Nicole's Contribution to the Foundations of the Calculus of Finite Differences.

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§ 1. While engaged in a study of the *Methodus Differentialis* of Jas. Stirling (1730) I have been struck by the fact that Nicole's * Papers on the same subject, printed in the Memoires de l'Academie Royale des Sciences (Paris), appear to form a fitting prelude to the work published by Stirling. The dates of Nicole's Papers are 1717, 1723, 1724, 1727, and it is almost certain that Stirling was well acquainted with their contents, for he remarks on page 24 of the *Methodus Differentialis* :—" Hac de re primus quod sciam egit D. Taylor in Methodo Incrementorum. Eadem etiam fusius et elegantissime traditur a D. *Nicol* in Actis Academiae Regiae Parisiensis."

As Nicole's Theorems are right in the line of progress, and are still of fundamental importance, it seems desirable that some account should be given of them, more especially as the introduction of the modern notation of the Calculus of Finite Differences effectively curtails and condenses his conclusions.

The demonstrations are frequently distinct from those given by Nicole, but in no case do they involve a principle not already used by him

It will appear that his work is almost entirely concerned with integral and rational functions to which the methods of Finite Differences may be applied with special facility. The note on the standard formulae of Interpolation has been added, though foreign to Nicole's memoirs, because it forms so elegant and useful an illustration of the application of Nicole's Functions.

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* Francois Nicole (1683–1758), described by Cantor as a pupil of Montmort, published at the age of nineteen "The Rectification of the Cissoid." In addition to his memoirs on finite differences, he also wrote several memoirs on geometry and probability.

§2. In the First Memoir, *Traité du Calcul des Differences Finies* (1717), Nicole proposes to expound certain ideas suggested to him by a perusal of the *Methodus Incrementorum* of Brook Taylor.

The Difference.

Consider a function, u_k, of the form

$$u_k = x(x+n)(x+2n)\ldots(x+\overline{p-1}n)$$

and write

$$u_{k+1} = (x+n)(x+2n)\ldots(x+pn),$$

found by substituting $x+n$ for x in u_k. Then the Difference

$$u_{k+1} - u_k = pn(x+n)(x+2n)\ldots(x+\overline{p-1}n). \quad \ldots\ldots\ldots(1)$$

(In modern notation

$$\Delta u_k = pn(x+n)\ldots(x+\overline{p-1}n), \text{ if } \Delta x = n).$$

Similarly, if

$$u_k = 1/x(x+n)\ldots(x+\overline{p-1}n)$$

and $$u_{k+1} = 1/(x+n)(x+2n)\ldots(x+pn),$$

then the Difference

$$u_k - u_{k+1} = pn/[x(x+n)\ldots(x+pn)]. \quad \ldots\ldots\ldots\ldots(2)$$

(In modern notation this would be

$$\Delta u_k = -pn/x(x+n)\ldots(x+pn),$$

but the notation introduced will not cause any confusion in what follows).

The Integral.

§3. The Integral is the inverse operation. Thus the Integral of the function $(x+n)(x+2n)\ldots(x+\overline{p-1}n)$ is equal to

$$x(x+n)\ldots(x+\overline{p-1}n)/pn, \quad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

because the Difference for the latter function is the former.

Similarly, the Integral of $1/x(x+n)\ldots(x+pn)$ is

$$1/pn\,x(x+n)\ldots(x+\overline{p-1}n). \quad \ldots\ldots\ldots\ldots\ldots\ldots(4)$$

Ex. 1.—To sum

$$F = 1.2 + 2.3 + \ldots + x(x+1). \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

If $F_k = 1.2 + \ldots + x(x+1)$

and $F_{k+1} = 1.2 + \ldots + x(x+1) + (x+1)(x+2)$

$F_{k+1} - F_k = (x+1)(x+2).$

Here $n=1$, $p=3$, and the Integral is $\frac{1}{3}x(x+1)(x+2)$.

Ex. 2.—To sum

$$S=1.4.7.10+4.7.10.13+\ldots+x(x+3)(x+6)(x+9). \quad \ldots\ldots(6)$$

Here $\Delta S=(x+3)(x+6)(x+9)(x+12)$, $n=3$, $p=5$.

Hence the Integral is

$$\tfrac{1}{15}x(x+3)\ldots(x+12).$$

This is not the correct sum. For when $x+3=1$, or $x=-2$, the Integral is

$$-\tfrac{1}{15}2.1.4.7.10,$$

whereas the sum should then be *zero*.

Hence the correct sum is

$$\tfrac{1}{15}x(x+3)\ldots(x+12)\;+\tfrac{1}{15}2.1.4.7.10. \quad \ldots\ldots\ldots\ldots\ldots\ldots(7)$$

Ex. 3.—To sum

$$\frac{1}{1.2}+\frac{1}{2.3}+\ldots+\frac{1}{x(x+1)}+\text{ etc.} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8)$$

Consider

$$u_k=\frac{1}{x(x+1)}+\frac{1}{(x+1)(x+2)}+\ldots ad\ \infty.$$

and

$$u_{k+1}=\frac{1}{(x+1)(x+2)}+\ldots ad\ \infty$$

$$\therefore\ u_k-u_{k+1}=\frac{1}{x(x+1)}.$$

Hence the corresponding Integral is $1/x$.

In particular, by putting $x=1$, we obtain Brouncker's result

$$1=\frac{1}{1.2}+\frac{1}{2.3}+\ldots ad\ \infty. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(9)$$

(Of course it follows that

$$\frac{1}{1.2}+\frac{1}{2.3}+\ldots+\frac{1}{x(x+1)}=1-\frac{1}{(x+1)}\ ).$$

The same example is taken by Stirling (Meth. Diff., page 23).

Ex. 4.—To sum

$$\frac{1}{1.3.5.7}+\frac{1}{3.5.7.9}+\ldots \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(10)$$

Denote the general term by

$$1/x(x+2)(x+4)(x+6).$$

If $F_k-F_{k+1}=1/x(x+2)(x+4)(x+6)$, we find $F_k=\tfrac{1}{6}/x(x+2)(x+4)$ as the Integral. To sum to infinity from the initial term, put $x=1\ \therefore$ sum $=\tfrac{1}{90}$.

Ex. 5.—To sum

$$\frac{4}{1.4.7.10.13.16}+\frac{49}{4.7\ldots19}+\frac{225}{7\ldots22}+\ldots. \quad \ldots\ldots\ldots\ldots\ldots(11)$$

Nicole's method is very instructive. He takes the general term to be

$$\tfrac{1}{36}(x+2)^2(x+3)^2/x(x+3)\ldots(x+15).$$

He writes the numerator of this fraction in the form

$$\tfrac{1}{36}\{A_0 + A_1 x + A_2 x(x+3) + A_3 x(x+3)(x+6) + A_4 x(x+3)(x+6)(x+9)\},$$

in which $A_0 \dots A_4$ are easily determined.

The general term may then be written

$$\tfrac{1}{36}\left\{\frac{A_0}{x(x+3)\dots(x+15)} + \dots + \frac{A_4}{(x+12)(x+15)}\right\}.$$

Hence the Integral is

$$\tfrac{1}{36}\left\{\frac{A_0}{15x(x+3)\dots(x+12)} \dots + \frac{A_4}{3(x+12)}\right\}. \quad \dots\dots\dots\dots(12)$$

To sum to infinity from the initial term put $x=1$ in (12).

The interest of this example lies in the fact that he represents an integral function of x, not as a sum of integral powers of x, but as a sum of factorials.

Such representations were freely used by Stirling to good purpose in the *Methodus Differentialis*.

The development of the principle here employed furnishes the material for Part II. of Nicole's *Traité* (1723).

PART II.—*Section* I.

§ 4. Being given

$$x(x+n)\dots(x+\overline{k-1}n),$$

to calculate

$$(x+m)(x+m+n)\dots(x+m+\overline{k-1}n) - x(x+n)\dots(x+\overline{k-1}n)$$

in the form

$$a_0 + a_1(x+n) + a_2(x+n)(x+2n) + \dots + a_{k-1}(x+n)\dots(x+\overline{k-1}n).$$

To attain his object Nicole employs a long inductive process. The desired result may, however, be found directly with the aid of the modern notation of finite differences and the application of Nicole's earlier calculations.

Assume the obviously possible identity

$$(x+m)(x+m+n)\dots(x+m+\overline{k-1}n)$$
$$= A_0 + A_1(x+n) + A_2(x+n)(x+2n) + \dots A_k(x+n)\dots(x+kn) \quad (13)$$

and calculate $A_0, A_1, \dots A_k$.

Put $x=-n$ in the identity (13).

$$\therefore \quad A_0 = (m-n)m(m+n)\dots(m+\overline{k-2}n).$$

Equate to each other the r^{th} differences of the two sides of the identity for $\Delta x = n$.

On the left side we obtain

$$n^r k(k-1)\dots(k-r+1)(x+m+rn)\dots(x+m+\overline{k-1}n);$$

and on the right

$$n^r r!\,A_r+(x+\overline{r+1}n)\,\phi(x)$$

where $\phi(x)$ is an integral function of (x).

Write $x+\overline{r+1}n=0$, *i.e.* $x=-(r+1)n$.

$$\therefore\ n^r r!\,A_r=n^r k(k-1)\dots(k-r+1)\times(m-n)\,m\,(m+n)\dots$$
$$\dots(m+\overline{k-r-2}n).$$

Hence $A_r={}_kC_r(m-n)\,m\,(m+n)\dots(m+\overline{k-r-2}n)$. .... (14)

Finally,
$$A_{k-2}={}_kC_2(m-n)\,m$$
$$A_{k-1}={}_kC_1(m-n)$$
$$A_k\ \ =1.$$

We thus find

$$(x+m)(x+m+n)\dots(x+m+\overline{k-1}n)$$
$$=(m-n)(m)(m+n)\dots(m+\overline{k-2}n)$$
$$+\sum_{r=1}^{r=k-1}{}_kC_r(m-n)(m)\dots(m+\overline{k-r-2}n)(x+n)(x+2n)\dots(x+rn)$$
$$+(x+n)(x+2n)+\dots(x+kn). \qquad (15)$$

Also
$$x(x+n)\dots(x+\overline{k-1}n)$$
$$=(x+n)(x+2n)\dots(x+kn)$$
$$-kn(x+n)(x+2n)\dots(x+\overline{k-1}n):$$

so that the difference may then be obtained in the desired form :—

$$(m-n)\,m\,(m+n)\dots+(m+\overline{k-2}n)$$
$$+\sum_{r=1}^{r=k-2}{}_kC_r(m-n)\dots(m+\overline{k-r-2}n)(x+n)\dots(x+rn)$$
$$+km(x+n)\dots(x+\overline{k-1}n). \qquad (16)$$

Cor. 1.—If $m=n$, there is a reduction to the first case considered.

Cor. 2.—If $n=0$, we find

$$(x+m)^k-x^k=\sum_{r=0}^{r=k-1}{}_kC_r\,m^{k-r}x^r. \qquad (17)$$

[*Cor. 3.* $(x+m)(x+m+n)\dots(x+m+\overline{k-1}n)$
$$=m(m+n)\dots(m+\overline{k-1}n)$$
$$+\sum_{r=1}^{r=k-1}{}_kC_r\,m(m+n)\dots(m+\overline{k-r-1})\times x(x+n)\dots(x+\overline{r-1}n)$$
$$+x(x+n)\dots(x+\overline{k-1}n)\,]. \qquad (18)$$

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Applications.

Ex. 1.—To find the integral corresponding to, say,

$$A+B(x+n)+C(x+n)(x+2n)+D(x+n)(x+2n)(x+3n)$$

when $\Delta x=m$, Nicole uses the method of indeterminate coefficients.

He assumes the integral to be of the form

$$a_1 x+a_2 x(x+n)+\ldots+a_4 x(x+n)(x+2n)(x+3n),$$

of which the difference for $\Delta x=m$ is then formed according to the rule given by (16). Corresponding coefficients equated to $A, B, \ldots$ furnish in succession the values of a_1, etc.

Ex. 2.—To sum $\quad 1.4.7.10+5.8.11.14+\ldots.$ .... (19)

Here $n=3$, $m=4$; and we may write the general term as

$$(x+n)(x+2n)(x+3n)(x+4n).$$

The integral is therefore of the form

$$F(x)=A_1 x+A_2 x(x+n)+\ldots A_5 x\ldots(x+4n).$$

The constants A are determined as in the preceding example.

To get the correct sum, add a constant C such that when $x+3=1$ we have $F(-2)+C=0$. For then the sum should be zero, since the difference reduces to the first term.

Thus the sum of three terms is $F(10)-F(-2)$.

§5. Memoir, Section II. (1723), and Memoir (1724) entitled *Addition aux deux Mémoires sur le Calcul des Differences Finies, imprimés l'année dernière*, deal with Inverse Factorial Series, and may be taken together.

In Section II. Nicole establishes the relation

$$\frac{1}{x(x+n)\ldots(x+kn-n)}-\frac{1}{(x+m)(x+m+n)\ldots.(x+m+kn-n)}$$
$$=km/(x+m-n)(x+m)\ldots(x+m+kn-n)$$
$$+\frac{k(k+1)}{1.2}m(m-n)/(x+m-2n)\ldots(x+m+kn-n)$$
$$+\frac{k(k+1)(k+2)}{1.2.3}m(m-n)(m-2n)/(x+m-3n)\ldots$$
$$+\text{ etc. }+{}_kH_r\, m(m-n)\ldots(n)/x(x+n) \qquad (20)$$

He assumes that m is a positive integral multiple of n, $m=rn$, so that the series contains only r terms.

Cor. 1.—If $m=n$ there is a reduction to (2).

Cor. 2.—By putting $n=0$ he obtains an infinity of terms, and writes the result

$$\frac{1}{x^k}-\frac{1}{(x+m)^k}={}_kH_1\frac{m}{(x+m)^{k+1}}+{}_kH_2\frac{m^2}{(x+m)^{k+2}}+\text{ etc. } \quad \ldots\text{(21)}$$

As a special case,

$$\frac{a}{(1-a)^k}=\frac{a}{(1-a)^k}-\frac{a}{(1-a+a)^k}+a$$

$$=a+\sum_{r=1}^{r=\infty}{}_kH_r\,a^{r+1}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(22)}$$

and he adds an independent proof of this result.

The memoir of 1724 furnishes another demonstration of the theorem of Section II., in which the restriction that r shall be an integer is removed, so that the series may have an infinity of terms. It is preceded by another expression for the difference, viz.:

$$\frac{1}{x(x+n)\ldots(x+kn-n)}-\frac{1}{(x+m)\ldots(x+m+kn-n)}$$

$$=\frac{k}{1}\cdot m/x(x+n)\ldots(x+kn)$$

$$-\frac{k(k+1)}{1\,.\,2}m(m-n)/x(x+n)\ldots(x+kn+n)$$

$$+\frac{k(k+1)(k+2)}{1\,.\,2\,.\,3}m(m-n)(m-2n)/x(x+n)\ \ldots\ (x+kn+2n)$$

$$-\text{ etc. } \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(23)}$$

Demonstration.

§ 6. Write, as Nicole does,

$$\frac{1}{x}-\frac{1}{x+m}=\frac{m}{x(x+m)},$$

$$=\frac{m(x+m-n)}{x(x+m)(x+m-n)},$$

$$=\frac{m}{(x+m-n)(x+m)}+\frac{m(m-n)}{(x+m-n)(x+m)x},$$

$$i.e. \quad +\frac{m(m-n)(x+m-2n)}{(x+m-2n)(x+m-n)(x+m)x}.$$

$$\therefore \quad \frac{1}{x} - \frac{1}{x+m}$$

$$= \frac{m}{(x+m-n)(x+m)} + \frac{m(m-n)}{(x+m-2n)(x+m-n)(x+m)}$$

$$+ m(m-n)(m-2n)/(x+m-3n)\ldots(x+m)$$

$$+ \text{ etc. } \quad \ldots\ldots\ldots (24)$$

The series terminating if m/n is a positive integer.

To the two sides of this identity apply the formula (2), and take the $(k-1)^{\text{th}}$ difference on each side, putting $\Delta x = n$. Divide on both sides by $(k-1)!\, n^{k-1}$, when we find

$$\frac{1}{x(x+n)\ldots(x+\overline{k-1}n)} - \frac{1}{(x+m)\ldots\ldots(x+m+\overline{k-1}n)}$$

$$= km/(x+m-n)(x+m)\ldots(x+m+kn-n)$$

$$+ \frac{k(k+1)}{1\,.\,2} m(m-n)/(x+m-2n)\ldots(x+m+kn-n) + \text{ etc.} \quad (25)$$

The process of taking the Differences seems so simple that one wonders that Nicole does not use it.

To satisfy modern requirements the interesting question of convergence should be discussed, but this we omit just as Nicole does.

§7. Again,

$$\frac{1}{x} - \frac{1}{x+m} = \frac{m}{x(x+m)}$$

$$= m(x+n+m-m)/x(x+n)(x+m)$$

$$= \frac{m}{x(x+n)} + \frac{m(n-m)}{x(x+n)(x+m)}$$

$$= \frac{m}{x(x+n)} + \frac{m(n-m)}{x(x+n)(x+2n)} + \frac{m(n-m)(2n-m)}{x(\ldots)(x+m)}$$

$$= \text{ etc. } \quad \ldots\ldots\ldots (26)$$

Apply to each side of this equation the formula (2), and take the $(k-1)^{\text{th}}$ difference for $\Delta x = n$; divide throughout by $n^{k-1}(k-1)!$ when (23) follows readily after a simple transformation.

[By writing in it $-m$ for m, and taking $n=1$, we get the formula

$$\frac{1}{(x-m)(x-m+1)\ldots(x-m+k-1)}=\frac{1}{x(x+1)\ldots(x+k-1)}$$

$$+\frac{km}{x(x+1)\ldots(x+k)}+{}_kH_2\,\frac{m(m+1)}{x(x+1)\ldots(x+k+1)}$$

$$+{}_kH_3\,\frac{m(m+1)(m+2)}{x(x+1)\ldots(x+k+2)}+\text{ etc.}]$$

Memoir, 1727.

§ 8. This, the last memoir of the set, takes up the summation of a variety of series as deduced from the chain of identities

$$\begin{aligned}\frac{1}{a-b}&=\frac{1}{a}+\frac{b}{a(a-b)}\\&=\frac{1}{a}+\frac{b}{a(a+c)}+\frac{b(b+c)}{a(a+c)(a-b)}\\&=\frac{1}{a}+\frac{b}{a(a+c)}+\frac{b(b+c)}{a(a+c)(a+d)}\\&\qquad+\frac{b(b+c)(b+d)}{a(a+c)(a+d)(a-b)}\\&=\text{etc.}\end{aligned}$$

Of this a particular case is

$$\frac{1}{a-b}=\frac{1}{a}+\frac{b}{a(a+1)}+\frac{b(b+1)}{a(a+1)(a+2)}+\text{ etc.,}\quad\ldots\ldots(27)$$

sometimes called Stirling's Series,* which is, of course, fundamental in §§ 6–7.

As I understand that this memoir has had attention drawn to it by Eneström (*Comptes Rendus* 103 (1886), pp. 523–5), it need not here be further discussed.

* Meth. Diff., p. 12.

Application of Nicole's Factorials to the Standard Formulae of Interpolation for a Function $f(x)$.

§ 9. Let $f(x)$ be an integral function of x of degree n. It may be represented in a great variety of ways in the form

$$f(x) = A_0 + A_1(x-\alpha_1) + A_2(x-\alpha_1)(x-\alpha_2) + \dots$$
$$\dots + A_n(x-\alpha_1)(x-\alpha_2)\dots(x-\alpha_n),$$

in which the coefficients may be found by division in any particular case.

For certain values of $\alpha_1, \dots, \alpha_n$ the expansion may be found very simply by Nicole's Functions—certainly in all cases corresponding to the standard formulae of interpolation, in which $\alpha_1 - \alpha_2 = \alpha_2 - \alpha_3 =$ etc.

The principal of these are the following :—

(i) $$A_0 + A_1 x + A_2 \frac{x(x-1)}{2!} + \frac{A_3}{3!} x(x-1)(x-2) + \text{ etc.}$$

(ii) $$A_0 + A_1 x + A_2 \frac{x(x+1)}{2!} + \frac{A_3}{3!} x(x+1)(x+2) + \text{ etc.}$$

(Newton).

(iii) $$C_0 + C_1 x + C_2 x^2 + (C_3 x + C_4 x^2)(x^2 - 1^2)$$
$$+ (C_5 x + C_6 x^2)(x^2 - 1^2)(x^2 - 2^2) + \text{etc.}$$

(Stirling I.).

(iv) $$C_0 + C_1 x + (C_2 + C_3 x)(x^2 - 1^2) + (C_4 + C_5 x)(x^2 - 1^2)(x^2 - 3^2)$$
$$+ (C_6 + C_7 x)(x^2 - 1^2)(x^2 - 3^2)(x^2 - 5^2) + \text{ etc.}$$

or

$$A_0 + A_1(x+1) + \left\{\frac{A_2}{2!} + \frac{A_3}{3!}(x+3)\right\}(x+1)(x-1)$$
$$+ \left\{\frac{A_4}{4!} + \frac{A_5}{5!}(x+5)\right\}(x+3)(x+1)(x-1)(x-3) + \text{ etc.}$$

(Stirling II.).

(v) $$A_0 + A_1 x + \frac{A_2}{2!} x(x-1) + \frac{A_3}{3!}(x+1)x(x-1) + \dots$$
$$\dots + A_{2r+1}\ {}_{x+r}C_{2r+1} + A_{2r+2}\ {}_{x+r}C_{2r+2} + \dots$$

(Gauss (*Encke*) I.).

(vi) $$A_0 + A_1 x + A_2\ {}_{x+1}C_2 + A_3\ {}_{x+1}C_3 + \dots$$
$$\dots + A_{2r}\ {}_{x+r}C_{2r} + A_{2r+1}\ {}_{x+r}C_{2r+1} + \dots$$

(Gauss (*Encke*) II.).

Now, on noting that $\alpha x+\beta x^2=x(\alpha+\beta n+\beta\overline{x-n})$, we may write (iii) in the form

$$A_0+A_1 x+\frac{A_2}{2\,!}x(x-1)+\left\{\frac{A_3}{3\,!}+\frac{A_4}{4\,!}(x-2)\right\}(x+1)x(x-1)+\ldots$$

$$\ldots+\left\{\frac{A_{2r+1}}{(2r+1)!}+\frac{A_{2r+2}}{(2r+2)!}(x-r-1)\right\}(x+r)\ldots(x-r)+\text{ etc.}$$

i.e. in a form equivalent to (v).

Likewise, since $\alpha x+\beta x^2=x(\alpha-\beta n+\beta\overline{x+n})$, it follows that (iii) may be written as in (vi). Thus, from our point of view, the Gaussian formulae differ only in a trifling detail from the older formula (iii) of Stirling.*

If these expressions are examined (with the alteration in (iii)), it will appear that they have the common property of consisting of a sum of Nicole's Functions, and that any factors in one such function are reproduced in all the following functions. It will then be seen that the coefficients may be easily expressed by the notation of Finite Differences.

§10. For example, take Gauss's Formula (v).

In $f(x)=A_0+A_1\,{}_xC_1+A_2\,{}_xC_2+$ etc. ....(28)

Put $x=0$. $\therefore\ A_0=f(0)$.

Take the first difference on each side of (28) for $\Delta x=1$.

$$\therefore\ \Delta f(x)=A_1+x\,\phi(x), \qquad \text{......................(29)}$$

where $\phi(x)$ is an integral function of x. Put $x=0$.

$$\therefore\ A_1=\Delta f(0). \qquad \text{.........................(30)}$$

Similarly,

$$\Delta^{2r}f(x)=A_{2r}+(x+r)\,\phi(x),$$

$$\therefore\ A_{2r}=\Delta^{2r}f(-r) \qquad \text{.........................(31)}$$

and

$$\Delta^{2r+1}f(x)=A_{2r+1}+(x+r)\,\phi(x),$$

$$\therefore\ A_{2r+1}=\Delta^{2r+1}f(-r). \qquad \text{.....................(32)}$$

In the case of No. (iv) we have to take $\Delta x=2$. We find

$$\Delta^{2r}f(x)=2^{2r}A_{2r}+(x+2r+1)\,\phi(x),$$

$$\therefore\ A_{2r}=\Delta^{2r}f(-2r-1)/2^{2r}, \qquad \text{.................(33)}$$

* Really due to Newton, see §14.

and

$$\Delta^{2r+1} f(x) = 2^{2r+1} A_{2r+1} + (x + 2r + 1)\, \phi(x),$$

$$\therefore\ A_{2r+1} = \frac{1}{2^{2r+1}}\ \Delta^{2r+1} f(-2r-1). \quad \ldots\ldots\ldots\ldots\ldots(34)$$

It is needless to do more than tabulate the results to be thus obtained:—

(i)′ $f(x) = f(0) + \sum_{r=1}^{r=n} \Delta^r f(0)\, {}_xC_r$.

(ii)′ $f(x) = f(0) + \sum_{r=1}^{r=n} \Delta^r f(-r)\, {}_xH_r$.

(iii)′ $f(x) = f(0) + \frac{1}{2\,!}[\, x\,\{\Delta f(0) + \Delta f(-1)\} + x^2\, \Delta^2 f(-1)\,]$

$$+ \sum_{r=1} \frac{1}{(2r+2)\,!}\,[(r+1)\, x\,\{\Delta^{2r+1} f(-r) + \Delta^{2r+1} f(-r-1)\}$$
$$+ x^2\, \Delta^{2r+2} f(-r-1)]\, (x^2 - 1^2)\,(x^2 - 2^2)\ldots(x^2 - r^2).$$

N.B.—For $\Delta^{2r+2} f(-r-1)$ may be written
$\Delta^{2r+1} f(-r) - \Delta^{2r+1} f(-r-1)$.

(iv)′ $f(x) = \frac{1}{2}\,\{f(0) + f(-1) + x\Delta f(-1)\}$

$$+ \sum_{r=1} \frac{1}{2^{2r+1}\,(2r+1)\,!}\,[(2r+1)\{\Delta^{2r} f(-2r-1) + \Delta^{2r} f(-2r+1)\}$$
$$+ x\Delta^{2r+1} f(-2r-1)]\,(x^2 - 1^2)\,(x^2 - 3^2)\ldots(x^2 - \overline{2r-1}^2).$$

N.B.—For $\Delta^{2r+1} f(-2r-1)$ may be written
$\Delta^{2r} f(-2r+1) - \Delta^{2r} f(-2r-1)$.

(v)′ $f(x) = f(0) + \Delta f(0)\, x + \Delta^2 f(-1)\, {}_xC_2 + \Delta^3 f(-1)\, {}_{x+1}C_3$
$+ \ldots + \Delta^{2r} f(-r)\, {}_{x+r-1}C_{2r} + \Delta^{2r+1} f(-r)\, {}_{x+r}C_{2r+1} +$ etc.

(vi)′ $f(x) = f(0) + \Delta f(-1)\, x + \Delta^2 f(-1)\, {}_{x+1}C_2 + \Delta^3 f(-2)\, {}_{x+1}C_3$
$+ \ldots + \Delta^{2r} f(-r)\, {}_{x+r}C_{2r} + \Delta^{2r+1} f(-r-1)\, {}_{x+r}C_{2r+1} +$ etc.

Since $\Delta^r f(x)$ is a linear homogeneous function of $f(x)$, $f(x+h), \ldots f(x+rh)$ for $\Delta x = h$, it is not difficult to see that $f(x)$ is supposed known for

$x = 0,\ 1,\ 2,\ \ldots\ n$	in (i)′
$x = 0,\ -1,\ -2,\ \ldots\ -n$	in (ii)′
$x = 0,\ -1,\ +1,\ -2,\ +2,\ \ldots$	in (iii)′, (v)′ and (vi)′
$x = -1,\ +1,\ -3,\ +3,\ \ldots$	in (iv)′.

Though the fundamental values chosen are not the same, the graphs of the corresponding functions are all identical; for the integral function of degree n is uniquely determined by assigning its values for $n+1$ values of its argument.

§ 11. It will now be obvious how to make up a great variety of similar formulae in which the coefficients may be simply expressed by the notation of Finite Differences.

e.g.

$$
\begin{aligned}
&A_0 + A_1 x + A_2 x(x-1)/2! + \ldots + A_{r+1}\, x(x-1)\ldots(x-r)/(r+1)! \\
&\quad + A_{r+2}(x+1)(x)\ldots(x-r)/(r+2)! \\
&\quad + A_{r+3}(x+2)(x+1)(x)(x-1)\ldots(x-r)/(r+3)! \\
&\quad + \text{etc.} \qquad\qquad (35)
\end{aligned}
$$

or

$$
\begin{aligned}
&A_0 + A_1 x + A_2 x(x+1)/2! \\
&\quad + x(x^2-1^2)\left[\frac{A_3}{3!} + \frac{x+2}{4!}A_4\right] + \ldots \\
&\quad + x(x^2-1^2)\ldots(x^2-r^2)\left[\frac{A_{2r+1}}{(2r+1)!} + \frac{x+r+1}{(2r+2)!}A_{2r+2}\right]
\end{aligned}
$$

+ terms in which the factors $x+r+2$, $x+r+3$, etc., are introduced in succession. .... (36)

The rule of formation is as follows :—

Consider the equidistant values of x

$$\alpha_{-3},\ \alpha_{-2},\ \alpha_{-1},\ \alpha,\ \alpha_1,\ \alpha_2,\ \alpha_3 \ldots$$

where $$\ldots \alpha - \alpha_{-1} = \alpha_1 - \alpha = \alpha_2 - \alpha_1, \text{ etc.}$$

Form the chain of factors

$$\ldots x-\alpha_{-1},\ x-\alpha,\ x-\alpha_1,\ x-\alpha_2, \text{ etc.}$$

Start with $x-\alpha$; then take either of the adjoining factors, say $x-\alpha_1$, then either $x-\alpha_2$ or $x-\alpha_{-1}$ adjoining, and use them in this order to form the succession of Nicole's Functions corresponding. There seems no reason why such a representation might not be of practical value in particular cases.

The Complementary Function.

§ 12. When $f(x)$ is not an integral function of x, if $F(x)$ is the integral function formed by the above process, then the identity

$$f(x) = F(x)$$

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is impossible in general, though true for at least the $n+1$ values of x chosen in succession. Let them be $\alpha_1, \alpha_2, \ldots \alpha_{n+1}$.

Then $$f(x) = F(x) + \phi(x)\,\Pi\,(x-\alpha_1)$$
would equally well represent the function, provided $\phi(x)$ is finite for $x = \alpha_1, \ldots, \alpha_{n+1}$.

Suppose the value of $f(x)$ is wanted for $x=\beta$, and that it satisfies the equation
$$f(\beta) = F(\beta) + R\,\Pi\,(\beta - \alpha)$$
where R is a constant.

Then $f(x) - F(x) - R\,\Pi\,(x-\alpha) = 0$ for the $n+2$ values $\alpha_1 \ldots \alpha_{n+1}, \beta$.

Hence, by Rolle's Theorem, for some intermediate value ξ of x we must have
$$f^{(n+1)}(\xi) - R\,(n+1)\,! = 0.$$
$$\therefore \quad R = f^{(n+1)}(\xi)/(n+1)\,!$$

Hence $$f(x) = F(x) + \Pi\,(x-\alpha)\,f^{(n+1)}(\xi)/(n+1)\,!$$
is also true for $x = \beta$.

The function $\Pi\,(x-\alpha) f^{(n+1)}(\xi)/(n+1)\,!$ is called the Complementary Function. The maximum error made by assuming $f(x) = F(x)$ will depend on the *maximum* value of
$$\Pi\,(x-\alpha) f^{(n+1)}(\xi)/(n+1)\,!$$
in the interval $\alpha_1 \ldots \alpha_{n+1}, \beta$.

The actual error may be much less.

If, as sometimes happens, the Complementary Function is negligible in the interval, we may for all practical purposes assume $f(x) = F(x)$. Otherwise this assumption may be quite illusory. As serious a source of error may arise from the fact that the values of $f(x)$ are not absolutely accurate as is assumed in the theory, but merely approximations.

§ 13. Some further examples of expansions by means of Nicole's Functions.

(1) If $f(x)$ is an integral function of x, to represent $f(x)$ in the form
$$A_0 + A_1 x + A_2 x^2 + (A_3 x + A_4 x^2)(x^2+1^2)$$
$$+ (A_5 x + A_6 x^2)(x^2+1^2)(x^2+2^2) + \text{ etc.}$$
or
$$A_0 + \left\{\frac{A_1}{1\,!} + \frac{A_2}{2\,!}(x-i)\right\} x + \sum_{r=1}\left\{\frac{A_{2r+1}}{(2r+1)\,!} + \frac{A_{2r+2}}{(2r+2)\,!}(x - ri - i)\right\}$$
$$\times (x^2+r^2)(\ldots)(x^2+2^2)(x^2+1^2)\,x.$$

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The sequence of values of x is

$$\ldots\ -3i,\ -2i,\ -i,\ 0,\ i,\ 2i,\ 3i,\ \ldots,$$

and we take $\Delta x = i$.

Hence

$$\begin{aligned} A_0 &= f(0) \\ iA_1 &= \Delta f(0) \\ i^2 A_2 &= \Delta^2 f(-i) \\ &\cdots\cdots\cdots\cdots \\ i^{2r+1} A_{2r+1} &= \Delta^{2r+1} f(-ri) \\ i^{2r+2} A_{2r+2} &= \Delta^{2r+3} f(-ri-i), \end{aligned}$$

and we find, after some reductions,

$$f(x) = f(0) + \sum_{r=0} \frac{(-1)^{r+1}}{(2r+2)!} \{(r+1)\, i\, [\Delta^{2r+1} f(-ri) + \Delta^{2r+1} f(-ri-i)] \\ + x\Delta^{2r+2} f(-ri-i)\}\, (x^2+r^2)\ldots(x^2+1^2)\, x.$$

It is not difficult to verify directly that

$$\Delta^{2r+1} f(-ri) + \Delta^{2r+1} f(-ri-i)$$

is purely imaginary, and $\Delta^{2r+2} f(-ri-i)$ is real, when $f(x)$ has real coefficients.

(2) To represent $f(x)$ in the form

$$A_0 + A_1 x + (A_2 + A_3 x)(x^2+1^2) + (A_4 + A_5 x)(x^2+1^2)(x^2+2^2) + \text{ etc.},$$

obtain $xf(x)$ as in the preceding example and divide out the factor x.

(3) $$f(x) = A_0 + A_1 x + (A_2 + A_3 x)(x^2+1^2) \\ + (A_4 + A_5 x)(x^2+1^2)(x^2+3^3) + \text{ etc.}$$

or $$= A_0 + A_1 x + A_2 x^2 + (A_3 x + A_4 x^2)(\overline{x-\alpha}^2 + \beta^2) \\ + (A_5 x + A_6 x^2)(\overline{x-\alpha}^2 + 4\beta^2)(\overline{x-\alpha}^2 + \beta^2) + \text{ etc.}$$

§ 14. Newton's Interpolation Formula (i), § 9, first occurs, I think, in the Principia Book III., Lemma V. The rest of this Lemma states the more general formula of parabolic interpolation for unequal increments of the variable. Professor Whittaker has drawn my attention to the fact that the so-called Stirling Formulae (iii) and (iv) were given by Newton in the short tractate *Methodus Differentialis* (1711) many years before Stirling used them (Phil. Trans., 1719, or Treatise, 1730) Newton there gives them as Casus (i) and Casus (ii) of Prop. III.

He also attaches two Casus to his Prop. IV., which, for equal increments, furnish very simple examples of my statement in § 11.

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Case (i).—Suppose $f(x)$, of even degree, given for the $2m+1$ values

$$\beta_m, \ldots, \beta_2, \beta_1, \alpha, \alpha_1, \ldots, \alpha_m.$$

Take these in the order $\alpha, \alpha_1, \beta_1, \alpha_2, \beta_2$, etc., and also in the order $\alpha, \beta_1, \alpha_1, \beta_2, \alpha_2$, etc., to furnish

$$f(x) = A_0 + A_1(x-a) + A_2(x-a)(x-a_1) + A_3(x-a)(x-a_1)(x-\beta_1) + \text{etc.}$$

and

$$f(x) = B_0 + B_1(x-a) + B_2(x-a)(x-\beta_1) + B_3(x-a)(x-\beta_1)(x-a_1) + \text{etc.}$$

Then it is easily found that

$$A_0 = B_0\,;\ A_2 = B_2\,;\ A_4 = B_4\,;\ \text{etc.},$$

so that on adding and dividing by 2 we find $f(x)$ in the form

$$f(x) = C_0 + C_1(x-a) + C_2(x-a)\left(x - \frac{a_1+\beta_1}{2}\right)$$
$$+ C_3(x-a)(x-a_1)(x-\beta_1) + C_4(x-a)(x-a_1)(x-\beta_1)\left(x - \frac{a_2+\beta_2}{2}\right) + \text{etc.}$$

Similarly, when $2m+2$ values are given

$$\beta_m \ldots \beta_1\, \beta,\ a\, a_1 \ldots a_m,$$

take these in the order

$$\alpha, \beta, a_1, \beta_1, a_2, \beta_2, \text{ etc.},$$

and then in the order

$$\beta, \alpha, \beta_1, \alpha_1, \beta_2, \alpha_2, \text{ etc.},$$

to obtain $f(x)$ in the forms

$$A_0 + A_1(x-a) + A_2(x-a)(x-\beta) + \text{ etc.}$$

and $$B_0 + B_1(x-\beta) + B_2(x-\beta)(x-a) + \text{ etc.}$$

Then $$A_1 = B_1,\ A_3 = B_3,\ \text{etc.},$$

and by adding we find

$$f(x) = C_0 + C_1\left(x - \frac{a+\beta}{2}\right) + C_2(x-a)(x-\beta) + \text{ etc.}$$

The expressions given by Newton have the same form, but for unequal increments. It is a simple matter of elementary algebra to show *à priori* that such expressions are possible. Nothing therefore seems to have been added to this form of interpolation since Newton's time, and Stirling's remark * in his Treatise, p. 107, applies to this day.

Addendum.

From the foregoing it seems natural to infer that Nicole was the first to introduce the Inverse Factorial Series. His first Memoir bears the date 30th January 1717. In the Philosophical Transactions for the months of July, August, and September 1717 (No. 353), there was published a Memoir entitled *De Seriebus Infinitis Tractatus. Pars Prima. Auctore Petro Remundo de Monmort, R.S.S. Una cum Appendice et Additamento* per D. Brook Taylor, R.S. Sec.

From this Memoir it is clear that both Montmort and Brook Taylor had at the same time been busy with similar ideas. The opening theorems are identical with those of §§ 2, 3 in Nicole's Memoir, save that the difference of $1/x\,(x+n)\ldots$ is taken with the negative sign as in modern notation. With respect to the summation of $\Sigma x\,(x+n)\,\ldots\,(x+\overline{p-1}n)$, Montmort remarks in Scholium I. to Prop. I. :—"*In hac propositione continetur particula quaedam Methodi Incrementorum de qua ante biennium librum edidit* D. Brook Taylor, Soc. Reg. Lond. Secr. *Librum ipsum adeat qui de ea methodo plura scire velit.*" Montmort had therefore received his inspiration from the same source as Nicole.

In Prop. II. he shews how to find $\Sigma\,\phi\,(x)/(x+n)\ldots(x+\overline{p-1}n)$ in the same way as Nicole, viz., by writing $\phi\,(x)$ in the form $A_0 + A_1\,x + A_2\,x\,(x+n) + \ldots$.

In Prop. V. he shows how to sum $\Sigma 1/(x+a)\,(x+b)\,(x+c)\ldots$, where a, b, $c\ldots$ are positive integers, by inserting the product $(x+a+1)\,(x+a+2)\ldots(x+b-1)\,(x+b+1)\ldots$, etc., and then proceeding as in Prop. II.

Scholium 3, attached to this proposition, gives the expansion of $1/(x+\alpha)$ in the form $A/x + B/x\,(x+1)$ + etc., commonly described as Stirling's Series.

*** "De descriptione Curvae Parabolici generis per data quotcunque puncta egerunt plures celebres Geometrae post *Newtonum*. Sed omnes eorum solutiones eaedem sunt cum hisce jam exhibitis ; quae quidem a *Newtonianis* vix discrepant, etc."**

It may also be noted that his Lemmas (1) and (2) are the Newtonian Formulae of Interpolation (i) and (ii), though he refers for proof to his own *Essai d'Analyse.*

Brook Taylor in his *Appendix* shows in a masterly manner how to deduce by his method of Increments the conclusions obtained by Montmort.*

In the summation of $\overset{\infty}{\Sigma}\, \phi(x)/(x+a)(x+b)\ldots$ he points out that the degree of $\phi(x)$ must be less by 2 than that of the denominator. He then represents the fraction $\phi(x)/(x+a)\ldots$ as a sum of partial fractions

$$A/(x+a)+B/(x+b)+\ldots$$

where

$$A+B+\ldots\equiv 0.$$

* Pierre Remond de Montmort, born at Paris in 1678, and left with a sufficient patrimony, devoted himself to the study of mathematics. In 1715, accompanied by the Abbé Conti, he visited London, and made the acquaintance of Newton and other scholars. During his stay in London he had more particularly the help of his compatriot Demoivre, who writes in his *Miscellanea Analytica,* p. 149:—"Habuit me comitem, interpretem, ductorem; apud Newtonum aliosque doctos viros admissus est, urbaniter ab iis exceptus, tandemque Societati Regiae annumeratus." His *Essai d'Analyse sur les jeux de Hazard* appeared in 1708 and in 1713. He died in 1719.