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TOTAL TIME ON TEST TRANSFORMS OF ORDER *n* AND THEIR IMPLICATIONS IN RELIABILITY ANALYSIS

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Abstract

In this paper we study the properties of total time on test transforms of order n and examine their applications in reliability analysis. It is shown that the successive transforms produce either distributions with increasing or bathtub-shaped failure rates or distributions with decreasing or upside bathtub-shaped failure rates. The ageing properties of the baseline distribution is compared with those of transformed distributions, and a partial order based on nth-order transforms and their implications are discussed.

Keywords: Total time on test transform; partial order; ageing concept; characterization

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1. Introduction

The concept of total time on test (TTT) transforms is well known for its applications in different fields of study such as reliability analysis (Lai and Xie (2006, p. 42)), econometrics (Pham and Turkkan (1994)), stochastic modeling (Vera and Lynch (2005)), tail orderings (Bartoszewicz (1996)), and ordering distributions (Kochar *et al.* (2002)). A major share of the literature on TTT is concerned with reliability problems that include characterization of ageing properties, model identification, tests of hypotheses, age replacement policies in maintenance, ordering life distributions, and defining new classes of life distributions. We refer the reader to Bergman and Klefsjö (1984), Bartoszewicz (1995), Haupt and Schabe (1997), Kochar *et al.* (2002), Li and Zou (2004), Ahmed *et al.* (2005), Li and Shaked (2007), Nanda and Shaked (2008), and the references therein for further details. For a random variable representing a lifetime, with distribution function $F(x)$ and survival function $\bar{F}(x)$, the function defined on [0, 1] by

$$H_F^{-1}(u) = \int_0^{F^{-1}(u)} \bar{F}(x)\,\mathrm{d}x \tag{1.1}$$

is called the TTT transform of F. Recently, Vera and Lynch (2005) introduced higher-order TTT transforms by applying definition (1.1) recursively to the transformed distributions. They used the dominance of the transformed models over the baseline distribution to develop a martingale-type structure between the population and the baseline model. The present paper aims to study the properties of these iterated TTT transforms in the context of reliability analysis. Our objective in attempting this generalization is twofold. Firstly, it is expected that the hierarchy

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of distributions generated by the iterative process will reveal more clearly the behavior of the reliability characteristics of the transformed models than in a single trial. Secondly, the results obtained in this approach that subsumes the results for TTT transforms of order one will contribute to new models and properties that could be useful in the analysis of lifetime data.

Most of the properties of (1.1) are studied in the literature in terms of the distribution function $F(x)$ and other reliability characteristics derived from it. An alternative but equivalent approach is to use the quantile function, defined for a right-continuous distribution function $F(x)$ as

$$Q(u) = \inf[x\colon F(x) \geq u], \qquad 0 \leq u \leq 1,$$

and the reliability properties based on $Q(u)$. The quantile-based definitions of the basic concepts such as the failure (reversed failure) rate, the mean and variance of residual life, etc., their properties, the identities connecting them, and their characterizations are discussed in Nair and Sankaran (2008). In the context of TTT transforms, a quantile-based approach is more tractable analytically and simpler than using the distribution function and related concepts. There exist many distributions with simple quantile functions, but whose distribution functions cannot be expressed in closed forms (e.g. lambda distributions). These can be brought into consideration as reliability models in the present form of analysis. Quantile-based measures are less influenced by extreme observations due to the presence of long-term survivors, which are quite likely when heavy-tailed distributions are used as models of lifetime data.

The work carried out in the rest of this paper consists of four sections. In Section 2 we present a quantile-based definition of TTT transforms of order n and derive some identities connecting the reliability functions of the baseline and transformed distributions. Characterizations of some quantile functions by properties of the nth-order transforms are discussed in Section 3. Comparison of the ageing properties of the original and transformed distributions are made in Section 4, and, finally, Section 5 contains a discussion of various order relations, their applications, and some concluding remarks.

2. Preliminary results

Let X be a nonnegative random variable with quantile function $Q(u)$, as defined in Section 1. We assume that the distribution function $F(x)$ of X is absolutely continuous and strictly increasing. Furthermore, following Parzen (1979), we define the density quantile function as $f(Q(u))$, where $f(x)$ is the density function of X, and we define the quantile density function as $q(u) = Q'(u)$, where the prime notation denotes differentiation. Obviously,

$$q(u) f(Q(u)) = 1.$$

Of particular interest to the present study is the hazard quantile function (equivalent to the failure rate)

$$H(u) = \frac{f(Q(u))}{1-u} = ((1-u)q(u))^{-1}$$

and the mean residual quantile function (equivalent to the mean residual life function)

$$M(u) = (1-u)^{-1} \int_u^1 (Q(p) - Q(u))\,\mathrm{d}p. \tag{2.1}$$

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The interpretations of $H(u)$ and $Q(u)$, and the proofs of the following identities are available in Nair and Sankaran (2008):

$$\begin{aligned} Q(u) &= \int_0^u ((1-p)H(p))^{-1}\,\mathrm{d}p \\ &= \mu - M(u) + \int_0^u (1-p)^{-1}M(p)\,\mathrm{d}p, \qquad \mu = \mathrm{E}(X), \end{aligned} \tag{2.2}$$

$$M(u) = (1-u)^{-1}\int_u^1 (H(p))^{-1}\,\mathrm{d}p, \quad \text{and} \quad (H(u))^{-1} = -\frac{\mathrm{d}}{\mathrm{d}u}((1-u)M(u)).$$

The definition of the TTT transform in (1.1) transforms to $T_1(u) = \int_0^u (1-p)q(p)\,\mathrm{d}p$, which satisfies the relationships $t_1(u) = T_1'(u) = (H(u))^{-1}$ and

$$T_1(u) = \mu - (1-u)M(u). \tag{2.3}$$

The above equations show that, given any of the functions $Q(u)$, $H(u)$, or $M(u)$, the other two can be determined uniquely. Since $T_1(0) = 0$, $T_1(1) = \mu$, and $T_1(u)$ is an increasing function of u, $T_1(u)$ is a quantile function with support $[0, \mu]$. Hence, $T_1(u)$ defines a proper distribution function on $[0, \mu]$, which is H_F defined in (1.1). Thus, there exists a transform of T_1 and its successive transforms, enabling us to define the TTT transform recursively starting from $Q(u)$.

Definition 2.1. The TTT transform of order n of a random variable X is defined recursively as

$$T_n(u) = \int_0^u (1-p)t_{n-1}(p)\,\mathrm{d}p, \qquad n = 1, 2, \dots, \tag{2.4}$$

with $T_0(u) = Q(u)$ and $t_n(u) = T_n'(u)$ provided that $\mu_{n-1} = \int_0^1 T_{n-1}(p)\,\mathrm{d}p < \infty$.

We denote by X_n the random variable with quantile function $T_n(u)$, mean μ_n, hazard quantile function $H_n(u)$, and mean residual quantile function $M_n(u)$. Differentiating (2.4) we have

$$t_n(u) = (1-u)t_{n-1}(u) = (H_{n-1}(u))^{-1} \tag{2.5}$$

and

$$t_n(u) = (1-u)^n t_0(u) = (1-u)^n q(u) = (1-u)^{n-1}(H(u))^{-1}. \tag{2.6}$$

From (2.5) and (2.6), we have the identity connecting the hazard quantile functions of X and X_n as

$$H(u) = (1-u)^n H_n(u), \qquad n = 0, 1, 2, \dots, \tag{2.7}$$

with $H_0(u) = H(u)$ representing $X_0 = X$.

Using (2.3), $T_n(u)$ and $M_n(u)$ are related by

$$T_{n+1}(u) = \mu_n - (1-u)M_n(u), \tag{2.8}$$

from which

$$t_{n+1}(u) = M_n(u) - (1-u)M_n'(u). \tag{2.9}$$

This along with $t_{n+1}(u) = (1-u)^n t_1(u)$ and (2.9) specified for $n = 0$ give the following relationship between the mean residual quantile functions of X and X_n:

$$M_n(u) - (1-u)M_n'(u) = (1-u)^n(M(u) - (1-u)M'(u)). \tag{2.10}$$

Some important life distributions along with the expressions for $Q(u)$, $H(u)$, and $t_n(u)$ are exhibited in Table 1 to enable calculation of the above functions for these distributions. These distributions, where discussed in the sequel, will have the same form as in Table 1.

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TABLE 1: Quantile-based functions of life distributions.

Distribution	$\bar{F}(x)$	$Q(u)$	$H(u)$	$t_n(u)$
Exponential	$e^{-\lambda x},\ x>0$	$-\lambda^{-1}\log(1-u)$	λ	$\lambda^{-1}(1-u)^{n-1}$
Lomax	$\left(1+\frac{x}{\alpha}\right)^{-c},\ x>0$	$\alpha((1-u)^{-1/c}-1)$	$c\alpha^{-1}(1-u)^{1/c}$	$\alpha c^{-1}(1-u)^{n-1/c-1}$
Rescaled beta	$\left(1-\frac{x}{R}\right)^{c},\ 0<x<R$	$R(1-(1-u)^{1/c})$	$cR^{-1}(1-u)^{-1/c}$	$Rc^{-1}(1-u)^{n+1/c-1}$
Weibull	$\exp\left[-\left(\frac{x}{\sigma}\right)\right]^{\lambda},\ x>0$	$\sigma(-\log(1-u))^{1/\lambda}$	$\lambda\sigma^{-1}(-\log(1-u))^{1-1/\lambda}$	$\sigma\lambda^{-1}(1-u)^{n-1}$ $\times(-\log(1-u))^{-1/\lambda-1}$
Half-logistic	$(1+c)(1+ce^{\lambda x})^{-1},\ x>0$	$\lambda^{-1}\log\left(\frac{c+u}{c(1-u)}\right)$	$\lambda(1+c)^{-1}(c+u)$	$(1+c)\lambda(c+u)^{-1}(1-u)^{n-1}$
Generalized beta	$(1-x^{\alpha})^{\theta},\ 0\le x\le 1$	$(1-(1-u)^{1/\theta})^{1/\alpha}$	$\alpha\theta(1-u)^{-1/\theta}$ $\times(1-(1-u)^{1/\theta})^{1/\alpha-1}$	$(\alpha\theta)^{-1}(1-u)^{n+1/\theta-1}$ $\times(1-(1-u)^{1/\theta})^{1/\alpha-1}$
Lambda	–	$cu^{\lambda_1}(1-u)^{-\lambda_2}$	$[cu^{\lambda_1-1}(\lambda_1(1-u)$ $+\lambda_2(u))]^{-1}(1-u)^{\lambda_2}$	$(1-u)^{n-\lambda_2-1}cu^{\lambda_1-1}$ $\times(\lambda_1(1-u)+\lambda_2(u))$
Burr	$(1+x^{\lambda})^{-1},\ x>0$	$u^{\alpha}(1-u)^{-\alpha}, \alpha=\lambda^{-1}$	$\alpha(1-u)^{-\alpha-1}u^{\alpha-1}$	$\alpha u^{\alpha-1}(1-u)^{n-\alpha-1}$

Remark 2.1. Definition 2.1 extends to negative integers as well. For example, $Q(u)$ can be considered as the transform of $T_{-1}(u)$, etc. In this backward recurrence,

$$t_{-n}(u) = (1-u)^{-n} q(u)$$

and

$$H(u) = (1-u)^{-n} H_n(u), \qquad n = 1, 2, 3 \ldots .$$

Equivalently, we can assume a given distributional form for X_n and revert to the distribution of X.

Remark 2.2. As seen from (2.6), the sequence $\langle H_n(u) \rangle$ increases for all positive n and decreases for negative n. Thus, the random variable X_n (or the nth-order transform) generates a distribution whose failure rate is larger or smaller than that of X_{n-1} according to whether n is positive or, respectively, negative.

3. Characterizations

Relationships (2.6)–(2.8) encourage us to seek mutual characterizations of the distributions of X and X_n by exploiting the functional forms of $Q(u)$, $H(u)$, $M(u)$, and their counterparts for X_n.

First we note that $T_n(u)$ characterizes the distribution of X. This follows from

$$t_n(u) = (1-u)^n q(u)$$

and

$$Q(u) = \int_0^u (1-p)^{-n} t_n(p) \, \mathrm{d}p.$$

Theorem 3.1. *The random variable X_n, $n = 1, 2, 3, \ldots$, has rescaled beta distribution if and only if X is distributed as either exponential, Lomax, or rescaled beta.*

Proof. From the expressions for the quantile density functions $t_n(u)$ of X_n in Table 1, the quantile function in the exponential case is

$$T_n(u) = \int_0^u t_n(p) \, \mathrm{d}p = (\lambda n)^{-1}(1 - (1-u)^n),$$

which is rescaled beta with parameters $((\lambda n)^{-1}, n^{-1})$ in the support of $[0, (n\lambda)^{-1}]$. Similar calculations show that, when X is Lomax, X_n is rescaled beta with parameters $[\alpha(nc-1)^{-1}, c(nc-1)^{-1}]$ in the support of $(0, \alpha(nc-1)^{-1})$, and, finally, when X is rescaled beta with parameters (R, c), X_n has the same distribution in $[0, R(1+nc)^{-1}]$ with parameters $[R(1+nc)^{-1}, c(1+nc)^{-1}]$. This proves the ‘if’ part.

To prove the converse, we assume that X_n is distributed as rescaled beta so that we can write

$$T_n(u) = R_n(1 - (1-u)^{1/c_n})$$

for some constants $R_n, c_n > 0$. Thus,

$$t_n(u) = \frac{R_n}{c_n}(1-u)^{1/c_n - 1}$$

or

$$\frac{R_n}{c_n}(1-u)^{1/c_n-1} = (1-u)^n q(u) \quad \text{for all } u.$$

This means that $(1-u)^n$ is a factor on the left-hand side and, therefore,

$$c_n^{-1} = k_n + n \quad \text{for some real } k_n.$$

Hence,

$$q(u) = (k_n + n)R_n(1-u)^{k_n-1}.$$

Since the left-hand side is independent of n, taking $n = 1$ we have

$$Q(u) = k_1^{-1} R_1(k_1+1)(1-(1-u)^{k_1}).$$

Hence, X follows the rescaled beta distribution in $(0, R_1 k_1^{-1}(k_1+1))$ for $k_1 > 0$ and the Lomax distribution for $-1 < k_1 < 0$. Finally, as $k_1 \to 0$, applying L'Hôpital's rule,

$$Q(u) \to R_1(-\log(1-u)),$$

which represents the exponential law. This completes the proof.

Our next characterization is by a relationship between the mean residual quantile functions of X and X_n.

Theorem 3.2. *The random variable X follows the generalized Pareto distribution with quantile function*

$$Q(u) = AB^{-1}((1-u)^{-B(B+1)^{-1}} - 1), \qquad B > -1,\ A > 0, \tag{3.1}$$

if and only if, for all $n = 0, 1, 2, \dots$ *and* $0 < u < 1$,

$$M_n(u) = (nB + n + 1)^{-1}(1-u)^n M(u). \tag{3.2}$$

Proof. Assuming that (3.2) holds, we have, on using (2.10) and simplifying,

$$BM(u) = (B+1)(1-u)M'(u)$$

or

$$\frac{M'(u)}{M(u)} = \frac{B}{(B+1)(1-u)}.$$

Integrating we have

$$M(u) = K(1-u)^{-B/(B+1)}. \tag{3.3}$$

Substituting (3.3) into (2.2) and noting that $\mu = M(0) = K$, we have the quantile function (3.1) with $K = A$. This proves the 'if' part. Next we assume that X has the distribution specified by (3.1). Then we have

$$q(u) = A(B+1)^{-1}(1-u)^{-B(B+1)^{-1}-1}.$$

Using (2.5), we obtain $t_n(u)$ and, hence,

$$T_n(u) = A((B+1)n - B)^{-1}(1-(1-u)^{n-B(B+1)^{-1}}).$$

From (2.1), replacing $Q(u)$ with $T_n(u)$ and simplifying,

$$M_n(u) = A(nB + n + 1)^{-1}(1 - u)^{n-B(B+1)^{-1}}.$$

The quantile function of X in (3.1) verifies

$$M(u) = A(1 - u)^{-B(B+1)^{-1}},$$

so that (3.2) holds and the proof is completed.

Remark 3.1. The random variable X_n, $n = -1, -2, \dots,$ has Lomax distribution if and only if X is distributed as either exponential of beta or Lomax. The negative transforms are as mentioned in Remark 2.1. The proof, being similar to that of Theorem 3.1, is not given here.

4. Ageing properties

It was seen in the last section that with each iteration of the TTT transform, the hazard quantile function increases for positive n and decreases for negative n. Another important aspect is to ascertain the ageing behavior in a particular iteration. We prove some general results about the ageing patterns of X_n in relation to X in this section.

Theorem 4.1. (i) *If X is IFR (increasing failure rate) then X_n is IFR for all n.*

(ii) *If X is DFR (decreasing failure rate) then X_n is DFR if $Q(u) \geq L(k, 1/n)$, is IFR if $Q(u) \leq L(k, 1/n)$, and is BS (bathtub shaped) if there exists a u_0 for which $Q(u) \geq L(k, 1/n)$ in $[0, u_0]$ and $Q(u) \leq L(k, 1/n)$ in $[u_0, 1]$. Here $L(\alpha, c)$ denotes the quantile function of the Lomax distribution in Table 1.*

Proof. Since $t_{n+1}(u) = (1 - u)^n t_1(u)$,

$$t'_{n+1}(u) = (1 - u)^{n-1}((1 - u)t'_1(u) - nt_1(u)), \tag{4.1}$$

where the prime notation denotes differentiation with respect to u.

From Barlow and Campo (1975), X is IFR or DFR if T_1 is concave or, respectively, convex. Hence, from (4.1),

$$\begin{aligned} X \text{ is IFR} &\implies t_1(u) \text{ is decreasing} \\ &\implies t'_{n+1}(u) < 0 \\ &\implies T_{n+1}(u) \text{ is concave} \\ &\implies X_n \text{ is IFR.} \end{aligned}$$

Similarly, when X is DFR, $T_1(u)$ is convex and accordingly

$$\begin{aligned} X_n \text{ is DFR} &\implies (1 - u)t'_1(u) \geq nt_1(u) \\ &\implies t_1(u) \geq k(1 - u)^{-n} \\ &\implies Q(u) \geq L\left(k, \frac{1}{n}\right), \\ X_n \text{ is IFR} &\implies (1 - u)t'_1(u) \leq nt_1(u) \\ &\implies t_1(u) \leq k(1 - u)^{-n} \\ &\implies Q(u) \leq L\left(k, \frac{1}{n}\right). \end{aligned}$$

The last part of (ii) follows from the above result and Theorem 4.1 of Haupt and Schabe (1997).

In a similar manner, by considering the backward iteration we have the following result.

Theorem 4.2. (i) *If X_n is DFR then X is DFR.*

(ii) *If X_n is IFR then X is IFR if $T_n(u) \le B(k(n+1)^{-1}, (n+1)^{-1})$, is DFR if $T_n(u) \ge B(k(n+1)^{-1}, (n+1)^{-1})$, and is UBS (upside bathtub shaped) if there exists a u_0 for which $T_n(u) \le B(k(n+1)^{-1}, (n+1)^{-1})$ in $[0, u_0]$ and $T_n(u) \ge B(k(n+1)^{-1}, (n+1)^{-1})$ in $[u_0, 1]$. Here $B(R, c)$ denotes the rescaled beta with parameters (R, c).*

The proof runs along the same lines as in Theorem 4.1 once we note that

$$t_1'(u) = (1-u)^{-n}(n(1-u)^{-1}t_{n+1}(u) + t_{n+1}'(u)),$$

and, therefore, the details are omitted.

Remark 4.1. The importance of Theorems 4.1 and 4.2 is that they help the construction of BS and UBS distributions by a simple mechanism. To obtain BS distributions, we need look only at DFR distributions for which $t_{n+1}(u)$ has a point of inflexion. Similarly, for obtaining UBS distributions, we look for IFR distributions of X and perform backward recurrence ($n < 0$) to reach a quantile density function that has an inflexion point. The procedure is illustrated in the following examples.

Example 4.1. The Weibull distribution (Table 1) has

$$t_n'(u) = \left(\frac{1}{\lambda} - 1\right) + (n-1)\log(1-u),$$

so that, when $\lambda \le 1$, T_{n+1} is convex on $[0, u_0]$ and concave on $[u_0, 1]$, where $u_0 = 1 - \exp[(\lambda - 1)/n\lambda]$. Hence, X_n has BS failure rate for $n \ge 1$. With increasing values of n, the change point in the failure rate becomes larger so that the range for which X_n is IFR increases. Note also that, for $\lambda \ge 1$ and every n, X_n continues to be IFR.

Similarly, when n is a negative integer, $T_{n+1}(u)$ is convex for $\lambda \ge 1$ in $[u_0, 1]$ and concave for $\lambda \ge 1$ in $[0, u_0]$, where $u_0 = 1 - \exp[(1-\lambda)/(n+2)\lambda]$. Thus, X_n is UBS for $n < 2$.

Example 4.2. The Burr distribution (Table 1) is DFR for $\alpha \ge 1$. In this case,

$$t_{n+1}'(u) = \alpha u^{\alpha-2}(1-u)^{n-\alpha-1}((\alpha-1) - u(n-1)),$$

showing that $u = (\alpha-1)/(n-1)$ is a point of inflexion when $\alpha \ge 1$. Thus, X_n is BS when $\alpha \ge 1$ and $n > 1$.

Example 4.3. The lambda distribution verifies

$$\begin{aligned} t_{n+1}'(u) &= Cu^{\lambda_1-1}(1-u)^{n-\lambda_2-1} \\ &\quad \times [\{(\lambda_1-\lambda_2)^2 + (2-n)\lambda_2 + (n-2)\lambda_1\}u^2 \\ &\qquad + \{-2\lambda_1^2 + 2\lambda_1\lambda_2 + (3-n)\lambda_1 - 2\lambda_2\}u + \lambda_1(\lambda_1-1)] \\ &= 0. \end{aligned}$$

Hence, X_n is UBS for $\lambda_1 = 1$, $\lambda_2 = 3$, and $n = 3$ with change point $u_0 = \frac{1}{4}$.

Remark 4.2. Haupt and Schabe (1997) proposed a method of constructing a BS distribution by choosing a twice differentiable function $\phi(u)$ satisfying $\phi(0) = 1, \phi(1) = 1$, and $0 \le \phi(u) \le 1$ with $\phi(u)$ having only one inflexion point u_0 such that it is convex on $[0, u_0]$ and concave on $[u_0, 1]$. Then the solution $F(t)$ of

$$\frac{\theta\phi'(F(t))\,\mathrm{d}F(t)}{1 - F(t)} = \mathrm{d}t, \qquad \theta = H_F^{-1} > 0, \tag{4.2}$$

is a BS distribution. Converting (4.2) in terms of quantile functions, the solution of (4.2) is a twice differentiable $T_n(u)$ for some n for which there is an inflexion point.

Theorem 4.3. (i) *X is DMRL (decreasing mean residual lifetime) implies that X_n is DMRL.*

(ii) *X_n is IMRL (increasing mean residual lifetime) implies that X is IMRL.*

Proof. We prove only (i), as the proof of (ii) is similar with negative n. Recall that X is DMRL if and only if $(1 - u)^{-1}(1 - \mu^{-1}T_1(u))$ is decreasing in u (Klefsjö (1982)) or, alternatively, $(1 - u)^{-1}(\mu - T_1(u))$ is decreasing, or

$$\mu - T_1(u) - (1 - u)t_1(u) \le 0.$$

Furthermore,

$$T_{n+1}(u) = \int_0^u (1 - p)^n t_1(p)\,\mathrm{d}p = (1 - u)^n T_1(u) + A(u),$$

where

$$A(u) = n\int_0^u (1 - p)^{n-1} T_1(p)\,\mathrm{d}p > 0 \quad \text{for all } u \in [0, 1].$$

Now,

$$\begin{aligned} \mu_n - T_{n+1}(u) - (1 - u)t_{n+1}(u) &= \mu_n - (1 - u)^n T_1(u) - A(u) - (1 - u)^{n+1} t_{n+1}(u) \\ &\le \mu_n - (1 - u)^n T_1(u) - (1 - u)^{n+1} t_{n+1}(u) \\ &\le \mu_1 - T_1(u) - (1 - u)t_1(u) \\ &\le 0 \quad \text{(since } X \text{ is DMRL).} \end{aligned}$$

Thus, X_n is DMRL.

Theorem 4.4. (i) *X is IFRA (increasing failure rate in average) implies that X_n is IFRA.*

(ii) *X_n is DFRA (decreasing failure rate in average) implies that X is DFRA.*

Proof. From Barlow and Campo (1975), X is IFRA if and only if $u^{-1}T_1(u)$ is decreasing for all u. This means that $t_1(u) \le u^{-1}T_1(u)$. We then have

$$\begin{aligned} t_{n+1}(u) - u^{-1}T_{n+1}(u) &= (1 - u)^n t_1(u) - u^{-1}(1 - u)^n T_1(u) - u^{-1}A(u) \\ &\le (1 - u)^n (t_1(u) - u^{-1}T_1(u)) \\ &\le (t_1 - u^{-1}T_1(u)) \\ &\le 0, \end{aligned}$$

which implies that X_n is IFRA.

Theorem 4.5. (i) *X is NBUE (new better than used in expectation) implies that X_n is NBUE.*

(ii) *X_n is NWUE (new worse than used in expectation) implies that X is NWUE.*

Proof. The random variable X is NBUE if and only if $\mu^{-1}T(u) > u$ for $u \in [0, 1]$ (Bergman (1977)). Hence,

$$\begin{aligned} u^{-1}T_n(u) - \mu_n &= u^{-1}((1-u)^n T_1(u) + A(u)) - \mu_n \\ &\geq u^{-1}(1-u)^n T_1(u) - \mu_1 \\ &= (1-u)^n (u^{-1}T_1(u) - \mu_1) \\ &\geq 0, \end{aligned}$$

which implies that X_n is NBUE.

Remark 4.3. The converse of the implications given in the above theorems does not hold in view of the characterizations given in Theorem 3.1 and Remark 3.1.

Comparison of other ageing concepts can also be facilitated in a like manner and some of these become evident from the order relations examined in the next section.

5. Order relations

In this section we discuss the implications of the results obtained so far in developing some order relations connecting the baseline and transformed distributions. Furthermore, a new partial order based on transforms of order n, which extends some of the existing results, is introduced.

Let X and Y be two nonnegative random variables with finite expectations, distribution functions $F(\cdot)$ and $G(\cdot)$, quantile functions $Q(u)$ and $R(u)$, and TTT transforms, $T(u)$ and $S(u)$, respectively. From (2.7),

$$H(u) = (1-u)^n H_n(u) \leq H_n(u),$$

and, therefore, we have $X_n \leq_{\text{hr}} X$, where '$\leq_{\text{hr}}$' denotes the hazard rate order.

It is said that X is smaller than Y in the dispersive order, $X \leq_{\text{disp}} Y$, if and only if

$$F^{-1}(u_2) - F^{-1}(u_1) \leq G^{-1}(u_2) - G^{-1}(u_1) \quad \text{for } 0 \leq u_1 \leq u_2 \leq 1,$$

which means that in our notation

$$\int_{u_1}^{u_2} q(p)\,\mathrm{d}p \leq \int_{u_1}^{u_2} r(p)\,\mathrm{d}p, \qquad r(u) = R'(u).$$

Setting $Y = X_n$ and $r(u) = t_n(u) = (1-u)^n q(u)$, we have $X_n \leq_{\text{disp}} X$.

It is said that X is smaller than Y in the convex transform order, $X \leq_{\text{cx}} Y$, if $G^{-1}F(x)$ is convex in the support of F. In terms of quantile density functions, this condition is equivalent to $r(u)/q(u)$ increasing in u. As before, with $Y = X_n$, the above ratio is decreasing in u and, hence, $X_n \leq_{\text{cx}} X$.

Using the definitions and implications of the hazard rate order, usual stochastic order, $X \leq_{\text{st}} Y$, mean residual life order, $X \leq_{\text{MRL}} Y$, variance residual life order, $X \leq_{\text{VRL}} Y$, harmonic mean residual life order, $X \leq_{\text{HMRL}} Y$, dilation order, $X \leq_{\text{dil}} Y$, increasing convex order, $X \leq_{\text{icv}} Y$, star order, $X \leq_{*} Y$, superadditive order, $X \leq_{\text{su}} Y$, excess wealth order,

$X \le_{\rm EW} Y$, decreasing mean residual life order, $X \le_{\rm DMRL} Y$, NBUE order, $X \le_{\rm NBUE} Y$, and Lorenz order, $X \le_{\rm Lorenz} Y$, discussed in Shaked and Shanthikumar (2007), we have the following chain of relationships:

$$\begin{array}{ccccccc} & & & & X_n \le_{\rm dil} X & & \\ & & & & \Uparrow & & \\ & & X_n \le_{\rm hr} X & \Longrightarrow & X_n \le_{\rm MRL} X & \Longrightarrow & X_n \le_{\rm VRL} X \\ & & \Downarrow & & \Downarrow & & \\ X_n \le_{\rm disp} X & \Longrightarrow & X_n \le_{\rm st} X & \Longrightarrow & X_n \le_{\rm HMRL} X & & \\ \Downarrow & & \Downarrow & & & & \\ X_n \le_{\rm EW} X & & X_n \le_{\rm icv} X & & & & \end{array}$$

and

$$\begin{array}{ccccc} X_n \le_{\rm cx} X & \Longrightarrow & X_n \le_{*} X & \Longrightarrow & X_n \le_{\rm su} X \\ \Downarrow & & & & \\ X_n \le_{\rm DMRL} X & \Longrightarrow & X_n \le_{\rm NBUE} X & \Longrightarrow & X_n \le_{\rm Lorenz} X. \end{array}$$

Definition 5.1. It is said that X is smaller than Y in the TTT transform of order n, written as $X \le_{\text{TTT-}n} Y$ (or, equivalently, $X_n \le_{\rm TTT} Y_n$), if $T_{n+1}(u) \le S_{n+1}(u)$ for all u in [0, 1], where $T_n(u)$ and $S_n(u)$ denote the TTT transforms of order n of X and Y, respectively.

First we note that, from the above definition,

$$X \le_{\text{TTT-}n} Y \iff T_{n+1}(u) \le S_{n+1}(u) \iff X_{n+1} \le_{\rm st} Y_{n+1}. \tag{5.1}$$

Hence, all the implications starting from the stochastic order in the chain presented above are implications of the TTT-n order. Furthermore, the usual TTT order between X and Y satisfies

$$X \le_{\rm TTT} Y \implies X \le_{\text{TTT-}n} Y,$$

as an extension of the result in Shaked and Shanthikumar (2007, Theorem 4B.29).

Another order of interest is the NBUE order defined as X is smaller than Y in the NBUE order, $X \le_{\rm NBUE} Y$, if

$$\frac{m(F^{-1}(u))}{l(G^{-1}(u))} \le \frac{{\rm E}(X)}{{\rm E}(Y)} \quad \text{for } u \in (0, 1), \tag{5.2}$$

where $m(\cdot)$ and $l(\cdot)$ are the mean residual life functions of X and Y, respectively. An equivalent statement of (5.2) is

$$\frac{T_1(u)}{{\rm E}(X)} \ge \frac{S_1(u)}{{\rm E}(Y)}. \tag{5.3}$$

Since $T_1(u)/{\rm E}(X)$ and $S_1(u)/{\rm E}(Y)$ are quantile functions of $X_1/{\rm E}(X)$ and $Y_1/{\rm E}(Y)$, successive application of (5.3) for $X_2, X_3, \ldots$ in the definition of the NBUE order gives

$$X \le_{\rm NBUE} Y \iff \frac{X_1}{{\rm E}(X)} \ge_{\rm st} \frac{Y_1}{{\rm E}(Y)} \iff \frac{X}{{\rm E}(X)} \ge_{\rm TTT} \frac{Y}{{\rm E}(Y)}$$

and

$$X_{n-1} \le_{\rm NBUE} Y_{n-1} \iff \frac{X_{n-1}}{{\rm E}(X_{n-1})} \ge_{\rm TTT} \frac{Y_{n-1}}{{\rm E}(Y_{n-1})} \iff \frac{X_n}{{\rm E}(X_{n-1})} \ge_{\rm st} \frac{Y_n}{{\rm E}(Y_{n-1})}.$$

These results extend Theorem 4B.26 of Shaked and Shanthikumar (2007). When X_{n-1} and Y_{n-1} have finite means and 0 as common left endpoints of their supports, then, for any $\phi(x)$ with $\phi(0) = 0$,

$$X \leq_{\text{TTT-}n} Y \quad \Longrightarrow \quad \phi(X) \leq_{\text{TTT-}n} \phi(Y).$$

From (5.1) and (5.3),

$$\frac{X}{\mathrm{E}(X)} \geq_{\text{TTT}} \frac{Y}{\mathrm{E}(Y)} \quad \Longrightarrow \quad \frac{X}{\mathrm{E}(X)} \geq_{\text{TTT-}n} \frac{Y}{\mathrm{E}(Y)}.$$

Note that the TTT transform of $X/\mathrm{E}(X)$ is the scaled TTT transform that is extensively used in many practical applications, including characterization of ageing classes.

An interesting property of the TTT order is that it is preserved under the minima of independent and identically distributed random variables (Kochar *et al.* (2002)). Following the lines of proof of this result, we prove a similar result for the TTT-n order.

Theorem 5.1. *Let $(X_1, X_2, \ldots, X_n)$ and $(Y_1, Y_2, \ldots, Y_n)$ be independent copies of two non-negative random variables X and Y that are identically distributed. Then, if $X \leq_{\text{TTT}-n} Y$, $V_n \leq_{\text{TTT}-n} W_n$, where $V_n = \min(X_1, X_2, \ldots X_n)$ and $W_n = \min(Y_1, Y_2, \ldots, Y_n)$.*

Proof. The survival functions of V_n and W_n are

$$\bar{F}_{V_n}(x) = (\bar{F}_X(x))^n \quad \text{and} \quad \bar{F}_{W_n}(x) = (\bar{F}_Y(x))^n.$$

To revert to quantile functions, we write $u_n = F_{V_n}(x)$ and $u = F_X(x)$ to obtain

$$u = 1 - (1 - u_n)^{1/n} \quad \text{and} \quad u_n = 1 - (1 - u)^n.$$

If $Q_n(u_n)$ is the quantile function of V_n, the quantile function $Q(u)$ of F is related to it as

$$Q(u) = Q_n(1 - (1 - u)^n)$$

or

$$q(u) = nq_n(1 - (1 - u)^n)(1 - u)^{n-1}.$$

Similarly, using the symbols r_n and r for the quantile density functions of W_n and Y,

$$r(u) = nr_n(1 - (1 - v)^n)(1 - v)^{n-1}, \qquad v = F_Y(x).$$

Since $X \leq_{\text{TTT-}n} Y$, we have, using the earlier notation,

$$T_{n+1}(u) \leq S_{n+1}(u),$$

and, hence,

$$\int_0^u (1 - p)^{n+1} q(p)\,\mathrm{d}p \leq \int_0^u (1 - p)^{n+1} r(p)\,\mathrm{d}p.$$

Applying Lemma A.2(b) of Kochar *et al.* (2002), we have

$$\int_0^u (1 - p)^{2n} q(p)\,\mathrm{d}p \leq \int_0^u (1 - p)^{2n} r(p)\,\mathrm{d}p.$$

Setting $y = 1 - (1 - p)^n$, we have

$$\int_0^{u_n} (1 - y)^{n+1} q_n(y)\,\mathrm{d}y \leq \int_0^{u_n} (1 - y)^{n+1} q_n(y)\,\mathrm{d}y,$$

which proves the result.

Another application of the TTT-n order concerns the proportional hazard models that extend the results of Li and Shaked (2007). Let $X(\theta)$ be a random variable of a proportional hazard model with survival function $[\bar{F}(x)]^{\theta}$, $\theta > 0$, corresponding to X with survival function $\bar{F}(x)$. Assume that the quantile functions of two models associated with $X(\theta)$ and $Y(\theta)$ are $Q_{\theta}(u)$ and $R_{\theta}(u)$ with respective quantile density functions $q_{\theta}(u)$ and $r_{\theta}(u)$. Retaining the previous notation, we can write

$$Q(u) = Q_{\theta}(1 - (1-u)^{\theta}) \quad \text{and} \quad Q_{\theta}(u) = Q(1 - (1-u)^{1/\theta}).$$

Theorem 5.2. *We have*

$$\begin{aligned} X \le_{\mathrm{TTT}-n} Y \quad &\Longrightarrow \quad X(\theta) \le_{\mathrm{TTT}-n} Y(\theta), \qquad \theta > 1, \\ X(\theta) \le_{\mathrm{TTT}-n} Y(\theta) \quad &\Longrightarrow \quad X \le_{\mathrm{TTT}-n} Y, \qquad \theta < 1. \end{aligned}$$

Proof. Taking $\theta > 1$, $X(\theta) \le_{\text{TTT-}n} Y(\theta)$ is the same as

$$\int_0^v (1-p)^{n+1} q_{\theta}(p)\,\mathrm{d}p \le \int_0^v (1-p)^{n+1} r_{\theta}(p)\,\mathrm{d}p, \tag{5.4}$$

where $v = 1 - (1-u)^{\theta}$. The last inequality reduces to

$$\begin{aligned} &\int_0^{1-(1-u)^{1/\theta}} q(1-(1-p)^{1/\theta})(1-(1-p)^{1/\theta})^{n+1}\theta^{-1}(1-p)^{\theta^{-1}-1}\,\mathrm{d}p \\ &\qquad \le \int_0^{1-(1-u)^{1/\theta}} r(1-(1-p)^{1/\theta})(1-(1-p)^{1/\theta})^{n+1}\theta^{-1}(1-p)^{\theta^{-1}-1}\,\mathrm{d}p, \end{aligned}$$

which is

$$\int_0^u (1-p)^{n+1} q(p)\,\mathrm{d}p \le \int_0^u (1-p)^{n+1} r(p)\,\mathrm{d}p. \tag{5.5}$$

Thus, if (5.5) holds then (5.4) applies, which is the first part of the theorem. The case in which $\theta < 1$ is similar.

To conclude, we note that the nth-order TTT transform presented here has helped to achieve a more explicit understanding of the effect of transforms on the properties of the baseline distribution. It generates new models that are more IFR or DFR and also BS or UBS from models in common use, and adds more flexibility to model choice by adopting quantile functions that do not convert into simple forms of distribution functions. The reliability properties and order relations extend the existing results and leave scope for new ageing classes. Sample counterparts *viz.* the nth-order TTT statistics along with their relationships with the TTT statistic of the original distribution, which is being investigated, can further strengthen the adaptability of the theoretical results in the present work.

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References

AHMED, I. A., LI, X. AND KAYID, M. (2005). The NBUT class of life distributions. *IEEE Trans. Reliab.* **54,** 396–401.

BARLOW, R. E. AND CAMPO, R. (1975). Total time on test processes and applications to failure data analysis. In *Reliability and Fault Tree Analysis*, eds R. E. Barlow *et al.*, SIAM, Philadelphia, pp. 451–481.

BARTOSZEWICZ, J. (1995). Stochastic order relations and the total time on test transform. *Statist. Prob. Lett.* **22,** 103–110.

BARTOSZEWICZ, J. (1996). Tail orderings and the total time on test transform. *Appl. Math.* **24,** 77–86.

BERGMAN, B. (1977). Crossings in the total time on test on test plot. *Scand. J. Statist.* **4,** 171–177.

BERGMAN, B. AND KLEFSJÖ, B. (1984). The total time on test concept and its use in reliability theory. *Operat. Res.* **32,** 596–606.

HAUPT, E. AND SCHABE, H. (1997). The TTT transformation and a new bathtab distribution model. *J. Statist. Planning Infer.* **60,** 229–240.

KLEFSJÖ, B. (1982). The HNBUE and HNWUE classes of life distributions. *Naval Res. Logistics* **29,** 331–344.

KOCHAR, S. C., LI, X. AND SHAKED, M. (2002). The total time on test transform and the excess wealth stochastic orders of distributions. *Adv. Appl. Prob.* **34,** 826–845.

LAI, C. D. AND XIE, M. (2006). *Stochastic Ageing and Dependence for Reliability*. Springer, New York.

LI, H. AND SHAKED, M. (2007). A general family of univariate stochastic orders. *J. Statist. Planning Infer.* **137,** 3601–3610.

LI, X. AND ZOU, M. (2004). Preservation of stochastic orders for random minima and maxima with applications. *Naval Res. Logistics* **51,** 332–334.

NAIR, N. U. AND SANKARAN, P. G. (2008). Quantile based reliability analysis. *Commun. Statist. Theory Meth.* **38,** 222–232.

NANDA, A. K. AND SHAKED, M. (2008). Partial ordering and aging properties of order statistics when sample size is random: a brief review. *Commun. Statist. Theory Meth.* **37,** 1710–1720.

PARZEN, E. (1979). Nonparametric statistical data modeling. *J. Amer. Statist. Assoc.* **74,** 105–131.

PHAM, T. G. AND TURKKAN, M. (1994). The Lorenz and the scaled total-time-on-test transform curves: a unified approach. *IEEE Trans. Reliab.* **43,** 76–84.

SHAKED, M. AND SHANTHIKUMAR, J. G. (2007). *Stochastic Orders*. Springer, New York.

VERA, F. AND LYNCH, J. (2005). K-mart stochastic modeling using iterated total time on test transforms. In *Modern Statistical and Mathematical Methods in Reliability*, World Scientific, Singapore, pp. 395–409.