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A NOTE ON THE IDEALS OF GROUPOID C^*-ALGEBRAS FROM SMALE SPACES

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In this note, we characterise completely the ideals of the groupoid C^*-algebra arising from the asymptotic equivalence relation on the points of a Smale space and show that the related Ruelle algebra is simple when the Smale space is topologically transitive.

1. INTRODUCTION

In [**4**], Putnam constructed several groupoid C^*-algebras from a Smale space by using such equivalence relations on the points of this space as the asymptotic equivalence, stable equivalence and unstable equivalence. Some important hyperbolic dynamical systems are Smale spaces, for example, subshifts of finite type, Anosov diffeomorphisms and solenoids, and so on [**4, 5**]. In the case of subshifts of finite type, the so-called Ruelle algebras arising from the stable equivalence relation and unstable equivalence relation are the stablised Cuntz–Kreieger algebras. Thus the Ruelle algebras from Smale spaces may be viewed as the "higher dimensional" generalisations of the Cuntz–Krieger algebras. In [**6**], the simplicity of the C^*-algebras from stable and unstable equivalence relations and the related Ruelle algebras were proved by I. Putnam and J. Spielberg when the Smale space is irreducible. But we still do not know whether the Ruelle algebra associated with the asymptotic equivalence relation is simple or not. In this note, we answer this question positively even under the weaker condition that the Smale space is topologically transitive; However, in this case, the groupoid C^*-algebra constructed directly from the asymptotic equivalence relation is not simple again. By the "Smale's spectral decomposition " in [**9**], we can characterise the ideals of this C^*-algebra completely.

For the definition of a Smale space, we refer to [**4, 5, 9**].

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DEFINITION 1.1: Let (X, d) be a compact metric space with a homeomorphism φ of X. We say that (X, d, φ) is a Smale space if there are constants $\lambda_0 \in (0,1)$ and $0 < \varepsilon_0$ and a continuous map

$$[\ ,\ ] : \{(x,y) \in X \times X \mid d(x,y) < \varepsilon_0\} \longrightarrow X$$

satisfying the following:

$$\begin{aligned}
[x,x] &= x, \\
[[x,y],z] &= [x,z], \\
[x,[y,z]] &= [x,z], \\
[\varphi x, \varphi y] &= \varphi([x,y])
\end{aligned}$$

for x, y, z in X, whenever both sides of the equations are defined. Let

$$\begin{aligned}
V^s(x,\varepsilon) &= \{y \in X \mid [x,y] = y,\ d(x,y) < \varepsilon\}, \\
V^u(x,\varepsilon) &= \{y \in X \mid [y,x] = y,\ d(x,y) < \varepsilon\}
\end{aligned}$$

for any $0 < \varepsilon \leqslant \varepsilon_0$. We also assume that

$$\begin{aligned}
d(\varphi y, \varphi z) &\leqslant \lambda_0 d(y,z), \quad \text{for} \quad y, z \in V^s(x,\varepsilon) \\
d(\varphi^{-1} y, \varphi^{-1} z) &\leqslant \lambda_0 d(y,z), \quad \text{for} \quad y, z \in V^u(x,\varepsilon).
\end{aligned}$$

We call a Smale space (X, d, φ) irreducible if φ is topologically mixing; that is, for every pair of non-empty open subsets U and V of X, there exists $N \geqslant 1$ such that for all $n \geqslant N$, $\varphi^n U \cap V$ is nonempty. Similarly, We call a Smale space (X, d, φ) topologically transitive if φ is topologically transitive; that is, for every pair of non-empty open subsets U and V of X and each integer $N \geqslant 0$, there exists some $n \geqslant N$ such that $\varphi^n U \cap V$ is nonempty.

If G is a second countable, locally compact groupoid, we denote by s (respectively, r) the source map (respectively, range map) of G, and let G^0 be the unit space of G. For the results on groupoids and groupoid C^*-algebras, we refer the readers to [**7**].

DEFINITION 1.2: Let (X, d, φ) be a Smale space, and let x, y be in X.

(1) x and y are called asymptotically equivalent if $\lim_{|n| \to \infty} d(\varphi^n x, \varphi^n y) = 0$;

(2) x and y are called stably equivalent if $\lim_{n \to +\infty} d(\varphi^n x, \varphi^n y) = 0$;

(3) x and y are called unstably equivalent if $\lim_{n \to -\infty} d(\varphi^n x, \varphi^n y) = 0$.

We denote the stable, unstable and asymptotic equivalence class of x by $V^s(x)$, $V^u(x)$ and $V^a(x)$ respectively. By the definitions, $V^a(x) = V^s(x) \cap V^u(x)$. $V^a(x)$ is countable and dense in X if φ is topologically mixing [**9**].

In [**8**] and [**9**], Ruelle proved that for every asymptotically equivalent pair (x, y), there is a conjugating homeomorphism $(\gamma, \mathcal{O})$; that is, there is a neighbourhood $\mathcal{O}$ of x and a homeomorphism $\gamma : \mathcal{O} \longrightarrow \gamma(\mathcal{O})$ such that $\gamma(x) = y$, $\lim_{|n|\to\infty} d(\varphi^n z, \varphi^n \gamma z) = 0$ uniformly for z in $\mathcal{O}$, and the germ of γ is uniquely determined by (x, y).

Define the set

$$G_a = \{(x, y) \mid x \text{ and } y \text{ are asymptotically equivalent}\}.$$

A base of the topology of G_a is given by the open sets

$$\{(z, \gamma z) \mid z \in \mathcal{O},\ (\mathcal{O}, \gamma) \text{ is a conjugating homeomorphism}\}.$$

Then, under this topology, G_a is a second countable, locally compact, Hausdorff, r-discrete, principal groupoid, and the counting measure is a Haar system [**4, 7, 8**]. Obviously, under the embedding map $x \in X \longrightarrow (x, x) \in G_a$, we can identify the unit space G_a^0 with X.

Define the map

$$\alpha : G_a \longrightarrow G_a,\ \alpha(x, y) = (\varphi x, \varphi y) \quad \text{for} \quad (x, y) \in G_a.$$

Then α is a groupoid isomorphism and preserves the Haar system. We form the semi-direct product by

$$G_a \times_\alpha \mathbb{Z} = \{(x, n, y) \mid n \in \mathbb{Z}, (\varphi^n x, y) \in G_a\},$$

with the multiplication and the inverse map given respectively by

$$(x, n, y)(x', n', y') = (x, n + n', y') \quad \text{if} \quad y = x'$$
$$(x, n, y)^{-1} = (y, -n, x).$$

Notice that the unit space $(G_a \times_\alpha \mathbb{Z})^0 = X$ by identifying $(x, 0, x)$ in $G_a \times_\alpha \mathbb{Z}$ with x in X.

The map sending $((x, y), n)$ in $G_a \times \mathbb{Z}$ to $(x, n, \varphi^n y)$ in $G_a \times_\alpha \mathbb{Z}$ is bijective, and we transfer the product topology from $G_a \times \mathbb{Z}$ over via this map. Then $G_a \times_\alpha \mathbb{Z}$ is a locally compact, Hausdorff, r-discrete groupoid, and the counting measure is a Haar system.

DEFINITION 1.3: Let A and R^a be the reduced groupoid C^*-algebra of G_a and $G_a \times_\alpha \mathbb{Z}$ respectively. Call the C^*-algebra R^a a Ruelle algebra arising from the asymptotic equivalence relation.

In [**4**] (or [**5**]), Putnam proved that the C^*-algebra A is simple whenever the Smale space is irreducible, but did not consider the simplicity of the Ruelle algebra R^a. Even though we have $R^a \cong A \times_\alpha \mathbb{Z}$, the simplicity of R^a is less obvious. In this note, we show that R^a is simple even under weaker conditions — the topologically transitive case.

For the reduced groupoid C^*-algebras and the related Ruelle algebras arising from stable equivalence and unstable equivalence, we refer the readers to [**4, 5, 6**].

2. Main Results

Throughout this note, we let (X, d, φ) be a Smale space.

LEMMA 2.1. *Let (Y, d) be a compact metric space without isolated points, and let ϕ be a homeomorphism of X. Suppose that ϕ is expansive, that is, there exists a constant $c > 0$ such that for x, y in X, $x \neq y$ implies $d(\phi^n x, \phi^n y) > c$ for some integer n. Then the set $\mathrm{Per}_n = \{x \in X \mid \varphi^n x = x\}$ is finite for each nonzero integer n.*

PROOF: Obviously, the homeomorphism ϕ^n is also expansive with the same constant c for every nonzero integer n. Hence we only prove that the set of fixed points $\mathrm{Per} = \{x \in Y \mid \phi x = x\}$ is finite.

Let x and y be two different elements in Per. Then $d(x, y) > c$. For otherwise, $d(\phi^k x, \phi^k y) \leqslant c$ for each integer k, and hence $x = y$ by the expansive property of ϕ, which is a contradiction. Hence the finiteness of the set of fixed points Per follows from the fact that Y is a compact space without isolate points. ∎

REMARK. From [**9**], for every Smale space (X, d, φ), the homeomorphism φ is expansive, hence the set Per_n is finite for each $n \neq 0$. The following lemma is similar to [**6**].

LEMMA 2.2. *Suppose that $\varphi^n x$ is in $V^a(x)$ for some x in X and a positive nonzero integer n. Then:*

(1) *The sequence $\{\varphi^{nk} x \mid k \geqslant 1\}$ converges an element in Per_n. Moreover, if $x_1 = \lim\limits_{k \to +\infty} \varphi^{nk} x$, then $x \in V^s(x_1)$.*

(2) *The sequence $\{\varphi^{-nk} x \mid k \geqslant 1\}$ converges to an element in Per_n. Moreover, if $x_2 = \lim\limits_{k \to +\infty} \varphi^{-nk} x$, then $x \in V^u(x_2)$.*

PROOF: We only prove (1).

Suppose that z is a limit point of the sequence $\{\varphi^{nk}(x) \mid k \geqslant 1\}$. Then $z = \lim\limits_{k \to +\infty} \varphi^{n n_k}(x)$, where $\{n_k\}$ is a subsequence of positive integers. Also since $\varphi^n x \in$

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$V^a(x)$, $\varphi^n z = \lim_{k\to+\infty} \varphi^{nn_k}(\varphi^n x) = \lim_{k\to+\infty} \varphi^{nn_k}(x) = z$. Hence $z \in \mathrm{Per}_n$. Thus, the limit points of $\{\varphi^{nk}(x) \mid k \geqslant 1\}$, which exist as X is compact, are contained in Per_n. In order to prove the convergence of $\{\varphi^{nk}x \mid k \geqslant 1\}$, we claim that there is at most one limit point of this sequence.

If not, let $\mathrm{Per}_n = \{x_1, x_2, \dots, x_m\}$, and choose an open neighbourhood U_i of x_i such that $\varphi^n U_i \cap U_j = \emptyset$ for $i \neq j$. Then the set

$$\left\{k \geqslant 1 \mid \varphi^{nk}x \notin \bigcup_{i=1}^{m} U_i\right\}$$

is finite. For otherwise, by the compactness of X, there must be a limit point z of $\{\varphi^{nk} \mid k \geqslant 1\}$ such that $z \notin \bigcup_{i=1}^{m} U_i$; which implies that $z \notin \mathrm{Per}_n$ and contradicts the above facts. Hence we can choose $K_0 > 0$ such that $\varphi^{nk}x \in \bigcup_{i=1}^{m} U_i$ for each $k \geqslant K_0$. But since $\varphi^n U_i \cap U_j = \emptyset$ for $i \neq j$, the $\varphi^{nk}x$ must be all in the same U_i for all $k \geqslant K_0$. It follows that the limit points of $\{\varphi^{nk}(x) \mid k \geqslant 1\}$ are contained in some U_i^-, the closure of U_i. Since $U_i^- \cap \mathrm{Per}_n = \{x_i\}$, the sequence $\{\varphi^{nk}(x) \mid k \geqslant 1\}$ has only one limit point x_i.

Let $x_1 = \lim_{k\to+\infty} \varphi^{nk}x$. Then $\varphi^l(x_1) = \lim_{k\to+\infty} \varphi^{nk}(\varphi^l x)$ for each $0 < l < n$ and by above paragraph, $\varphi^n x_1 = x_1$. It follows that $\lim_{k\to+\infty} d(\varphi^k x, \varphi^k x_1) = 0$. Hence $x \in V^s(x_1)$.

Similarly, we can prove (2). ∎

Lemma 2.3. *Suppose that the Smale space (X, φ) is irreducible. Then the set*

$$\{x \in X \mid \varphi^n(x) \in V^a(x) \text{ for some nonzero integer } n\}$$

is countable.

Proof: It is well known that for any x and y in X, the set $V^s(x) \cap V^u(y)$ is at most countable if (X, φ) is irreducible. For, if $V^s(x) \cap V^u(y)$ is non-empty, choose an element z in $V^s(x) \cap V^u(y)$. Then $V^s(x) \cap V^u(y) \subseteq V^a(z)$, which is countable if φ is topologically mixing.

Suppose $\varphi^n x \in V^a(x)$ for some x in X and a nonzero integer n. We can assume that $n > 0$, by replacing n with $-n$ if n is a negative integer. From Lemma 2.2, there are x_1 and x_2 in Per_n such that $x \in V^s(x_1) \cap V^u(x_2)$. Hence the set

$$\{x \in X \mid \varphi^n(x) \in V^a(x) \text{ for some nonzero integer } n\}$$

is contained in the set

$$\bigcup_{n=1} \Big(\bigcup_{x_i, x_j \in \mathrm{Per}_n} V^s(x_i) \cap V^u(x_j) \Big),$$

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which is countable from Lemma 2.1 and the above paragraph. This completes the proof. ∎

Recall that a locally compact groupoid G is called essentially principal if for every invariant closed subset F of its unit space, the set of u in F whose isotropy group $G(u)$ is reduced to $\{u\}$ is dense in F, where $G(u) = r^{-1}(u) \cap s^{-1}(u)$. F is called invariant if $F = r(s^{-1}(F))$ ([**7**]).

Proposition 2.4. *Suppose that the Smale space (X, φ) is irreducible. Then the groupoid $G_a \times_\alpha \mathbb{Z}$ is essentially principal.*

Proof: Observe that $G_a \subseteq G_a \times_\alpha \mathbb{Z}$ by identifying (x, y) in G_a with $(x, 0, y)$ in $G_a \times_\alpha \mathbb{Z}$. Notice that $V^a(x)$ is dense in $G_a^0 \equiv X$ for each $x \in X$ when (X, φ) is irreducible. Hence $G_a \times_\alpha \mathbb{Z}$ is a minimal groupoid, that is, the unit space $(G_a \times_\alpha \mathbb{Z})^0$ has no nontrivial invariant open subset, [**7**]. So we only consider $F = (G_a \times_\alpha \mathbb{Z})^0 \equiv X$.

Let x be in X. Then the isotropy group $(G_a \times_\alpha \mathbb{Z})(x)$ contains properly the one-point set $\{x\}$ if and only if there exists a nonzero integer n such that $\varphi^n x \in V^a(x)$. Hence from Lemma 2.3, the set of x in X whose isotropy group $(G_a \times_\alpha \mathbb{Z})(x)$ contains properly the one-point set $\{x\}$ is countable, and then its compliment is dense in X as X is a compact space without isolate points. Thus $G_a \times_\alpha \mathbb{Z}$ is essentially principal. ∎

Lemma 2.5. *Let the Smale space (X, φ) be irreducible. Then the groupoid G_a and $G_a \times_\alpha \mathbb{Z}$ are amenable in the sense of Renault.*

Proof: Let H be the product groupoid $G_s \times G_u$, where G_s and G_u are the locally compact principal groupoids defined by the stable and unstable equivalence relations, respectively. Then the unit space of H is $X \times X$ and the diagonal $\triangle = \{(x, x) \mid x \in X\}$ are abstract transversals in the sense of Muhly, Renault and Williams [**3**]. Hence the reduced groupoid $H_\triangle^\triangle$ is equivalent to G_a. Since G_s and G_u are amenable ([**5**]), by [**1**, Theorem 2.2.13], we have that G_a is amenable, and then that $G_a \times_\alpha \mathbb{Z}$ is amenable by [**7**]. ∎

By the above arguments, we know that $G_a \times_\alpha \mathbb{Z}$ is a r-discrete, essentially principal and amenable groupoid without nontrivial invariant open subsets. As noted in [**7**], we have:

Theorem 2.6. *Suppose that the Smale space (X, φ) is irreducible. Then the Ruelle algebra R^a is simple.*

For the rest of this note, we characterise the ideals of the C^*-algebra $C_r^*(G_a)$ and prove the simplicity of the Ruelle algebra R^a in the topologically transitive case.

Smale's spectral decomposition. ([**9**]) Let (X, φ) be a topologically transitive Smale space. Then X is the union of m disjoint compact subsets X_i which are cyclically permuted by φ such that $\varphi^m|_{X_i}$ is topologically mixing. These properties determine uniquely the X_i, m.

REMARK. From [8] (or [2]), the metric d in the definition of the Smale space could be replaced by a Hölder equivalent metric such that φ and φ^{-1} are Lipschitz. Let $\varphi_i = \varphi^m|_{X_i}$. Then, from [9], (X_i, φ_i) is an irreducible Smale space and the following lemma is straightforward.

LEMMA 2.7. *Let x and y be in X. Then x and y are asymptotically equivalent in (X, φ) if and only if both x and y are in the same X_i, and x and y are asymptotically equivalent in (X_i, φ_i). Hence, $G_a = \bigcup_{i=1}^{m} G_a^{(i)}$, where*

$$G_a^{(i)} = \left\{(x, y) \mid x \text{ and } y \text{ are asymptotically equivalent in } (X_i, \varphi_i)\right\}$$

for $i = 1, \ldots, m$.

LEMMA 2.8. *Let (X, φ) be a topologically Smale space and let $X_1, \ldots, X_m$ be the compact subsets of X in the Smale spectral decomposition. Then $X_1, \ldots, X_m$ are all the nontrivial invariant closed subsets of the unit space $(G_a)^0$, and they are all minimal in the sense that they could not properly contain any invariant closed nonempty subset of G_a^0.*

PROOF: From Lemma 2.7, each X_i is an invariant closed subset in $(G_a)^0$. If E is any invariant closed nonempty subset of $(G_a)^0$, then there exists some i $(1 \leqslant i \leqslant m)$ such that $E \cap X_i$ is nonempty. Observe that $E \cap X_i$ is an invariant closed subset of $\left(G_a^{(i)}\right)^0$, the unit space of the groupoid $G_a^{(i)}$. Also since (X_i, φ_i) is irreducible, we have that $G_a^{(i)}$ is minimal, and so $E \cap X_i = X_i$ which implies that $X_i \subseteq E$. This completes the proof. ∎

THEOREM 2.9. *Suppose that the Smale space (X, φ) is topologically transitive. Then*

$$C_r^*(G_a) = C_r^*\left(G_a^{(1)}\right) \oplus C_r^*\left(G_a^{(2)}\right) \oplus \cdots \oplus C_r^*\left(G_a^{(m)}\right),$$

where $G_a^{(i)}$ is the groupoid from the asymptotic equivalence relation in the irreducible Smale space (X_i, φ_i) given in the Smale spectral decomposition of (X, φ).

PROOF: We first observe that $G_a^{(i)}$ is a closed subgroupoid of G_a for each i. If $(x, y) \in G_a$ but $(x, y) \notin G_a^{(i)}$, then there exists a conjugating homeomorphism $(\mathcal{O}, \gamma)$ determined by the asymptotic equivalence pair (x, y) in (X, φ). Let $\mathcal{O}' = \mathcal{O} \cap (X - X_i)$. Then $U = \left\{(z, \gamma z) \mid z \in \mathcal{O}'\right\}$ is an open neighbourhood of (x, y) in G_a and is contained in $G_a - G_a^{(i)}$, which implies that $G_a^{(i)}$ is closed in G_a. Hence G_a is the disjoint union of the closed subgroupoid $G_a^{(i)}, i = 1, 2, \ldots, m$. Note that G_a is amenable because of the amenability of $G_a^{(i)}$. Then $C_r^*(G_a) = C_r^*\left(G_a^{(1)}\right) \oplus C_r^*\left(G_a^{(2)}\right) \oplus \cdots \oplus C_r^*\left(G_a^{(m)}\right)$. ∎

REMARK. There are two kinds of topologies on every $G_a^{(i)}$. One is the topology defined from the conjugating homeomorphism on the Smale space (X_i, φ_i), as defined on G_a; the other is the relative topology as a subgroupoid of G_a. It is not difficult to check that these two topologies are consistent. Hence every C^*-algebra $C_r^*\left(G_a^{(i)}\right)$ is simple. By the above theorem, we have that every ideal of $C_r^*(G_a)$ is a direct sum of some $C_r^*\left(G_a^{(i)}\right)$, $i = 1, \dots, m$.

LEMMA 2.10. *Suppose that (X, φ) is a topologically transitive Smale space. Then there are no nontrivial invariant closed subsets in $(G_a \times_\alpha \mathbb{Z})^0$, the unit space of $G_a \times_\alpha \mathbb{Z}$; that is, $G_a \times_\alpha \mathbb{Z}$ is a minimal groupoid.*

PROOF: Let $X_1, X_2, \dots, X_m$ be the compact subsets of X in the Smale spectral decomposition of (X, φ). For each i, choose x in X_i and y in X_{i-1} such that $x = \varphi(y)$ (if i=1, then let $X_0 = X_m$). Then $(y, 1, x) \in G_a \times_\alpha \mathbb{Z}$ and $d(y, 1, x) = x \in X_i$, however $r(y, 1, x) = y \notin X_i$, which implies that X_i is not invariant in $(G_a \times_\alpha \mathbb{Z})^0$.

Suppose that E is an arbitrary invariant closed non-empty subset of $(G_a \times_\alpha \mathbb{Z})^0$. Then there are $1 \leqslant i_1 < i_2 < \dots < i_k \leqslant m$ such that $E \cap X_{i_j} \neq \emptyset$ for each $j = 1, \dots, k$, and $E = \bigcup_{j=1}^{k} \left(E \cap X_{i_j}\right)$. We first prove that $E \cap X_{i_j} \neq \emptyset$ implies that $E \cap X_{i_j} = X_{i_j}$.

In fact, let $u = (x, n, y)$ be in $G_a^{(i_j)} \times_\alpha \mathbb{Z}$ such that $s(u) = y$ is in $E \cap X_{i_j}$. Then $\left(\varphi_{i_j}^n(x), y\right)$ is in $G_a^{(i_j)}$. Since y is in X_{i_j}, we have that x in X_{i_j} by Lemma 2.7. Set $u' = (x, mn, y)$ and notice the invariance of E. Then $u' \in G_a \times_\alpha \mathbb{Z}$ and $s(u') = y \in E$, hence $x \in E$. Thus we have prove that $r(u) = x$ is in $E \cap X_{i_j}$, which implies that $E \cap X_{i_j}$ is an invariant closed subset of the unit space $\left(G_a^{(i_j)} \times_\alpha \mathbb{Z}\right)^0$. Since φ_{i_j} is topologically mixing, $G_a^{(i_j)} \times_\alpha \mathbb{Z}$ is minimal, hence $E \cap X_{i_j} = X_{i_j}$. Thus $E = \bigcup_{j=1}^{k} X_{i_j}$.

We claim that $E = X$. Otherwise, from above two paragraphs, there exists some j $(1 \leqslant j \leqslant m)$ such that $X_{j-1} \cap E = \emptyset$ and $X_j \subseteq E$ (if $j = 1$, then let $X_0 = X_m$). Let $y \in X_{j-1}$, $x \in X_j$ such that $\varphi(y) = x$. Then $(y, 1, x) \in G_a \times_\alpha \mathbb{Z}$ and $s(y, 1, x) = x \in E$, but $r(y, 1, x) = y \notin E$, which contradicts with the invariance of E. Hence $E = X$. This completes the proof. ∎

LEMMA 2.11. *Suppose that (X, φ) is topologically transitive. Then $G_a \times_\alpha \mathbb{Z}$ is essentially principal.*

PROOF: Using the above lemma, we consider the invariant closed subset $F = (G_a \times_\alpha \mathbb{Z})^0 \equiv X$ of the unit space $(G_a \times_\alpha \mathbb{Z})^0$. Let $u = (x, 0, x) \in (G_a \times_\alpha \mathbb{Z})^0$, without loss in generality, let $x \in X_i$. Observe that $(G_a \times_\alpha \mathbb{Z})(u) \neq \{u\}$ if and only if there exists a nonzero integer k such that $\varphi_i^k(x)$ and x are asymptotically equivalent in (X_i, φ_i) if and only if $\left(G_a^{(i)} \times_\alpha \mathbb{Z}\right)(u) \neq \{u\}$. From Lemma 2.3 and the proof of Proposition 2.4,

$$\left\{u \in (G_a^{(i)} \times_\alpha \mathbb{Z})^0 \mid (G_a^{(i)} \times_\alpha \mathbb{Z})(u) \neq \{u\}\right\}$$

is countable, and so

$$\{u \in (G_a \times_\alpha \mathbb{Z})^0 \mid (G_a \times_\alpha \mathbb{Z})(u) \neq \{u\}\}$$

is countable, hence its compliment is dense in $(G_a \times_\alpha \mathbb{Z})^0$. Thus $G_a \times_\alpha \mathbb{Z}$ is essentially principal. ∎

By above arguments, we have proved that $G_a \times_\alpha \mathbb{Z}$ is a second countable, amenable, essentially principal, r-discrete minimal groupoid. Then by [**7**], we have:

THEOREM 2.12. *If (X, φ) is a topologically transitive Smale space, then the Ruelle algebra R^a is simple.*

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