16

Unpolarized lepton-hadron scattering

16.1 Moment sum rules

We shall consider the previous lepton-hadron unpolarized process studied in Section 15.3 governed by the T-product of two electromagnetic currents. The general Lorentz decomposition of the hadronic tensor has the form:

$$
\begin{aligned}
J_\mu(x)J_\nu(0) = &(\partial_\mu \partial'_\nu - g_{\mu\nu})\mathcal{O}_L(x) \\
&+ (g_{\mu\lambda}\partial_\rho \partial'_\nu + g_{\rho\nu}\partial_\mu \partial'_\lambda - g_{\mu\lambda} g_{\rho\nu} \partial \cdot \partial - g_{\mu\nu}\partial_\lambda \partial'_\rho)\mathcal{O}_2^{\lambda\rho}(x) \\
&+ i\epsilon_{\mu\nu\lambda\rho}\partial^\lambda \mathcal{O}_3^\rho(x) \\
&+ i(\epsilon_{\mu\nu\lambda\rho}\partial\cdot\partial' - \epsilon_{\mu\sigma\lambda\rho}\partial_\nu \partial'^\sigma + \epsilon_{\nu\sigma\lambda\rho}\partial_\mu \partial'^\sigma)\mathcal{O}_4^{\lambda\rho}(x) \,,
\end{aligned}
\tag{16.1}
$$

where $\partial_\mu \equiv \partial/\partial x_\mu$ and $\mathcal{O}_i$ are suitable bilocal operators, where $\mathcal{O}_L$ corresponds to the longitudinal structure functions $W_2 - 2xW_1$ defined in Eq. (15.36). The operators $\mathcal{O}_{3,4}$ do not contribute to the unpolarized process. Using the result in Eq. (15.56), one can write an OPE for the invariants. In the QCD deep inelastic scattering region, one can neglect quark mass corrections such that we have a good realization of the $SU(n)_f$ flavour symmetry. For the case $n_f = 2$ here (isospin symmetry), the electromagnetic current corresponds to the third component of $SU(2)$ such that the product $J(x)J(0)$ and the associate composite operators $\mathcal{O}$ belong to the representations:

$$
3 \otimes 3 = 1 \oplus 3 \oplus 5 \,. \tag{16.2}
$$

Therefore the lowest twist ($\tau = 2$) gauge invariant operators which dominate the light-cone expansion are, the non-singlet ($\lambda_a/2$ is the $SU(n)_f$ flavour matrix):

$$
\mathcal{O}^{(i)}_{NS,\mu_1\cdots\mu_k} = \frac{i^{k-1}}{k!}\left\{\bar\psi \frac{\lambda_a}{2}\gamma_{\mu_1} D_{\mu_2}\cdots D_{\mu_k}\psi + \text{permutations}\right\} \,, \tag{16.3}
$$

and singlet operators which mix under renormalizations:

$$
\begin{aligned}
\mathcal{O}^{(i)}_{S,\mu_1\cdots\mu_k} &= \frac{i^{k-1}}{k!}\{\bar\psi\gamma_{\mu_1} D_{\mu_2}\cdots D_{\mu_k}\psi + \text{permutations}\} \,, \\
\mathcal{O}^{(i)}_{g,\mu_1\cdots\mu_k} &= 2\frac{i^{k-2}}{k!} Tr\left\{G_{\mu_1\alpha} D_{\mu_1}\cdots D_{\mu_k} G^\alpha_{\mu_k} + \text{permutations}\right\} \,.
\end{aligned}
\tag{16.4}
$$

We have omitted terms containing $g_{\mu\nu}$, the so-called trace terms. Substituting Eq. (15.56) into Eq. (16.1), one can deduce in momentum space:

$$
\begin{aligned}
T_{\mu\nu} &= i \int d^4x\, e^{iqx} \langle p|\mathcal{T} J_\mu(x) J_\nu(0)|p\rangle \\
&= -(g_{\mu\nu}q^2 - q_\mu q_\nu) \sum_{i,n} \langle p|\mathcal{O}^{(i)}_{L,\mu_1\cdots\mu_n}(0)|p\rangle C^{(i)}_{L,n}(-q^2) q^{\mu_1}\cdots q^{\mu_n} \left(\frac{-q^2}{2}\right)^{-n-1} \\
&\quad + (g_{\mu\lambda}q_\rho q_\nu + g_{\rho\nu}q_\mu q_\lambda - q^2 g_{\mu\lambda}g_{\rho\nu} - g_{\mu\nu}q_\lambda q_\rho) \\
&\quad \times \sum_{i,n} \langle p|\mathcal{O}^{(i)\lambda\rho}_{2,\mu_1\cdots\mu_n}(0)|p\rangle C^{(i)}_{2,n}(-q^2) q^{\mu_1}\cdots q^{\mu_n} \left(\frac{-q^2}{2}\right)^{-n-1} ,
\end{aligned}
\tag{16.5}
$$

where we have defined the Fourier transform of the coefficient functions:

$$
\begin{aligned}
C^{(i)}_{L,n}(-q^2) q^{\mu_1}\cdots q^{\mu_n} \left(\frac{-q^2}{2}\right)^{-n-1} &= i \int d^4x\, e^{iqx}\, x^{\mu_1}\cdots x^{\mu_n} C^{(i)}_{L,n}(x^2) , \\
C^{(i)}_{2,n+2}(-q^2) q^{\mu_1}\cdots q^{\mu_n} 2 \left(\frac{-q^2}{2}\right)^{-n-2} &= i \int d^4x\, e^{iqx}\, x^{\mu_1}\cdots x^{\mu_n} C^{(i)}_{2,n}(x^2) ,
\end{aligned}
\tag{16.6}
$$

and we have used the simplified notation:

$$
\langle p|\mathcal{T} J_\mu(x) J_\nu(0)|p\rangle \equiv \frac{1}{2}\sum_\lambda \langle p;\lambda|\mathcal{T} J_\mu(x) J_\nu(0)|\lambda; p\rangle . \tag{16.7}
$$

Using the tensor structures:

$$
\begin{aligned}
\langle p|\mathcal{O}^{(i)}_{L,\mu_1\cdots\mu_n}(0)|p\rangle &= \hat{\mathcal{O}}_{L,n} p_{\mu_1}\cdots p_{\mu_n} + \cdots \\
\langle p|\mathcal{O}^{(i)\lambda\rho}_{2,\mu_1\cdots\mu_n}(0)|p\rangle &= \hat{\mathcal{O}}_{2,n+2} p^\lambda p^\rho p_{\mu_1}\cdots p_{\mu_n} + \cdots ,
\end{aligned}
\tag{16.8}
$$

where $\hat{\mathcal{O}}_i$ are reduced matrix elements not calculable in perturbation theory, and we have omitted terms containing $g_{\mu\nu}$, we finally deduce:

$$
T_{\mu\nu} = 2\omega^n \sum_{i,n\ \text{even}} e_{\mu\nu} C^{(i)}_{L,n}(-q^2)\hat{\mathcal{O}}^{(i)}_{L,n} - d_{\mu\nu} C^{(i)}_{2,n}(-q^2)\hat{\mathcal{O}}^{(i)}_{2,n} , \tag{16.9}
$$

with:

$$
\begin{aligned}
e_{\mu\nu} &\equiv g_{\mu\nu} - q_\mu q_\nu/q^2 , \\
d_{\mu\nu} &\equiv g_{\mu\nu} - q^2 p_\mu p_\nu/(p\cdot q)^2 - (p_\mu q_\nu + p_\nu q_\mu)/(p\cdot q) ,
\end{aligned}
\tag{16.10}
$$

where $\omega^{-1} \equiv Q^2/(2p\cdot q)$ is the Bjorken variable. Because of crossing symmetry:

$$
T_{\mu\nu}(\omega) = T_{\mu\nu}(-\omega) , \tag{16.11}
$$

the sum runs only over even n. The unphysical relation in Eq. (16.9) ($0 \le \omega \le 1$) can be converted to a physical one $\omega \ge 1$ by using a Cauchy integral to both sides of Eq. (16.9). Since $T_{\mu\nu}$ is an analytic function in the complex ω plane with branch cuts along the real

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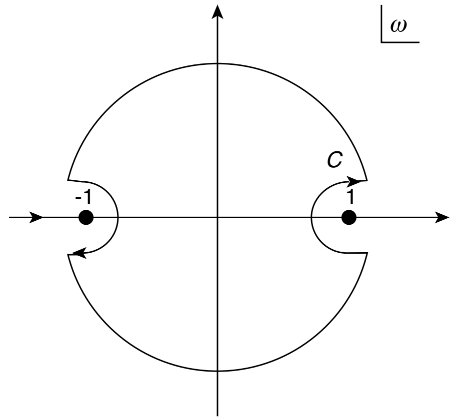

Fig. 16.1. Integration contour.

axis for $\omega \leq -1$ and $\omega \geq 1$, as shown in Fig. 16.1, it obeys the dispersion relation:

$$T_{\mu\nu} = \frac{1}{\pi}\left\{\int_{Q^2/2}^{\infty} - \int_{-\infty}^{-Q^2/2}\right\} \frac{d\nu'}{\nu' - \nu}\mathrm{Im}T_{\mu\nu}(Q^2, \nu) \; + \text{subtractions} \; . \tag{16.12}$$

Using the Cauchy integration to both sides of Eq. (16.9) along the contour in Fig. 16.1, one obtains:

$$\frac{1}{2i\pi}\oint_C \frac{T_{\mu\nu}}{\omega^n} = \frac{2}{\pi}\int_1^{\infty}\frac{d\omega}{\omega^n}\mathrm{Im}T_{\mu\nu} = 2\int_0^1 dx\, x^{n-2}W_{\mu\nu} \; , \tag{16.13}$$

where we have used the definitions in Eqs. (15.34) and (15.35) and the crossing symmetry in Eq. (16.11).

Noting that:

$$\oint_C d\omega\, \omega^{m-n} = \delta_{m,n-1} \; , \tag{16.14}$$

one can write:

$$T_{\mu\nu} = 2\sum_i e_{\mu\nu}C^{(i)}_{L,n-1}(-q^2)\hat{\mathcal{O}}^{(i)}_{L,n-1} - d_{\mu\nu}C^{(i)}_{2,n-1}(-q^2)\hat{\mathcal{O}}^{(i)}_{2,n-1} \; . \tag{16.15}$$

Equating Eqs. (16.15) and (16.13), one can deduce the *moment sum rules* for the structure functions [226]:

$$\begin{aligned}
\mathcal{M}^{(n)}_L(Q^2) &\equiv \int_0^1 dx\, x^{n-2}F_L(x, Q^2) = \sum_i C^{(i)}_{L,n}(-q^2)\hat{\mathcal{O}}^{(i)}_{L,n} \; , \\
\mathcal{M}^{(n)}_2(Q^2) &\equiv \int_0^1 dx\, x^{n-2}F_2(x, Q^2) = \sum_i C^{(i)}_{2,n}(-q^2)\hat{\mathcal{O}}^{(i)}_{2,n} \; ,
\end{aligned} \tag{16.16}$$

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where the structure functions $F_L \equiv F_2 - 2xF_1$ (*longitudinal structure functions*) and F_2 are defined through:

$$W_{\mu\nu} = \omega[e_{\mu\nu}F_L + d_{\mu\nu}F_2] , \tag{16.17}$$

and are related to the $W_{1,2}$ in Eq. (15.36) as:

$$\begin{aligned} F_L(x, Q^2) &= -W_1(\nu, Q^2) + \left(1 + \frac{\nu^2}{Q^2}\right) W_2(\nu, Q^2) , \\ F_2(x, Q^2) &= \frac{\nu}{M_p^2} W_2(\nu, Q^2) . \end{aligned} \tag{16.18}$$

The coefficient functions $C^{(i)}_{L,n}$ and $C^{(i)}_{2,n}$ in Eq. (16.16) are of short-distance nature and are calculable using perturbative QCD. The reduced matrix elements $\hat{\mathcal{O}}^{(i)}_{L,n}$ and $\hat{\mathcal{O}}^{(i)}_{2,n}$ are of long-distance nature and cannot be calculable. They can be determined experimentally, which can be done by measuring the moments in Eq. (16.16) at a fixed Q_0^2 and solve it for the reduced matrix elements. In practice, the moments are not very convenient as they are expressed in such a way that direct predictions of the structure functions cannot be made. Instead, one can take their inverse Mellin transform, which can be obtained by analytically continuing from integer n to complex n following the Carlson theorem [227].

One gets:

$$F_{L;2}(x, Q^2) = \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} dn\, \zeta^{1-n} C^{(i)}_{L;2,n}(Q^2)\hat{\mathcal{O}}^{(i)}_{L;2,n} , \tag{16.19}$$

where C is an arbitrary real positive constant. Assuming (for simplifying the discussion) that only one operator contributes to the moment, we can suppress the index i. Therefore, one can deduce from the moments in Eq. (16.16):

$$\hat{\mathcal{O}}_{L;2,n} = \frac{1}{C_{L;2,n}\left(Q_0^2\right)} \int_0^1 dx\, x^{n-2} F_{L;2}\left(x, Q_0^2\right) , \tag{16.20}$$

which, when inserted into Eq. (16.19), gives after rearranging the integral:

$$F_{L;2}(x, Q^2) = \int_x^1 \frac{dy}{y} K\left(\frac{y}{x}, Q^2, Q_0^2\right) F_{L;2}\left(y, Q_0^2\right) , \tag{16.21}$$

where the kernel function is:

$$K\left(z, Q^2, Q_0^2\right) = \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} dn\, z^{1-n} \frac{C_{L;2,n}(Q^2)}{C_{L;2,n}\left(Q_0^2\right)} . \tag{16.22}$$

Equation (16.21) expresses that once we know the structure function at a given Q_0^2 for all x $(0 < x < 1)$, one can predict its value at another Q^2 using a perturbative QCD calculation of the kernel function K.

16.2 RGE for the Wilson coefficients

The Q^2-dependence of the structure functions is completly contained into the one of the Wilson coefficients. As the electromagnetic current is not renormalized, the anomalous dimension of the composite operators should be cancelled by the one of the Wilson coefficients. Using the discussions in Chapter 11, we can write the RGE for the Wilson coefficients:

$$\left\{\nu\frac{\partial}{\partial\nu}+\beta(\alpha_s)\alpha_s\frac{\partial}{\partial\alpha_s}-\sum_j\gamma_m(\alpha_s)m_j\frac{\partial}{\partial m_j}-\gamma_n^{(i)}\right\}C_n^{(i)}(-q^2)=0\,. \tag{16.23}$$

where $\gamma_n^{(i)}$ is the anomalous dimension of the composite operators $\hat{\mathcal{O}}_n^{(i)}$, which can be proven to be gauge invariant such that the gauge-dependent term in the RGE is absent here. In the case of non-singlet structure functions, we have only one operator. In the case of singlet operators, we have coupled RGE due to the mixing of the two operators presented previously in Eq. (16.4). In this case, one should understand the anomalous dimension as a 2×2 matrix and the Wilson coefficient as a two-component vector. The solution to the RGE is:

$$C_n^{(i)}(Q^2/\nu^2,\alpha_s,m)=C_n^{(i)}(1,\bar{\alpha}_s(t),\bar{m}(t))\exp\left[-\int_0^t dt'\gamma_n^{(i)}[\bar{g}(t')]\right]\,, \tag{16.24}$$

where $t=1/2\log(Q^2/\nu^2)$. One can also rewrite the solution as:

$$C_n^{(i)}(Q^2)=C_n^{(i)}(1,\bar{\alpha}_s(t))\exp\left[-\int_{\alpha_s}^{\bar{\alpha}_s}dg\frac{\gamma_n^{(i)}(g)}{\beta(g)}\right]\,, \tag{16.25}$$

where the β function has been defined in Chapter 11 (Table 11.1):

$$\beta=\beta_1\left(\frac{\alpha_s}{\pi}\right)+\beta_2\left(\frac{\alpha_s}{\pi}\right)^2+\cdots\,. \tag{16.26}$$

16.3 Anomalous dimension of the non-singlet structure functions

In the following, one can safely suppress the index i because in the non-singlet case, only one operator dominates the light-cone expansion. Therefore:

$$\gamma_{NS,n}^{(i)}\equiv\gamma_{NS,n}=\gamma_n^0\left(\frac{\alpha_s}{\pi}\right)+\gamma_n^1\left(\frac{\alpha_s}{\pi}\right)^2+\cdots \tag{16.27}$$

In the following, we shall explicitly discuss the evaluation of γ_n^0. It comes from the Feynman diagrams in Fig. 16.2.

Using the Feynman rules given in Appendix E for the composite operators, Fig. 16.2a gives in the massless case and in the Feynman gauge:

$$V_{ij}^{(a)}=i^5g^2\sum_{a,l}\frac{\lambda_{il}^a}{2}\frac{\lambda_{lj}^a}{2}\int\frac{d^Nk}{(2\pi)^N}\frac{\gamma^\mu\hat{k}\hat{\Delta}(\Delta\cdot k)^{n-1}\hat{k}\gamma^\nu}{k^4}\frac{(-g_{\mu\nu})}{(p-k)^2}\,. \tag{16.28}$$

The relevant contribution to the anomalous dimension is the divergent part of the coefficient of $(\Delta\cdot p)^{n-1}\hat{\Delta}$. Using standard Feynman parametrization and shift of momentum

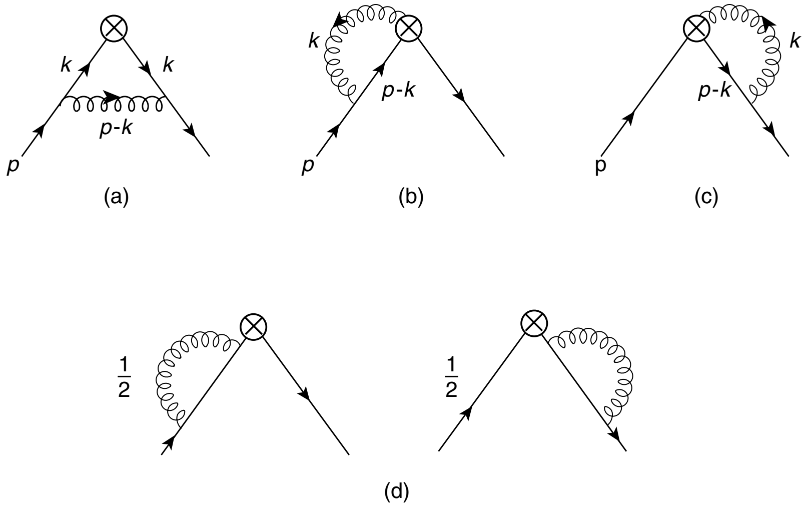

Fig. 16.2. Diagrams involved in the evaluation of γ_n^0.

(see Appendix F), the divergent part is:

$$V_{ij}^{(a)}\big|_{\epsilon \text{ pole}} = ig^2\delta_{ij}C_F \int_0^1 dx(1-x)\int \frac{d^Nk}{(2\pi)^N}\frac{\mathcal{N}}{[k^2+p^2x(1-x)]^3} \, , \tag{16.29}$$

where:

$$\mathcal{N} = -\frac{2k^2}{N}\gamma^\alpha\gamma^\beta\hat{\Delta}\gamma_\beta\gamma_\alpha x^{n-1}\hat{\Delta}(\Delta\cdot p)^{n-1} \, . \tag{16.30}$$

Therefore:

$$V_{ij}^{(a)}\big|_{\epsilon \text{ pole}} = \left(\frac{\alpha_s}{\pi}\right)\frac{2}{\hat{\epsilon}}\frac{C_F}{4}\frac{2}{n(n+1)}\hat{\Delta}(\Delta\cdot p)^{n-1} \, , \tag{16.31}$$

where $C_F = (N_c^2-1)/2N_c$ for $SU(N)_c$ and:

$$\frac{2}{\hat{\epsilon}} \equiv \frac{2}{\epsilon} + \log 4\pi - \gamma_E \, . \tag{16.32}$$

Figures 16.2b and c give the same result. It reads:

$$V_{ij}^{(b)} = V_{ij}^{(c)} = -i^3g^2C_F\delta_{ij}\int \frac{d^Nk}{(2\pi)^N}\frac{\Delta^\mu\hat{\Delta}\left[\sum_{n=0}^{n-2}(\Delta\cdot p)^l[\Delta\cdot(p+k)]^{n-l-2}\right](\hat{p}\hat{k})\gamma_\mu}{k^2(k+p)^2} \, . \tag{16.33}$$

The pole part of the coefficient of $(\Delta\cdot p)^{n-1}\hat{\Delta}$ is:

$$\begin{aligned} V_{ij}^{(b)}\big|_{\epsilon \text{ pole}} &= 2ig^2C_F\delta_{ij}\hat{\Delta}\int_0^1 dx\int\frac{d^Nk}{(2\pi)^N}\frac{\sum_{n=0}^{n-2}(\Delta\cdot p)^l(\Delta\cdot k + x\Delta k)^{n-l-1}}{[k^2+p^2x(1-x)]^2} \\ &= -\left(\frac{\alpha_s}{\pi}\right)\frac{2}{\hat{\epsilon}}\frac{C_F}{2}\delta_{ij}(\Delta\cdot p)^{n-1}\hat{\Delta}\left\{\int_0^1 dx\sum_{l=1}^{n-1}x^l = \sum_{l=2}^{n}\frac{1}{l}\right\} \, . \end{aligned} \tag{16.34}$$

The diagrams in Fig. 16.2d give the same contributions as the fermion wave function renormalization constant Z_{2F} defined in Eqs. (9.22) and (9.29). In the Feynman gauge, it gives:

$$V_{ij}^{(d)}\Big|_{\epsilon \text{ pole}} = -\left(\frac{\alpha_s}{\pi}\right)\frac{2}{\hat{\epsilon}}\frac{C_F}{4}\delta_{ij}(\Delta\cdot p)^{n-1}\hat{\Delta} \tag{16.35}$$

Adding the different contributions, one obtains the renormalization constant defined as:

$$Z_n^{NS} \equiv 1 + \frac{V_{ij}^{(a+b+c+d)}\Big|_{\epsilon \text{ pole}}}{(\Delta\cdot p)^{n-1}\hat{\Delta}} . \tag{16.36}$$

Using the definition of the anomalous dimension:

$$\gamma_n = \frac{\nu}{Z}\frac{dZ}{d\nu} \equiv \text{coefficient of } -\left(\frac{1}{\hat{\epsilon}}\right) , \tag{16.37}$$

one obtains the result:

$$\gamma_n^0 = \frac{C_F}{2}\left[1 - \frac{2}{n(n+1)} + 4\sum_{l=2}^{n}\frac{1}{l}\right] , \tag{16.38}$$

or equivalently:

$$\gamma_n^0 = \frac{C_F}{2}\left[4S_{1,n} - 3 - \frac{2}{n(n+1)}\right] , \tag{16.39}$$

with

$$S_{1,n} \equiv \sum_{l=1}^{n}\frac{1}{l} . \tag{16.40}$$

The expression of $S_{1,n}$ can be analytically continued to complex n thanks to the Carlson theorem [227] which we have used previously when taking the inverse Mellin transform. In this case, one can write:

$$S_{1,n} = n\sum_{k=1}^{\infty}\frac{1}{k(k+n)} = \psi(n+1) + \gamma_E \ : \quad \psi(z) \equiv \frac{d\log\Gamma(z)}{dz} . \tag{16.41}$$

where the expression of γ_n^1 is also known [228] and corrected in [232]. At this order, the problem of even (resp. odd) structure functions arises. The corresponding anomalous dimensions are $\gamma_n^{1,\pm}$. They read:

$$\begin{aligned}
\gamma_n^{1,\pm} = &\frac{32}{9}S_{1,n}\left[67 + \frac{8(2n+1)}{n^2(n+1)^2}\right] - 64S_{1,n}S_{2,n} - \frac{32}{9}[S_{2,n} - S^{\pm}_{2,n/2}]\left[2S_{1,n} - \frac{1}{n(n+1)}\right] \\
&- \frac{128}{9}\tilde{S}_n^{\pm} + \frac{32}{3}S_{2,n}\left[\frac{3}{n(n+1)} - 7\right] + \frac{16}{9}S^{\pm}_{3,n/2} - 28 \\
&- 16\frac{151n^4 + 260n^3 + 96n^2 + 3n + 10}{9n^3(n+1)^3} \\
&\pm \frac{32}{9}\frac{(2n^2+2n+1)}{n^3(n+1)^3} + \frac{32n_f}{27}\left[6S_{2,n} - 10S_{1,n} + \frac{3}{4} + \frac{11n^2+5n-3}{n^2(n+1)^2}\right],
\end{aligned} \tag{16.42}$$

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where:

$$S^{+}_{l,n/2} = S_{l,n/2} \,, \qquad S^{-}_{l,n/2} = S_{l,(n-1)/2} \,, \qquad \tilde{S}^{\pm}_{n} = -\frac{5}{8}\zeta_3 \mp \sum_{k=1}^{\infty} \frac{(-1)^k}{(k+n)^2} S_{l,n+k} \,. \tag{16.43}$$

16.4 Strategy for obtaining the Wilson coefficients

The main task in perturbative QCD is to calculate the Wilson coefficients. This can be simplified by the key observation that they are independent of the states which sandwich the light-cone expansion of the T-product of the electromagnetic current for the forward Compton amplitude $T_{\mu\nu}$. For instance, instead of taking proton states, one could consider quark or gluon Green's function with the insertion of the T-product of electromagnetic current. In the case of quark fields, the truncated (quark external line) Green's function reads:

$$\Gamma_{\mu\nu}(q,p)_{\text{trunc}} = i \int d^4x d^4x_1 d^4x_2 \, e^{-qx+p(x_1-x_2)} \langle 0|\mathcal{T} J_\mu(x) J_\nu(0) \psi(x_1)\bar{\psi}(x_2)|0\rangle \,, \tag{16.44}$$

where p is the quark momentum. Repeating the same reasoning as in the previous section, one can write the OPE analogous to the one in Eq. (16.9):

$$\Gamma_{\mu\nu}(q,p)_{\text{trunc}} = 2\omega^n \sum_{i,n \text{ even}} e_{\mu\nu} C^{(i)}_{L,n}(-q^2) \hat{\mathcal{O}}^{(i,\text{pert})}_{L,n} - d_{\mu\nu} C^{(i)}_{2,n}(-q^2) \hat{\mathcal{O}}^{(i,\text{pert})}_{2,n} \,, \tag{16.45}$$

where the Wilson coefficients are the same as in Eq. (16.9) but the 'composite operators' $\hat{\mathcal{O}}^{(i,\text{pert})}_{L:2,n}$ are calculable in perturbative QCD. The strategy is to calculate $\Gamma_{\mu\nu}(q,p)_{\text{trunc}}$ and $\hat{\mathcal{O}}^{(i,\text{pert})}_{L:2,n}$ in perturbation theory and then deduce the Wilson coefficients order by order of perturbative QCD.

16.4.1 Non-singlet part of the Bjorken sum rule

In the non-singlet part of the Bjorken sum rule, the Wilson coefficients can be expressed as:

$$C_{n,NS}(1,\alpha_s(Q^2)) = C^0_{n,NS}\left\{1 + C^1_{n,NS}\left(\frac{\alpha_s}{\pi}\right) + \cdots\right\} \tag{16.46}$$

For their evaluation, we shall consider the quark Green's functions:

$$T_{\mu\nu}(q,\psi) = i \int d^4x \, e^{iqx} \langle \psi|\mathcal{T} J_\mu(x) J_\nu(0)|\psi\rangle \,, \tag{16.47}$$

which has the decomposition:

$$T_{\mu\nu}(q) = e_{\mu\nu} T_L + d_{\mu\nu} T_2 \,, \tag{16.48}$$

where $e_{\mu\nu}$ and $e_{\mu\nu}$ have been defined in Eq. (16.10). We shall also use:

$$\hat{\mathcal{O}}^{(i,\text{pert})}_{L:2,n} p^{\mu_1} \cdots p^{\mu_n} = \langle \psi|\mathcal{O}^{(i),\mu_1\cdots\mu_n}_{L:2,n}|\psi\rangle \,. \tag{16.49}$$

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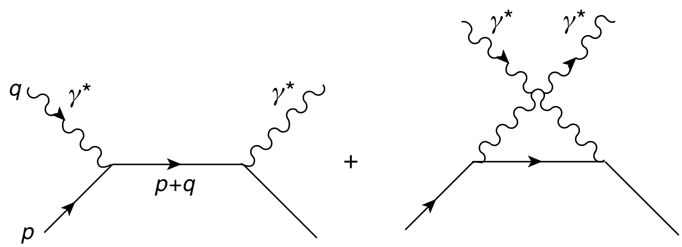

Fig. 16.3. Tree-level diagram for a photon-quark scattering.

The quark tree-level diagram shown in Fig. 16.3 leads to the amplitude:

$$T^0_{\mu\nu} = Q_\psi^2 \frac{1}{2} \sum_\lambda \bar{u}_\lambda(p) \left[\gamma_\mu \frac{1}{\hat{p}+\hat{q}-m} \gamma_\nu + \gamma_\mu \frac{1}{\hat{p}-\hat{q}-m} \gamma_\nu \right] u_\lambda(p) \,. \tag{16.50}$$

where $u_j(p)$ is the quark spinor, and Q_ψ is its charge in units of e. Introducing the Bjorken variables, one has:

$$T^0_{\mu\nu} = Q_\psi^2 d_{\mu\nu} \left(\frac{1}{x-1} - \frac{1}{x+1} \right) = 2 Q_q^2 d_{\mu\nu} \sum_{n=2,4\cdots} \left(\frac{1}{x} \right)^n \,, \tag{16.51}$$

where $d_{\mu\nu}$ has been defined in Eq. (16.10). Then, ones find:

$$T^0_2 = Q_\psi^2 \frac{2}{x^2-1} \,, \qquad T^0_L = 0 \,. \tag{16.52}$$

These results are already known from the free-field theory discussed in the beginning of this chapter. Solving the RGE for the Wilson coefficient, one obtains the modification due to QCD at leading order:

$$C_{2,n}(Q^2) \sim \left(\log \frac{Q^2}{\Lambda^2} \right)^{\gamma_n^0/2\beta_1} \,, \tag{16.53}$$

showing that the naïve Bjorken scaling is modified by the running coupling of QCD to leading order. To second order, one has [233]:

$$C^1_{n,NS} = \frac{C_F}{4} \left(2S^2_{1,n} + 3S_{1,n} - 2S_{2,n} - \frac{2S_{1,n}}{n(n+1)} + \frac{3}{n} + \frac{4}{n+1} + \frac{2}{n^2} - 9 \right) \,. \tag{16.54}$$

Therefore, to second order, the non-singlet moments read:

$$\begin{aligned} \mathcal{M}_n^{NS}(Q^2) \; &= \left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)} \right)^{\gamma_n^0/\beta_1} \left(\frac{1+\beta_2/\beta_1(\alpha_s(Q^2)/\pi)}{1+\beta_2/\beta_1\left(\alpha_s(Q_0^2)/\pi\right)} \right)^{-p_n} \\ &\times \left(\frac{1+C^1_{NS,n}(\alpha_s(Q^2)/\pi)}{1+C^1_{NS,n}\left(\alpha_s(Q_0^2)/\pi\right)} \right) \mathcal{M}_n^{NS}(Q_0^2) \,, \end{aligned} \tag{16.55}$$

where:

$$p_n = \gamma_n^1/\beta_2 - \gamma_n^0/\beta_1 \; . \tag{16.56}$$

This relation is well verified experimentally and used to measure the QCD coupling α_s.

16.4.2 Callan–Gross scaling violation

To leading order, the longitudinal structure function, coming from the diagram in Fig. 16.3, vanishes being defined as $F_2 - 2xF_1$. In the following, we analyze the structure function to order α_s.

Non-singlet part

To order α_s, the non-singlet part comes from the diagram in Fig. 16.4.

The analysis is simplified by noting that T_L is the only amplitude multiplied by $q_\mu q_\nu$. The amplitude from the *direct* diagram is:

$$\begin{aligned} T^{ij}_{\mu\nu}\big|_{\text{dir}} &= -iC_F\delta_{ij}g^2\frac{1}{4}\sum_\sigma \bar{u}(p,\sigma) \\ &\times \int \frac{d^Nk}{(2\pi)^N}\frac{\gamma_\alpha(\hat{p}+\hat{k})\gamma^\mu(\hat{p}+\hat{k}+\hat{q})\gamma^\nu(\hat{p}+\hat{k})\gamma^\alpha}{(p+k)^4(p+k+q)^2k^2}u(p,\sigma) \; . \end{aligned} \tag{16.57}$$

Using:

$$\sum_\sigma \bar{u}(p,\sigma)\mathcal{M}u(p,\sigma) = Tr[\hat{p}\mathcal{M}] \; , \tag{16.58}$$

and extracting term proportionnal to $q^\mu q^\nu$, one obtains after usual manipulations:

$$T_L^{NS}\big|_{\text{dir}} = \left(\frac{\alpha_s}{\pi}\right)C_F\frac{2}{x}\int_0^1 ydy\int_0^1 dz\frac{y(1-yz)}{[y-[1-(1-y-yz)/x]]^2} \; , \tag{16.59}$$

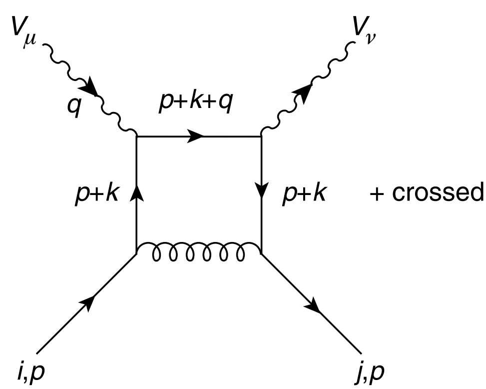

Fig. 16.4. Diagrams contributing to F_L^{NS}.

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Expanding in powers of $1/x$ and integrating, one obtains:

$$T_L^{NS}\big|_{\rm dir} = \left(\frac{\alpha_s}{\pi}\right) C_F \sum_{n=1}^{\infty} \frac{1}{n+1}\left(\frac{1}{x}\right)^n . \tag{16.60}$$

The *crossed* diagram doubles the even n contribution and cancels the odd one. Then, one finally obtains:

$$T_L^{NS} = 2\left(\frac{\alpha_s}{\pi}\right) C_F \sum_{n={\rm even}}^{\infty} \frac{1}{n+1}\left(\frac{1}{x}\right)^n . \tag{16.61}$$

Comparing with Eqs. (16.52) and (16.16), one can deduce the *scaling violation* QCD correction to the Callan–Gross relation:

$$\mathcal{M}_{L,n}^{NS} = \delta_L^{NS}\left(\frac{\alpha_s}{\pi}\right)\frac{C_F}{n+1}\mathcal{M}_{2,n}^{NS} , \tag{16.62}$$

where for ep scattering $\delta_L^{NS} = 1/6$. Taking the Mellin transforms, one can derive the non-singlet part of the structure functions:

$$F_L^{NS}(x, Q^2) = \int_x^1 dy\, C_{NS}^L(y, Q^2) F_2^{NS}\left(\frac{x}{y}, Q^2\right) , \tag{16.63}$$

where:

$$C_L^{NS}(y, Q^2) = C_F x(\alpha_s(Q^2)/\pi) + \mathcal{O}\left(\alpha_s^2\right) , \tag{16.64}$$

where the α_s^2 correction has been evaluated in [235].

Singlet part

The calculation of the singlet part is similar to that for the non-singlet. To the quark diagram in Fig. 16.3, one has to add the gluonic diagram in Fig. 16.5.

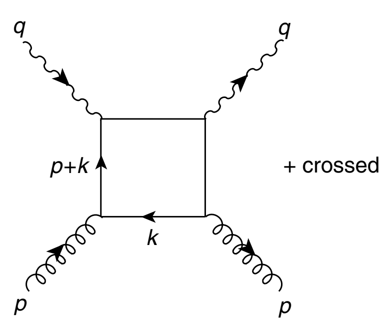

Fig. 16.5. Diagrams contributing to the gluon component of the structure function.

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For electron-proton scattering, the singlet structure function can be decomposed as:

$$F_L^S(x, Q^2) = \int_x^1 dy \, \left\{ C_S^L(y, Q^2) F_2^S\left(\frac{x}{y}, Q^2\right) + C_G^L(y, Q^2) F_G^S\left(\frac{x}{y}, Q^2\right) \right\} , \qquad (16.65)$$

where:

$$C_S^L(x, Q^2) \equiv C_{NS}^L + C_{QS}^L ,$$
$$C_{QS}^L(x, Q^2) = C_{QS}^{1,L} \left(\frac{\alpha_s(Q^2)}{\pi}\right)^2 ,$$
$$C_G^L(x, Q^2) = 4n_f T_R x(1-x) \left(\frac{\alpha_s(Q^2)}{\pi}\right) + C_G^{1,L} \left(\frac{\alpha_s(Q^2)}{\pi}\right)^2 . \qquad (16.66)$$

C_{NS}^L has been defined in Eq. (16.64). The coefficients $C_{QS}^{1,L}$ and $C_G^{1,L}$ have been evaluated in [235–237]. The full longitudinal structure function is the sum of the non-singlet and quark singlet components.

It is given by:

$$F_L \equiv F_2 - 2xF_3 = F_L^S + F_L^{NS} . \qquad (16.67)$$

16.5 Singlet anomalous dimensions and moments

The singlet calculations are more involved than the case of non-singlet and longitudinal structure functions. The corresponding anomalous dimension is a 2×2 matrix because of the mixing of the operators in Eq. (16.4). Using an expansion of the anomalous dimension and Wilson coefficient function:

$$\gamma_n = \gamma_{0n} \left(\frac{\alpha_s}{\pi}\right) + \gamma_{1n} \left(\frac{\alpha_s}{\pi}\right)^2 + \cdots$$
$$C_n^{(i)}(1, \alpha_s(Q^2)) = C_{n,i}^0 \left\{ 1 + C_{n,i}^1 \left(\frac{\alpha_s}{\pi}\right) + \cdots \right\} \qquad (16.68)$$

To leading order,

$$C_n^{(i)}(1, \alpha_s(Q^2)) = C_{n,j}^0 \left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)}\right)_{ij}^{\gamma_{0n}/\beta_1} \qquad (16.69)$$

where the indices $i, j \equiv q, g$ indicate quark and gluon composite operators respectively. The calculation of $C_{n,i}^0$ is very analogous to the non-singlet case by considering the forward Compton amplitude sandwiched between two quark states for $C_{n,q}^0$ and two gluon states for $C_{n,g}^0$. One obtains to this order:

$$C_{n,q}^0 = \begin{cases} 1 & \text{for } \mathrm{C_{n,2}} \\ 0 & \text{for } \mathrm{C_{n,L}} \end{cases} . \qquad (16.70)$$

Since the gluon does not couple to the photon to lowest order, one obtains:

$$C_{n,g}^0(Q^2) = 0 . \qquad (16.71)$$

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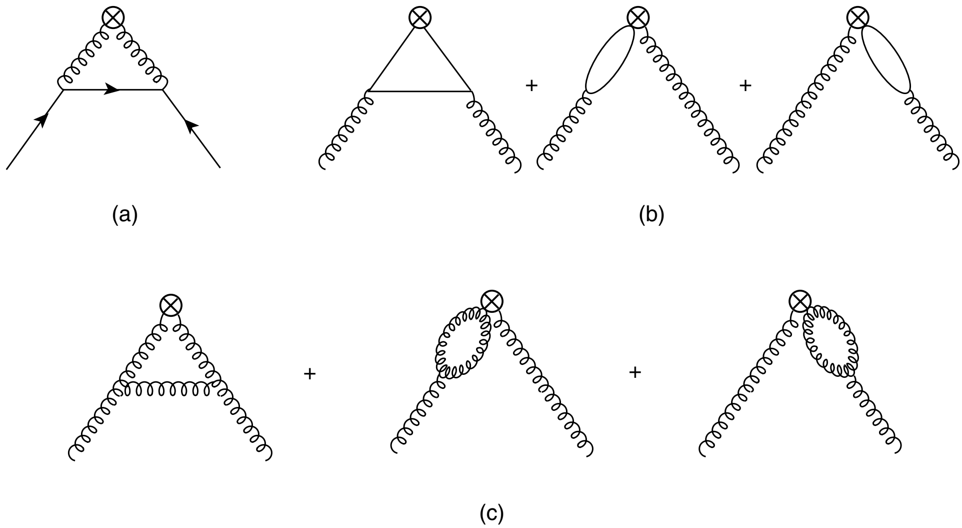

Fig. 16.6. Diagrams contributing to the singlet anomalous dimensions.

The anomalous dimension matrix reads to leading order:

$$\gamma_{0n} = \begin{pmatrix} \gamma_{0n}^{qq} & \gamma_{0n}^{qg} \\ \gamma_{0n}^{gq} & \gamma_{0n}^{gg} \end{pmatrix} . \tag{16.72}$$

The diagrams contributing to the anomalous dimensions are given in Fig. 16.6, in addition to the contribution from the diagrams in Fig. 16.2. The results are [168,234]:

$$\begin{aligned}
\gamma_{0n}^{qq} &= \frac{C_F}{2}\left[1 - \frac{2}{n(n+1)} + 4\sum_{j=2}^{n}\frac{1}{j}\right] , \\
\gamma_{0n}^{qg} &= -2n_f T_R \frac{n^2+n+2}{n(n+1)(n+2)} , \\
\gamma_{0n}^{gq} &= -C_F \frac{n^2+n+2}{n(n^2-1)} , \\
\gamma_{0n}^{gg} &= 2\left[C_G\left(\frac{1}{12} - \frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)} + \sum_{j=2}^{n}\frac{1}{j}\right) + T_R\frac{n_f}{3}\right] ,
\end{aligned} \tag{16.73}$$

where $C_F = (N_c^2 - 1)/2N_c,\ T_R = 1/2$ and $C_G = N_c$ for $SU(N)_c$. To this order, the moments in Eq. (16.16) read:

$$\begin{aligned}
\mathcal{M}_{2,n}(Q^2) &= \sum_i C_{n,i}^0 \left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)}\right)_{iq}^{\gamma_{0n}/\beta_1} , \\
\mathcal{M}_{L,n}(Q^2) &= 0 .
\end{aligned} \tag{16.74}$$

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In order to make a comparison with experiment, it is convenient to diagonalize the anomalous dimension matrix γ_{on}. On this basis, one can write:

$$\mathcal{M}_{2,n}(Q^2) = C^0_{+,n}\left(\log\frac{Q^2}{\Lambda^2}\right)^{-\gamma^+_{0n}/2\beta_1} + C^0_{-,n}\left(\log\frac{Q^2}{\Lambda^2}\right)^{-\gamma^-_{0n}/2\beta_1} , \qquad (16.75)$$

with:

$$\gamma^\pm_{0n} = \frac{1}{2}\left[\gamma^{qq}_{0n} + \gamma^{gg}_{0n} \pm \sqrt{\left(\gamma^{qq}_{0n} + \gamma^{gg}_{0n}\right)^2 + 4\gamma^{qg}_{0n}\gamma^{gq}_{0n}}\,\right] . \qquad (16.76)$$

To the next order, the expressions of the anomalous dimensions are known and the Wilson coefficients read [233]:

$$C^1_{n,q} = C^1_{n,NS} = \frac{C_F}{4}\left(2S^2_{1,n} + 3S_{1,n} - 2S_{2,n} - \frac{2S_{1,n}}{n(n+1)} + \frac{3}{n} + \frac{4}{n+1} + \frac{2}{n^2} - 9\right) ,$$
$$C^1_{n,g} = T_F n_f\left(-\frac{1}{n} + \frac{1}{n^2}\frac{6}{n+1} - \frac{6}{n+2} - S_{1,n}\frac{n^2+n+2}{n(n+1)(n+2)}\right) . \qquad (16.77)$$

To this order, the moments in the singlet case have more involved expressions, because of the mixing of operators. We refer the readers to, for example, the papers in [228,232,233], the review in [49] and book [46] for some expositions of this case. Finally, the expressions of few moments including three-loop corrections have been evaluated in [238].