation, but the normalization is not. If we multiply T_R^a and f_{abc} by any constant factor, the algebra relation will still be true. The normalization of the generators is fixed here by relating the trace of a product of generators to the quadratic Casimir invariant:

$$\mathrm{Tr}\left(T_R^a T_R^b\right) = I_2(G_R)\delta^{ab}, \tag{8.49}$$

where I_2 is called the Dynkin index for the representation G_R. The Dynkin index may also be written as

$$I_2(G_R) = \frac{d_R}{d_G} C_2(G_R) \tag{8.50}$$

where d_R is the dimension (number of rows/columns) of the generators in the representation G_R, and d_G is the dimension of the group. $C_2(G_R)$ is the quadratic Casimir invariant for the group in the representation, G_R: $C_2(G_R)$ and $I_2(G_R)$ are constants which are listed in tables for various representations of Lie groups [96]. d_G is the same as the dimension of the adjoint representation of the algebra G_{adj}, by definition of the adjoint representation. Note, therefore, that $I_2(G_{\mathrm{adj}}) = C_2(G_{\mathrm{adj}})$.

The normalization is not completely fixed by these conditions, since one does not know the value of the Casimir invariant *a priori*. Moreover, Casimir invariants are often defined with inconsistent normalizations, since their main property of interest is their ability to commute with other generators, rather than their absolute magnitude. The above relations make the Casimir invariants consistent with the generator products. To complete the normalization, it is usual to define the length of the longest roots or eigenvalues of the Lie algebra as 2. This fixes the value of the Casimir invariants and thus fixes the remaining values. For most purposes, the normalization is not very important as long as one is consistent, and most answers can simply be expressed in terms of the arbitrary value of $C_2(G_R)$. Thus, during the course of an analysis, one should not be surprised to find generators and Casimir invariants changing in definition and normalization several times. What is important is that, when comparisons are made between similar things, one uses consistent conventions of normalization and definition.

8.5.2 Adjoint transformations and unitarity

A Lie algebra is formed from the d_G matrices T^a which generate a Lie group. These matrices are $d_R \times d_R$ matrices which act on the vector space, which has been denoted *representation space*. In addition, the d_G generators which fulfil the algebra condition form a basis which spans the group space. Since the group is formed from the algebra by exponentiation, both a Lie algebra A and its group G live on the vector space referred to as group space. In the case of the adjoint representation $G_R = G_{\text{adj}}$, the group and representation spaces coincide ($d_G = d_R$, $a, b, c \leftrightarrow A, B, C$). The adjoint representation is a direct one-to-one mapping of the algebra properties into a set of matrices. It is easy to show that the structure constants themselves form a representation of the group which is adjoint. This follows from the Jacobi identity in eqn. (8.48). Applying the algebra relation (8.47) to eqn. (8.48), we have

$$[T^a, -\mathrm{i} f^{bcd} T^d] + [T^b, -\mathrm{i} f^{cad} T^d] + [T^c, -\mathrm{i} f^{abd} T^d] = 0. \tag{8.51}$$

Using it again results in

$$\left[-f^{bcd} f^{ade} - f^{cad} f^{bde} - f^{abd} f^{cde} \right] T^e = 0. \tag{8.52}$$

Then, from the coefficient of T^e, making the identification,

$$\left[T^a \right]_{BC} \equiv \mathrm{i} f^a_{BC} \tag{8.53}$$

it is straightforward to show that one recovers

$$[T^a, T^b] = -\mathrm{i} f^{abd} T^d. \tag{8.54}$$

Thus, the components of the structure constants are the components of the matrices in the adjoint representation of the algebra. The representation is uniquely identified as the adjoint since all indices on the structure constants run over the dimension of the group $a, b = 1, \ldots, d_G$.

The group space to which we have been alluding is assumed, in field theory, to be a Hilbert space, or a vector space with a positive definite metric. Representation space does not require a positive definite metric, and indeed, in the case of groups like the Lorentz group of spacetime symmetries, the metric in representation space is indefinite. The link between representation space and group space is made by the adjoint representation, and it will prove essential later to understand what this connection is.

Adjoint transformations can be understood in several ways. Suppose we take a group vector v^a which transforms by the rule

$$v'^a = {U_{\text{adj}}}^a{}_b \, v^b, \tag{8.55}$$

where

$$U_{\text{adj}} = \exp\left(\mathrm{i}\theta^a \, T^a_{\text{adj}} \right). \tag{8.56}$$

It is also possible to represent the same transformation using a complete set of arbitrary matrices to form a basis for the group space. For the matrices we shall choose the generators T_R, is an arbitrary representation

$$V_R = v^a \, T^a_R. \tag{8.57}$$

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If we assume that the v^a in eqns. (8.55) and (8.57) are the same components, then it follows that the transformation rule for V_R must be written

$$V'_R = v^{a\prime} T^a_R = U_R^{-1} \, V_R \, U_R, \tag{8.58}$$

where

$$U_R = \exp\left(\mathrm{i}\theta^a \, T^a_R\right). \tag{8.59}$$

This now has the appearance of a similarity transformation on the group space. To prove this, we shall begin with the assumption that the field transforms as in eqn. (8.58). Then, using the matrix identity

$$\exp(A) B \exp(-A) = B + [A, B] + \frac{1}{2!}[A, [A, B]] + \frac{1}{3!}[A, [A, [A, B]]] + \cdots, \tag{8.60}$$

it is straightforward to show that

$$V_R{}' = v^a \Big\{ \delta^a_{\ r} - \theta_b f^{ab}_{\ \ r} + \frac{1}{2}\theta_b \theta_c f^{ca}_{\ \ s} f^{bs}_{\ \ r} + - \frac{1}{3!}\theta_b \theta_c \theta_d f^{da}_{\ \ q} f^{cq}_{\ \ p} f^{bp}_{\ \ r} + \cdots \Big\} T^r_R, \tag{8.61}$$

where the algebra commutation relation has been used. In our notation, the generators of the adjoint representation may written

$$(T^a_{\text{adj}})^b_{\ c} = \mathrm{i} f^{ab}_{\ \ c}, \tag{8.62}$$

and the structure constants are real. Eqn. (8.61) may therefore be identified as

$$V_R{}' = v^a (U_{\text{adj}})^a_{\ b} T^b_R, \tag{8.63}$$

where

$$U_{\text{adj}} = \exp(\mathrm{i}\theta^a T^a_{\text{adj}}). \tag{8.64}$$

If we now define the components of the transformed field by

$$V_R{}' = v'^a T^a_R, \tag{8.65}$$

in terms of the original generators, then it follows that

$$v'^a = (U_{\text{adj}})^a_{\ b} v^b. \tag{8.66}$$

We can now think of the set of components, v^a and v'^a, as being grouped into d_G-component column *vectors* $\mathbf{v}$ and $\mathbf{v}'$, so that

$$\mathbf{v}' = U_{\text{adj}} \mathbf{v}. \tag{8.67}$$

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Thus, we see that the components of a group vector, v^a, always transform according to the adjoint representation, regardless of what type of basis we use to represent them. To understand the significance of this transformation rule, we should compare it with the corresponding tensor transformation rule in representation space. If we use the matrix

$$U_R = [U_R]^A{}_B \tag{8.68}$$

where $A, B = 1, \ldots, d_R$, as a transformation of some representation space vector ϕ^A or tensor $[V_R]^A{}_B$, then, by considering the invariant product

$$\phi^\dagger V_R \phi \to (U\phi)^\dagger \, U V_R U^{-1} \, (U\phi), \tag{8.69}$$

we find that the transformation rule is the usual one for tensors:

$$\phi^A = U^A{}_B \, \phi^B \tag{8.70a}$$

$$V_{AB} = U^A{}_C U^B{}_D \, V_{CD}. \tag{8.70b}$$

The transformation rule (8.58) agrees with the rule in eqn. (8.70b) provided

$$U^\dagger = U^{-1}. \tag{8.71}$$

This is the unitary property, and it is secured in field theory also by the use of a Hilbert space as the group manifold. Thus, the form of the adjoint transformation represents unitarity in the field theory, regardless of the fact that the indices A, B might have an indefinite metric.

The object V_R, which transforms like $U^{-1} V U$, signifies a change in the disposition of the system. This form is very commonly seen; for example, in dynamical changes:

$$\begin{aligned} \partial_\mu \phi \to \partial_\mu (U\phi) &= (\partial_\mu U)\phi + U(\partial_\mu \phi) \\ &= U(\partial_\mu + \Gamma_\mu)\phi \end{aligned} \tag{8.72}$$

where

$$\Gamma_\mu = U^{-1} \partial_\mu U. \tag{8.73}$$

This object is usually called a ‘connection’, but, in this context, it can be viewed as an expression of a change in the dynamical configuration, of the internal constraints on a system. In the following two chapters, we shall see examples of these transformations, when looking at the Lorentz group and gauge symmetries in particular.

8.5.3 Casimir invariants

From the Lie algebra relation in eqn. (8.47), it is straightforward to show that the quadratic sum of the generators commutes with each individual generator:

$$\begin{aligned}[T^a, T^b T^b] &= T^a\, T^b T^b - T^b T^b\, T^a \\ &= T^a\, T^b T^b - T^b\,(T^a T^b + \mathrm{i} f^{abc} T^c) \\ &= [T^a, T^b]\, T^b - \mathrm{i} f^{abc}\, T^b T^c \\ &= -\mathrm{i}\, f^{abc}\,[T^c T^b + T^b T^c] \\ &= 0. \end{aligned} \tag{8.74}$$

The last line follows since the bracket is a symmetric matrix, whereas the structure constants are anti-symmetric. In fact, the quadratic sum of the generators is proportional to the identity matrix. This follows also from Schur's lemma:

$$T^a T^a = \frac{1}{d_G}\, C_2(G_R)\, \mathbf{I}_R, \tag{8.75}$$

or

$$f^a_{\ bc}\, f^{dbc} = -\frac{1}{d_G}\, C_2(G_{\mathrm{adj}}) \delta^{ad}. \tag{8.76}$$

8.5.4 Sub-algebra

Just as groups have sub-groups, algebras have sub-algebras. A sub-set, H, of an algebra, A, is called a linear sub-algebra of A if H is a linear sub-space of the group space and is closed with respect to the algebra relation. i.e. for any matrix elements of the sub-algebra h_1, h_2 and h_3, one has

$$[t_1, t_2] = -\mathrm{i} f_{12}{}^{3} t_3. \tag{8.77}$$

This is a non-Abelian sub-algebra. Sub-algebras can also be Abelian:

$$[h_1, h_2] = 0. \tag{8.78}$$

8.5.5 The Cartan sub-algebra

The Cartan sub-algebra is an invariant sub-algebra whose elements generate the *centre* of a Lie group when exponentiated. This sub-algebra has a number of extremely important properties because many properties of the group can be deduced directly from the sub-set of generators which lies in the Cartan sub-algebra.

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The generators of the Cartan sub-algebra commute with one another but not necessarily with other generators of the group. Since the Cartan sub-algebra generates the centre of the group (the maximal Abelian sub-group) under exponentiation, Schur's lemma tells us that the group elements found from these are diagonal and proportional to the identity matrix. The Cartan sub-algebra is the sub-set the group generators T^a which are simultaneously diagonalizable in a suitable basis. In other words, if there is a basis in which one of the generators, T^a, is diagonal, then, in general, several of the generators will be diagonal in the same basis. One can begin with a set of generators, T^a_R, in a representation, G_R, and attempt to diagonalize one of them using a similarity transformation:

$$T^{a\prime} \to \Lambda\, T^a_R\, \Lambda^{-1}. \tag{8.79}$$

The same transformation, Λ, will transform a fixed number of the matrices into diagonal form. This number is always the same, and it is called the *rank* of the group or rank(G). The diagonalizable generators are denoted H^i, where $i = 1, \ldots, \mathrm{rank}(G)$. These form the Cartan sub-algebra. Note that, in the case of the fundamental representation of $SU(2)$, the third Pauli matrix is already diagonal. This matrix is the generator of the Cartan sub-algebra for $SU(2)$ in the $d_R = 2$ representation. Since only one of the generators is diagonal, one concludes that the rank of $SU(2)$ is 1.

8.5.6 Example of diagonalization

The simplest example of a Cartan sub-algebra may be found in the generators of the group $SO(3)$ in the fundamental representation, or identically of $SU(2)$ in the adjoint representation. These matrices are well known as the generators of rotations in three dimensions, and are written:

$$T^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix}$$

$$T^2 = \begin{pmatrix} 0 & 0 & \mathrm{i} \\ 0 & 0 & 0 \\ -\mathrm{i} & 0 & 0 \end{pmatrix}$$

$$T^3 = \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{8.80}$$

To find a basis which diagonalizes one of these generators, we pick T^1 to diagonalize, arbitrarily. The self-inverse matrix of eigenvectors for T^1 is easily found. It is given by

$$\Lambda = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \\ 0 & \frac{i}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix}. \tag{8.81}$$

Constructing the matrices $\Lambda^{-1} T^a \Lambda$, one finds a new set of generators,

$$T^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$
$$T^2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & i \\ 1 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}$$
$$T^3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & i & 1 \\ -i & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{8.82}$$

Since only one of these is diagonal, rank rank $SU(2) = 1$. Equally, we could have chosen to diagonalize a different generator. This would then have had the same eigenvalues, and it would have been the generator of the Cartan sub-algebra in an alternative basis. None of the generators are specially singled out to generate the sub-algebra. The diagonalizability is an intrinsic property of the algebra.

8.5.7 Roots and weights

The *roots* and *weights* of algebra representations are proportional to eigenvalues of the Cartan sub-algebra generators for different d_R. The roots are denoted α^A and the weights are denoted λ^A. Because the algebra relation ensures exactly d_G independent vectors on the group space, there are d_G independent eigenvalues to be found from the generators.[1] We shall explore the significance of these eigenvalues in the next section.

[1] This might seem confusing. If one has rank(G) simultaneously diagonalizable $d_R \times d_R$ matrices, then it seems as though there should be $d_R \times$ rank(G) eigenvalues to discern. The reason why this is not the case is that not all of the generators are independent. They are constrained by the algebra relation. The generators are *linearly independent* but constrained through the quadratic commutator condition

For generators of the Cartan sub-algebra, H_R^i, in a representation G_R, the *weights* are eigenvalues:

$$H_R^i = \begin{bmatrix} \lambda_1^i & & & \\ & \lambda_2^i & & \\ & & \lambda_3^i & \\ & & & \ddots \end{bmatrix}. \tag{8.83}$$

The name *root* is reserved for an eigenvalue of the adjoint representation:

$$H_{\text{adj}}^i = \begin{bmatrix} \alpha_1^i & & & \\ & \alpha_2^i & & \\ & & \alpha_3^i & \\ & & & \ddots \end{bmatrix}. \tag{8.84}$$

The significance of the adjoint representation is that it is a direct one-to-one mapping of intrinsic algebra properties. The roots have a special significance too: the algebra can be defined purely in terms of its roots. The diagonal basis we have referred to above is a step towards showing this, but to see the true significance of the root and weights of an algebra, we need to perform another linear transformation and construct the Cartan–Weyl basis.

8.5.8 The Cartan–Weyl basis

The Cartan–Weyl basis is one of several bases in which the generators of the Cartan sub-algebra are diagonal matrices. To construct this basis we can begin from the diagonal basis, found in the previous section, and form linear combinations of the remaining non-diagonal generators. The motivation for this requires a brief theoretical diversion.

Suppose that Θ and Φ are arbitrary linear combinations of the generators of a Lie algebra. This would be the case if Θ and Φ were non-Abelian gauge fields, for instance

$$\begin{aligned} \Theta &= \theta_a\, T^a \\ \Phi &= \phi_a\, T^a, \end{aligned} \tag{8.85}$$

where $a = 1, \ldots, d_G$. Then, consider the commutator eigenvalue equation

$$[\Theta, \Phi] = \alpha\Phi, \tag{8.86}$$

where α is an eigenvalue for the 'eigenvector' Φ. If we write this in component form, using the algebra relation in eqn. (8.47), we have

$$\theta^a \phi^b\, f_{abc} T^c = \alpha\, \phi_l T^l. \tag{8.87}$$

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Now, since the T^a are *linearly* independent we can compare the coefficients of the generators on the left and right hand sides:

$$(\phi^a f_{ab}{}^c - \alpha\, \delta_b{}^c)\phi^b = 0. \tag{8.88}$$

This equation has non-trivial solutions if the determinant of the bracket vanishes, and thus we require

$$\det \left| \phi^a f_{ab}{}^c - \alpha\, \delta_b{}^c \right| = 0. \tag{8.89}$$

For a d_G dimensional Lie algebra this equation cannot have more than d_G independent roots, α. Cartan showed that if one chooses Θ so that the secular equation has the maximum number of different eigenvalues or roots, then only zero roots $\alpha = 0$ can be degenerate (repeated). If $\alpha = 0$ is r-fold degenerate, then r is the rank of the semi-simple Lie algebra.

The generators associated with zero eigenvalues are denoted H^i, where $i = 1, \ldots, \mathrm{rank}(G)$ and they satisfy

$$[\theta^j H^j, H^i] = 0, \tag{8.90}$$

i.e. they commute with one another. The remaining generators, which they do not commute with are written E_α, for some non-zero α, and they clearly satisfy

$$[\theta^j H^j, E_\alpha] = \alpha\, E_\alpha. \tag{8.91}$$

We can think of the roots or eigenvalues as vectors living on the invariant sub-space spanned by the generators H^i. The components can be found by allowing the H^i to act on the E_α. Consider

$$\begin{aligned} [\theta^j H^j, [H_i, E_\alpha]] &= [\theta^j H^j, H_i E_\alpha] - [\theta^j H^j, E_\alpha H_i] \\ &= \alpha [H^i, E_\alpha]. \end{aligned} \tag{8.92}$$

This result can be interpreted as follows. If E_α is an 'eigenvector' associated with the eigenvalue α, then there are $\mathrm{rank}(G)$ eigenvectors $[H^i, E_\alpha]$ belonging to the same eigenvalue. The eigenvectors must therefore each be proportional to E_α:

$$[H^i, E_\alpha] = \alpha^i\, E_\alpha, \tag{8.93}$$

and the components of the vector are defined by

$$\alpha = \alpha^i\, \theta^i. \tag{8.94}$$

This relation defines the components of a *root vector* on the invariant Cartan sub-space. Comparing eqn. (8.93) with the algebra relation in eqn. (8.47),

$$f_{ia}{}^b = \alpha_i\, \delta_a{}^b. \tag{8.95}$$

Finally, by looking at the Jacobi identity,

$$[\Theta, [E_\alpha, E_\beta]] + [E_\alpha, [E_\beta, \Theta]] + [E_\beta, [\Theta, E_\alpha]] = 0, \tag{8.96}$$

we find that

$$[\Theta, [E_\alpha, E_\beta]] = (\alpha + \beta)[E_\alpha, E_\beta]. \tag{8.97}$$

This means that $[E_\alpha, E_\beta]$ is the eigenvector associated with the root $(\alpha + \beta)$, provided that $\alpha + \beta \neq 0$. If $\alpha + \beta = 0$ then, since the zero eigenvalues are associated with H^i, we must have

$$\begin{aligned} [E_\alpha, E_{-\alpha}] &= f_{\alpha,-\alpha}{}^{i}\, H_i \\ &= \alpha_i\, H^i. \end{aligned} \tag{8.98}$$

This shows how the E_α act as stepping operators, adding together solutions to the eigenvalue equation. It also implies that if there is a zero root, then there must be *pairs* of roots α, $-\alpha$. In summary,

$$[H^i\,, E_\alpha\,] = \alpha^i\ E_\alpha$$
$$[E_\alpha, E_{-\alpha}] = \alpha^i\ \ H_i$$

What is the physical meaning of the root vectors? The eigenvalue equation is an equation which tells us how many ways one generator of transformations maps to itself, up to a scalar multiple under the action of the group. The H are invariant sub-spaces of a symmetry group because they only change the magnitude of a symmetry state, not its character. In other words, the Cartan sub-algebra represents the number of simultaneous labels which can be measured or associated with a symmetry constraint. Labels represent physical properties like spin, momentum, energy, etc. The stepping operators for a given representation of the group determine how many independent values of those labels can exist based on symmetry constraints. This is the number of weights in a stepping chain. In the case of rotations, the root/weight eigenvalues represent the spin characteristics of particles. A system with one pair of weights (one property: rotation about a fixed axis) in a $d_R = 2$ representation can only be in a spin up or spin down state because there are only two elements in the stepping chain. A $d_R = 3$ representation has three elements, so the particle can have spin up down or zero etc.

The Chevalley normalization of generators is generally chosen so as to make the magnitude of the longest root vectors equal to $(\alpha, \alpha) = \sqrt{\alpha^a \alpha^a} = 2$.

8.5.9 Group vectors and Dirac notation

In quantum mechanics, Dirac introduced a notation for the eigenvectors of an operator using *bra* and *ket* notation. Dirac's notation was meant to emphasize the role of eigenvectors as projection operators which span a vector space. Dirac's notation is convenient since it is fairly intuitive and is widely used in the physics literature. An eigenvector is characterized by a number of labels, i.e. the eigenvalues of the various operators which share it as an eigenvector. If we label these eigenvalues $\alpha, \beta, \ldots$ and so on, then we may designate the eigenvectors using a field or eigenfunction

$$\psi_{\alpha_i,\beta_j,\ldots} \tag{8.99}$$

or in Dirac notation as a *ket*:

$$|\alpha_i, \beta_j, \ldots\rangle. \tag{8.100}$$

Notice that, in Dirac's notation, the redundant symbol ψ is removed, which helps to focus one's attention on the relevant labels: the eigenvalues themselves. The operators which have these eigenfunctions as simultaneous eigenvectors then produce:

$$\begin{aligned} A_i\ \psi_{\alpha_i,\beta_j,\ldots} &= \alpha_i\ \psi_{\alpha_i,\beta_j,\ldots} \\ B_j\ \psi_{\alpha_i,\beta_j,\ldots} &= \beta_j\ \psi_{\alpha_i,\beta_j,\ldots} \qquad (i, j \text{ not summed}), \end{aligned} \tag{8.101}$$

or, equivalently,

$$\begin{aligned} A_i\,|\,\alpha_i, \beta_j, \ldots\rangle &= \alpha_i\,|\,\alpha_i, \beta_j, \ldots\rangle \\ B_j\,|\,\alpha_i, \beta_j, \ldots\rangle &= \beta_j\,|\,\alpha_i, \beta_j, \ldots\rangle \qquad (i, j \text{ not summed}). \end{aligned} \tag{8.102}$$

In most physical problems we are interested in group spaces with a positive definite metric, i.e. Hilbert spaces. In that case, the dual vectors are written as a Hermitian conjugate:

$$\psi^\dagger_{\alpha_i,\beta_j,\ldots} \tag{8.103}$$

or in Dirac notation as a *bra*:

$$\langle\alpha, \beta, \ldots|. \tag{8.104}$$

The length of a vector is then given by the inner product

$$\langle\alpha_i, \beta_j|\alpha_k, \beta_l\rangle = \psi^\dagger_{\alpha_i,\beta_j}\psi_{\alpha_k,\beta_l} = \delta_{ik}\delta_{jl} \times \text{length}. \tag{8.105}$$

The eigenvectors with different eigenvalues are orthogonal and usually normalized to unit length.

The existence of simultaneous eigenvalues depends on the existence of commuting operators. Operators which do not commute, such as x^i, p^j and group generators, T^a, T^b, can be assigned eigenvectors, but they are not all linearly independent; they have a projection which is a particular group element:

$$\langle x|p\rangle = \mathrm{e}^{\mathrm{i}\,p\,x/\hbar}. \tag{8.106}$$

8.5.10 Example: rotational eigenvalues in three dimensions

In this section, we take a first look at the rotation problem. We shall return to this problem in chapter 11 in connection with angular momentum and spin. The generators of three-dimensional rotations are those of $SO(3)$, or equivalently $su(2)$ in the adjoint representation. The generators are already listed in eqns. (8.80). We define

$$\begin{aligned} T^2 &= T^a T^a \\ E_\pm &= T_2 \mp \mathrm{i} T_3 \\ H &= T_1. \end{aligned} \tag{8.107}$$

In this new basis, the generators satisfy the relation

$$[H, E_\pm] = \pm E_\pm. \tag{8.108}$$

The stepping operators are Hermitian conjugates:

$$E_+^\dagger = E_-. \tag{8.109}$$

The generator H labels a *central* generator, or invariant sub-space, and corresponds to the fact that we are considering a special axis of rotation. The eigenvalues of the central generator H are called its *weights* and are labelled Λ_c

$$H\,|\Lambda_c\rangle = \Lambda_c|\Lambda_c\rangle. \tag{8.110}$$

$|\Lambda_c\rangle$ is an eigenvector of H with eigenvalue Λ_c. The value of the quadratic form, T^2, is also interesting because it commutes with H and therefore has its own eigenvalue when acting on H's eigenfunctions, which is independent of c. It can be evaluated by expressing T^2 in terms of the generators in the new basis:

$$\begin{aligned} E_+E_- &= T_2^2 + T_3^2 - \mathrm{i}[T_2, T_3] \\ E_-E_+ &= T_2^2 + T_3^2 + \mathrm{i}[T_2, T_3], \end{aligned} \tag{8.111}$$

so that, rearranging and using the algebra relation,

$$\begin{aligned} T^2 &= E_-E_+ + T_1^2 - \mathrm{i}[T_2, T_3] \\ &= E_-E_+ + T_1^2 - \mathrm{i}(-\mathrm{i}T_1) \\ &= E_-E_+ + H(H+1), \end{aligned} \tag{8.112}$$

where we have identified $T_1 = H$ in the last line. By the analogous procedure with $\pm$ labels reversed, we also find

$$T^2 = E_+E_- + H(H-1). \tag{8.113}$$

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These forms allow us to evaluate the eigenvalues of T^2 for two of the eigenfunctions in the full series. To understand this, we note that the effect of the $E_\pm$ generators is to generate new solutions step-wise, i.e. starting with an arbitrary eigenfunction $|\Lambda_c\rangle$ they generate new eigenfunctions with new eigenvalues. This is easily confirmed from the commutation relation in eqn. (8.108), if we consider the 'new' eigenvector $E_\pm|\Lambda_c\rangle$ from $|\Lambda_c\rangle$ and try to calculate the corresponding eigenvalue:

$$\begin{aligned} H\ E_\pm\ |\Lambda_c\rangle &= (E_\pm H + [H, T_\pm])\,|\Lambda_c\rangle \\ &= (E_\pm H \pm E_\pm)\,|\Lambda_c\rangle \\ &= (\Lambda_c \pm 1)\ E_\pm\,|\Lambda_c\rangle. \end{aligned} \tag{8.114}$$

We see that, given any initial eigenfunction of H, the action of $E_\pm$ is to produce a new eigenfunction with a new eigenvalue, which differs by ± 1 from the original, up to a possible normalization constant which would cancel out of this expression:

$$E_\pm\,|\Lambda_c\rangle \propto |\Lambda_c \pm 1\rangle. \tag{8.115}$$

Now, the number of solutions cannot be infinite because the Schwarz (triangle) inequality tells us that the eigenvalue of T^2 (whose value is not fixed by the eigenvalue of H, since T^2 and T^a commute) must be bigger than any of the individual eigenvalues T^a:

$$\left\langle \Lambda_c | E_+E_- + E_-E_+ + H^2 | \Lambda_c \right\rangle > \langle \Lambda_c | H^2 | \Lambda_c \rangle, \tag{8.116}$$

so the value of H acting on $|\Lambda_c\rangle$ must approach a maximum as it approaches the value of T^2 acting on $|\Lambda_c\rangle$. Physically, the maximum value occurs when all of the rotation is about the $a = 1$ axis corresponding to our chosen Cartan sub-algebra generator, $T_1 = H$.

In other words, there is a highest value, Λ_{max}, and a lowest eigenvalue, Λ_{min}. Now eqns. (8.112) and (8.113) are written in such a way that the first terms contain $E_\pm$, ready to act on any eigenfunction, so, since there is a highest and lowest eigenvalue, we must have

$$\begin{aligned} E_+\,|\Lambda_{\text{max}}\rangle &= 0 \\ E_-\,|\Lambda_{\text{min}}\rangle &= 0. \end{aligned} \tag{8.117}$$

Thus,

$$T^2|\Lambda_{\text{max}}\rangle = \Lambda_{\text{max}}(\Lambda_{\text{max}} + 1)\ |\Lambda_{\text{max}}\rangle, \tag{8.118}$$

and

$$T^2|\Lambda_{\text{min}}\rangle = \Lambda_{\text{min}}(\Lambda_{\text{min}} - 1)\ |\Lambda_{\text{min}}\rangle. \tag{8.119}$$

From these two points of reference, we deduce that

$$\Lambda_{\rm max}(\Lambda_{\rm max}+1) = \Lambda_{\rm min}(\Lambda_{\rm min}-1). \tag{8.120}$$

This equation has two solutions, $\Lambda_{\rm min} = \Lambda_{\rm max} + 1$ (which cannot exist, since there is no solution higher than $\Lambda_{\rm max}$ by assumption), and

$$\Lambda_{\rm max} = -\Lambda_{\rm min}, \tag{8.121}$$

thus

$$T^2 = \Lambda_{\rm max}(\Lambda_{\rm max}+1)\,\mathbf{I}. \tag{8.122}$$

The result means that the value T^2 is fixed by the maximum value which H can acquire. Strangely, the value is not $\Lambda^2_{\rm max}$ (all rotation about the 1 axis), which one would expect from the behaviour of the rotation *group*. This has important implications for quantum mechanics, since it is the algebra which is important for angular momentum or spin. It means that the total angular momentum can never be all in one fixed direction. As $\Lambda_{\rm max} \to \infty$ the difference becomes negligible.

The constant of proportionality in eqn. (8.115) can now be determined from the Hermitian property of the stepping operators as follows. The squared norm of $E_+|\Lambda_c\rangle$ may be written using eqn. (8.112)

$$\begin{aligned}
|E_+|\Lambda_c\rangle|^2 &= \langle\Lambda_c|E_-E_+|\Lambda_c\rangle \\
&= \langle\Lambda_c|T^2 - H(H+1)|\Lambda_c\rangle \\
&= \Lambda_{\rm max}(\Lambda_{\rm max}+1) - \Lambda_c(\Lambda_c+1) \\
&= (\Lambda_{\rm max}-\Lambda_c)(\Lambda_{\rm max}+\Lambda_c+1).
\end{aligned} \tag{8.123}$$

Thus,

$$\begin{aligned}
E_+|\Lambda_c\rangle &= \sqrt{(\Lambda_{\rm max}-\Lambda_c)(\Lambda_{\rm max}+\Lambda_c+1)}\,|\Lambda_c+1\rangle \\
E_-|\Lambda_c\rangle &= \sqrt{(\Lambda_{\rm max}+\Lambda_c)(\Lambda_{\rm max}-\Lambda_c+1)}\,|\Lambda_c-1\rangle.
\end{aligned} \tag{8.124}$$

Eqn. (8.121), taken together with eqn. (8.114), implies that the eigenvalues are distributed symmetrically about $\Lambda_c = 0$ and that they are separated by integer steps. This means that the possible values are restricted to

$$\Lambda_c = 0, \pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \pm 2, \ldots, \pm\Lambda_{\rm max}. \tag{8.125}$$

There are clearly $2\Lambda_{\rm max} + 1$ possible solutions. In the study of angular momentum, $\Lambda_{\rm max}$, is called the *spin* up to dimensional factors ($\hbar$). In group theory, this is referred to as the highest weight of the representation. Clearly, this single value characterizes a key property of the representation.

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What the above argument does not tell us is the value of $\Lambda_{\max}$. That is determined by the dimension of the irreducible representation which gives rise to rotations. In field theory the value of $\Lambda_{\max}$ depends, in practice, on the number of spacetime indices on field variables. Since the matrices for rotation in three spatial dimensions are fixed by the spacetime dimension itself, the only freedom left in transformation properties under rotations is the number of spacetime indices which can be operated on by a rotational transformation matrix. A scalar (no indices) requires no rotations matrix, a vector (one index) requires one, a rank 2-tensor requires two and so on. The number of independently transforming components in the field becomes essentially blocks of $2\Lambda_{\max} + 1$ and defines the spin of the fields.

8.6 Examples of discrete and continuous groups

Some groups are important because they arise in field theory with predictable regularity; others are important because they demonstrate key principles with a special clarity.

8.6.1 $GL(N, C)$: the general linear group

The group of all complex $N \times N$, non-singular matrices forms a group. This group has many sub-groups which are important in physics. Almost all physical models can be expressed in terms of variables which transform as sub-groups of this group.

(1) Matrix multiplication combines non-singular matrices into new non-singular matrices.

(2) Matrix multiplication is associative.

(3) The identity is the unit matrix

$$I = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ 0 & \vdots & 1 & 0 \\ 0 & \dots & 0 & 1 \end{pmatrix}. \tag{8.126}$$

(4) Every non-singular matrix has an inverse, by definition.

The representation space of a collection of matrices is the vector space on which the components of those matrices is defined. Since matrices normally multiply vectors, mapping one vector, v^A, onto another vector, v'^A,

$$v_A \rightarrow v'^A = U_{AB}\, v^B, \tag{8.127}$$

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it is normal to think of these matrices as acting on group vectors. In field theory, these transformations are especially important since the group vectors are multiplets of fields, e.g.

$$\phi(x)_A = \begin{pmatrix} \phi_1(x) \\ \phi_2(x) \\ \vdots \\ \phi_{d_R}(x) \end{pmatrix}, \tag{8.128}$$

where d_R is the dimension of the representation, or the size of the $d_R \times d_R$ matrices. Note: the dimension of a representation (the number of components in a multiplet) is not necessarily the same as the dimension of the group itself. For example: a three-dimensional vector ($d_R = 3$) might be constrained, by some additional considerations, to have only an axial symmetry (group dimension $d_G = 1$, a single angle of rotation); in that case one requires a 3×3 representation of a one-dimensional group, since vectors in three dimensions have three components.

8.6.2 $U(N)$: unitary matrices

$U(N)$ is the set of all unitary matrices of matrix dimension N. An $N \times N$ unitary matrix satisfies

$$U^\dagger U = (U^{\mathrm{T}})^* U = I, \tag{8.129}$$

where I is the $N \times N$ unit matrix, i.e. $U^\dagger = U^{-1}$. When $n = 1$, the matrices are single-component numbers. An $N \times N$ matrix contains N^2 components; however, since the transpose matrix is related to the untransposed matrix by eqn. (8.129), only half of the off-diagonal elements are independent of one another. Moreover, the diagonal elements must be real in order to satisfy the condition. This means that the number of independent *real* elements in a unitary matrix is $(N^2 - N)/2$ complex plus N real means N^2 real numbers. This is called the dimension of the group. $U(N)$ is non-Abelian for $U > 1$.

8.6.3 $SU(N)$: the special unitary group

The special unitary group is the sub-group of $U(N)$ which consists of all unitary matrices with unit determinant. Since the requirement of unit determinant is an extra constraint on the all of the independent elements of the group (i.e. the product of the eigenvalues), this reduces the number of independent elements by one compared with $U(N)$. Thus the dimension of $SU(N)$ is $N^2 - 1$ real components. $SU(N)$ is non-Abelian for $N > 1$. $SU(N)$ has several simple properties:

$$\begin{aligned} C_2(G_{\mathrm{adj}}) &= N \\ d_G &= N^2 - 1 \\ d_F &= N \\ C_2(G_f) &= \frac{N^2-1}{2N}, \end{aligned} \tag{8.130}$$

where $C_2(G)$ is the quadratic Casimir invariant in representation G, d_G is the dimension of the group, and d_F is the dimension of the fundamental representation $R \to F$.

8.6.4 *SU*(2)

The set of 2×2 unitary matrices with unit determinant has $N^2 - 1 = 3$ elements for $n = 2$. Up to similarity transformations, these may be written in terms of three real parameters $(\theta_1, \theta_2, \theta_2)$:

$$g_1 = \begin{pmatrix} \cos\left(\frac{1}{2}\theta_1\right) & \mathrm{i}\sin\left(\frac{1}{2}\theta_1\right) \\ \mathrm{i}\sin\left(\frac{1}{2}\theta_1\right) & \cos\left(\frac{1}{2}\theta_1\right) \end{pmatrix} \tag{8.131a}$$

$$g_2 = \begin{pmatrix} \cos\left(\frac{1}{2}\theta_2\right) & \sin\left(\frac{1}{2}\theta_2\right) \\ -\sin\left(\frac{1}{2}\theta_2\right) & \cos\left(\frac{1}{2}\theta_2\right) \end{pmatrix} \tag{8.131b}$$

$$g_3 = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\frac{1}{2}\theta_3} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\frac{1}{2}\theta_3} \end{pmatrix}. \tag{8.131c}$$

These matrices are the exponentiated Pauli matrices $\mathrm{e}^{\frac{\mathrm{i}}{2}\sigma_i}$. Using this basis, any element of the group may be written as a product of one or more of these matrices with some θ_i.

8.6.5 *U*(1): *the set of numbers* $z : |z|^2 = 1$

The set of all complex numbers $U = \mathrm{e}^{\mathrm{i}\theta}$ with unit modulus forms an Abelian group under multiplication:

(1) $\mathrm{e}^{\mathrm{i}\theta_1}\,\mathrm{e}^{\mathrm{i}\theta_2} = \mathrm{e}^{\mathrm{i}(\theta_1+\theta_2)}$.

(2) $(\mathrm{e}^{\mathrm{i}\theta_1}\,\mathrm{e}^{\mathrm{i}\theta_2})\,\mathrm{e}^{\mathrm{i}\theta_3} = \mathrm{e}^{\mathrm{i}\theta_1}\,(\mathrm{e}^{\mathrm{i}\theta_2}\,\mathrm{e}^{\mathrm{i}\theta_3})$.

(3) $\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{e}^{\mathrm{i}0} = \mathrm{e}^{\mathrm{i}\theta}$.

(4) $U^{-1} = U^*$ since $\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{e}^{-\mathrm{i}\theta} = \mathrm{e}^{\mathrm{i}0} = 1$.

The representation space of this group is the space of complex scalars Φ, with constant modulus:

$$\Phi^*\Phi \to (U\Phi)^* U\Phi = \Phi^*\, U^* U\; \Phi = \Phi^*\Phi. \tag{8.132}$$

This group is important in electromagnetism; it is this symmetry group of complex phases which is connected to the existence of a conserved electrical charge.

8.6.6 Z_N: the Nth roots of unity

The Nth roots of unity form a sub-group of $U(1)$. These complex numbers may be written in the form $\exp(2\pi \mathrm{i} \frac{p}{N})$, for $p = 0, \ldots, N-1$. The group Z_N is special because it is not infinite. It has exactly N discrete elements. The group has the topology of a circle, and the elements may be drawn as equi-distant points on the circumference of the unit circle in the complex plane. Z_N is a modulo group. Its elements satisfy modulo N arithmetic by virtue of the multi-valuedness of the complex exponential. The group axioms are thus satisfied as follows:

(1) $\exp\left(2\pi \mathrm{i}\, \frac{p}{N}\right) \exp\left(2\pi \mathrm{i}\, \frac{p'}{N}\right) = \exp\left(2\pi \mathrm{i}\, \frac{p+p'}{N}\right) = \exp\left(2\pi \mathrm{i}\left[\frac{p+p'}{N} + m\right]\right)$, where N, m, p are integers;

(2) follows trivially from $U(1)$;

(3) follows trivially from $U(1)$;

(4) the inverse exists because of the multi-valued property that

$$\exp\left(-2\pi \mathrm{i}\frac{p}{N}\right) = \exp\left(2\pi \mathrm{i}\frac{N-p}{N}\right). \tag{8.133}$$

Thus when $p = N$, one arrives back at the identity, equivalent to $p = 0$.

The representation space of this group is undefined. It can represent translations or shifts along a circle for a complex scalar field. Z_2 is sometimes thought of as a reflection symmetry of a scalar field, i.e. $Z_2 = \{1, -1\}$ and $\phi \to -\phi$. An action which depends only on ϕ^2 has this symmetry.

Usually Z_N is discussed as an important sub-group of very many continuous Lie groups. The presence of Z_N as a sub-group of another group usually signifies some multi-valuedness or redundancy in that group. For example, the existence of a Z_2 sub-group in the Lie group $SU(2)$ accounts for the double-valued nature of electron spin.

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8.6.7 $O(N)$: the orthogonal group

The orthogonal group consists of all matrices which satisfy

$$U^{\mathrm{T}} U = I \tag{8.134}$$

under normal matrix multiplication. In other words, the transpose of each matrix is the inverse matrix. All such matrices are real, and thus there are $(N^2 - N)/2 + n = N(N+1)/2$ real components in such a matrix. This is the dimension of the group. The orthogonal group is non-Abelian for $N > 2$ and is trivial for $n = 1$.

The special orthogonal group is the sub-group of $O(N)$ which consists of matrices with unit determinant. This reduces the dimension of the group by one, giving $N(N-1)/2$ independent components.

8.6.8 $SO(3)$: the three-dimensional rotation group

This non-Abelian group has three independent components corresponding to rotations about three-independent axes in a three-dimensional space. The group elements may be parametrized by the rotation matrices g_i about the given axis i:

$$U_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & \sin\theta_1 \\ 0 & -\sin\theta_1 & \cos\theta_1 \end{pmatrix} \tag{8.135}$$

$$U_y = \begin{pmatrix} \cos\theta_2 & 0 & -\sin\theta_2 \\ 0 & 1 & 0 \\ \sin\theta_2 & 0 & \cos\theta_2 \end{pmatrix} \tag{8.136}$$

$$U_z = \begin{pmatrix} \cos\theta_3 & \sin\theta_3 & 0 \\ -\sin\theta_3 & \sin\theta_3 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{8.137}$$

The representation space of this group is a three-dimensional Euclidean space and the transformations rotate three-dimensional vectors about the origin, preserving their lengths but not their directions. Notice that these matrices do not commute; i.e. a rotation about the x axis followed by a rotation about the y axis, is not the same as a rotation about the y axis followed by a rotation about the x axis.

8.6.9 $SO(2)$: the two-dimensional rotation group

This group has only one element, corresponding to rotations about a point in a plane. Any element of $SO(2)$ may be written in the form

$$U = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}. \tag{8.138}$$

The representation space of this group is a two-dimensional Euclidean space, and the transformation rotates two-component vectors about the origin. Notice how the matrices parametrizing $SO(3)$ are simply rotations of $SO(2)$ embedded in a three-dimensional framework.

8.7 Universal cover groups and centres

We know that groups can contain other groups, as sub-groups of the whole, and therefore that some are larger than others. The *universal cover group* is defined to be a simply connected group which contains an image of every point in a given Lie group. If we consider an arbitrary Lie group, in general it will have companion groups which are locally the same, but globally different. The best known example of this is the pair $SU(2)$ and $SO(3)$, which are locally isomorphic, but globally different. In fact $SU(2)$ contains two images of $SO(3)$ or covers it twice, or contains two equivalent copies of it. Taking this a step further, if three groups have the same local structure, then they will all be sub-groups of the universal cover groups.

If we begin with a Lie algebra, it is possible to exponentiate the generators of the algebra to form group elements:

$$\Theta = \theta^A T^A \rightarrow G = e^{i\Theta}. \tag{8.139}$$

The group formed by this exponentiation is not unique; it depends on the particular representation of the algebra being exponentiated. For instance, the 2×2 representation of $SU(2)$ exponentiates to $SU(2)$, while the 3×3 representation of $SU(2)$ exponentiates to $SO(3)$. Both of these groups are locally isomorphic but differ in their centres. In the case of $SU(2)$ and $SO(3)$, we can relate them by factorizing out the centre of the universal cover group,

$$SU(2)/Z_2 = SO(3). \tag{8.140}$$

From Schur's lemma, we know that the centre of a group is only composed of multiples of the identity matrix, and that, in order to satisfy the rules of group multiplication, they must also have modulus one. It follows from these two facts that any element of the centre of a group can be written

$$g_c = \exp(\pm 2\pi i\, q/N)\, \mathbf{I}, \qquad q = 0, \ldots, N-1. \tag{8.141}$$

These elements are the Nth roots of unity for some N (in principle infinite, but in practice usually finite). If we start off with some universal cover group then, whose centre is Z_N, there will be many locally isomorphic groups which can be found by factoring out sub-groups of the centre. The largest thing one can divide out is Z_N itself, i.e. the whole centre. The group formed in this way is called the adjoint group, and it is generated by the *adjoint representation*:

$$\frac{\text{group}}{\text{centre of group}} = \text{adjoint group}. \tag{8.142}$$

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Table 8.1. Some common Lie algebras and groups.

Algebra	Centre	Cover
A_N	Z_N	$SU(N-1)$
B_N	Z_2	$SO(2N+1)$
C_N	Z_2	$Sp(2N)$
D_N	Z_4 (Nodd)	$SO(2N)$
	$Z_2 \times Z_2$ (Neven)	
E_6	Z_3	E_6
G_2, F_4, E_8	Z_3	

But it is not necessary to factor out the entire centre, one can also factor out a sub-group of the full centre; this will also generate a locally isomorphic group. For example, $SU(8)$ has centre Z_8. We can construct any of the following locally isomorphic groups:

$$SU(8) \qquad SU(8)/Z_8 \qquad SU(8)/Z_4 \qquad SU(8)/Z_2. \tag{8.143}$$

Some well known Lie groups are summarized in table 8.1.

8.7.1 Centre of $SU(N)$ is Z_N

$SU(N)$ is a simply connected group and functions as its own universal cover group. As the set of $N \times N$ matrices is the fundamental, defining representation, it is easy to calculate the elements of the centre. From Schur's lemma, we know that the centre must be a multiple of the identity:

$$g_c = \alpha \, \mathbf{I}_N. \tag{8.144}$$

where $\mathbf{I}_N$ is the $N \times N$ identity matrix. Now, $SU(N)$ matrices have unit determinant, so

$$\det I_N = \alpha^N = 1. \tag{8.145}$$

Thus, the solutions for α are the Nth roots of unity, Z_N.

8.7.2 Congruent algebras: N-ality

Since roots and weights of representations can be drawn as vectors in the Cartan sub-space, different representations produce similar, but not identical, patterns. Elements E_α of the algebra step through chains of solutions, creating a laced lattice-work pattern. Representations which exponentiate to the same group have patterns which are *congruent* to one another [124].

Congruence is a property of discrete sets. The correct terminology is 'congruent to x modulo m'. The property is simplest to illustrate for integers. x is said to be conjugate to y modulo m if $y - x$ is an integer multiple of m:

$$y = x + km, \tag{8.146}$$

for integer k, m. Congruence modulo m is an equivalence relation, and it sorts numbers into classes or congruent sets. The patterns made by congruent sets can be overlain consistently. The equivalence class, E_x, is the set of all integers which can be found from x by adding integer multiples m to it:

$$\begin{aligned} E_x &= \{x + km \mid \text{integer } k\} \\ &= \{\ldots, -2m + x, -m + x, x, x + m, x + 2m, \ldots\}. \end{aligned} \tag{8.147}$$

There are exactly m different congruence classes modulo m, and these partition the integers; e.g. for $m = 4$, we can construct four classes:

$$\begin{aligned} E_0 &= \{\ldots, -8, -4, 0, 4, 8, \ldots\} \\ E_1 &= \{\ldots, -7, -3, 1, 5, 9, \ldots\} \\ E_2 &= \{\ldots, -6, -2, 2, 6, 10, \ldots\} \\ E_3 &= \{\ldots, -5, -1, 3, 7, 11, \ldots\}. \end{aligned} \tag{8.148}$$

Lie algebra representations can also be classified into congruence classes. Historically, congruence classes of $SU(N)$ modulo N are referred to as N-ality as a generalization of 'triality' for $SU(3)$. Each congruence class has a label q; $q = 0$ corresponds to no centre, or the adjoint congruence class. The well known algebras contain the following values [56]:

$$q = \sum_{k=1}^{n} \alpha_k \qquad (\text{mod}\, n+1) \qquad \text{for } A_n \tag{8.149}$$

$$q = \alpha_n \qquad (\text{mod}\, 2) \qquad \text{for } B_n \tag{8.150}$$

$$q = \alpha_1 + \alpha_3 + \alpha_5 \qquad (\text{mod}\, 2) \qquad \text{for } C_n \tag{8.151}$$

$$q = \alpha_1 - \alpha_2 + \alpha_4 - \alpha_5 \qquad (\text{mod}\, 3) \qquad \text{for } E_6 \tag{8.152}$$

$$q = \alpha_4 + \alpha_6 + \alpha_7 \qquad (\text{mod}\, 2) \qquad \text{for } E_7 \tag{8.153}$$

$$q = 0 \qquad \text{for all representations of } E_7, E_8, F_4, G_2. \tag{8.154}$$

In the special case of D_n, the congruence classes require classification by a two-component vector:

$$\begin{aligned} q_1 &= (\alpha_{n-1} + \alpha_n, 2\alpha_1 + \alpha_3 + \cdots \\ &\qquad + 2\alpha_{n-2} + (n-2)\alpha_{n-1} + n\alpha_n + \cdots) \qquad (\text{mod } 2) \qquad \text{odd } n \\ q_2 &= (\alpha_{n-1} + \alpha_n, 2\alpha_1 + 2\alpha_3 + \cdots \\ &\qquad + 2\alpha_{n-3} + (n-2)\alpha_{n-1} + n\alpha_n) \qquad (\text{mod } 4) \qquad \text{even } n. \end{aligned} \tag{8.155}$$

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The congruence is modulo the order of the centre. The algebra D_n requires a two-dimensional label, since its centre is two-dimensional. E_7, E_8, F_4, and G_2 all have trivial centres, thus they all lie in a single class congruent to the adjoint.

8.7.3 Simple and semi-simple Lie algebras

A Lie algebra is *simple* if it has no proper invariant sub-algebras; i.e if only one element (the identity) commutes with every other in the group. A simple algebra is necessarily semi-simple. A semi-simple Lie algebra can be written in block-diagonal form, as a direct sum of invariant sub-algebras, each of which is a simple Lie algebra

$$A = A_1 \oplus A_2 \oplus A_3 \oplus \cdots A_N, \tag{8.156}$$

i.e. it factorizes into block-diagonal form with simple blocks. A semi-simple algebra has no Abelian invariant sub-algebras.

8.8 Summary

The existence of a symmetry in a physical system means that it is possible to re-label parameters of a model without changing its form or substance. Identify the symmetries of a physical system and one can distinguish between the freedom a system has to change and the constraints which hold it invariant: symmetries are thus at the heart of dynamics and of perspective.

Symmetries form *groups*, and can therefore be studied with the group theory. Since a symmetry means that some quantity R_ξ does not change, when we vary the action with respect to a parameter ξ, *conservation of* R_ξ is also linked to the existence of the symmetry. All of the familiar conservation laws can be connected to fundamental symmetries.

In the case of electromagnetism, Lorentz covariance was exposed just by looking at the field equations and writing them in terms of $(3+1)$ dimensional vectors. The chapters which follow examine the transformations which change the basic variables parametrizing the equations of motion, and the repercussions such transformations have for covariance.