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ON KRULL–SCHMIDT FINITELY ACCESSIBLE CATEGORIES

SEPTIMIU CRIVEI

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Abstract

Let $\mathcal{C}$ be a finitely accessible additive category with products, and let $(U_i)_{i\in I}$ be a family of representative classes of finitely presented objects in $\mathcal{C}$ such that each object U_i is pure-injective. We show that $\mathcal{C}$ is a Krull–Schmidt category if and only if every pure epimorphic image of the objects U_i is pure-injective.

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1. Introduction

Accessible categories in the sense of [1] and, in particular, finitely accessible categories, have been reconsidered in the 1990s due to their important applications in different areas of mathematics, and especially in model theory and homotopy theory. Any finitely accessible additive category $\mathcal{C}$ may be embedded as a full subcategory of the category $\mathrm{Mod}(A)$ of unitary right modules over the functor ring A of $\mathcal{C}$ such that the pure exact sequences in $\mathcal{C}$ are those which become exact sequences in $\mathrm{Mod}(A)$ through the embedding (for example, see [1, 6, 17]). Then $\mathcal{C}$ may be seen as being equivalent to the full subcategory of the category $\mathrm{Mod}(A)$ consisting of flat right A-modules. This equivalence offers the main technique for translating properties of modules over the functor ring of $\mathcal{C}$ to properties of the finitely accessible category $\mathcal{C}$. Such a method has been used for studying various properties of finitely accessible categories, especially related to purity, but not restricted to it: pure-semisimplicity [18], locally finite representation type [9], Krull–Schmidt property [4] or existence of flat covers [8], to mention just few of them.

A fundamental result in module theory is the classical Osofsky theorem, which characterises semisimple rings as those rings for which every cyclic module is injective, or equivalently, every finitely generated module is injective [15]. Since its appearance in the 1960s, several generalisations have been considered in the literature.

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Some of the most important ones have been its module counterpart called the Osofsky–Smith theorem [16], a ring version by Gómez Pardo and Guil Asensio [12] where injectivity is replaced by pure-injectivity, and some categorical version for locally finitely generated Grothendieck categories given by Gómez Pardo *et al.* [11]. Also, in a recent paper, Crivei *et al.* [7] discussed the Osofsky–Smith theorem in locally finitely generated Grothendieck categories.

The present paper has the following generalisation of the Osofsky theorem as starting point: if R is a right pure-injective ring, then R is semiperfect if and only if every pure epimorphic image of R is right pure-injective [12, Corollary 2.3]. As noted in [12], a ring whose cyclic right modules are injective is von Neumann regular, and the Osofsky theorem follows immediately from the previous result. We shall be interested in establishing a similar theorem in finitely accessible additive categories with products. We shall first generalise the above characterisation to the case of a ring with enough idempotents, and then we shall use functor ring techniques to pass to finitely accessible additive categories. Recall that a category is called semiperfect if every finitely generated object has a projective cover. Let us note that in finitely accessible additive categories there are enough pure-injective objects (see [13]), while nontrivial projective objects (and so projective covers) might be missing completely, even in the case of a Grothendieck category. For instance, if Q is the infinite quiver $\bullet \to \bullet \to \cdots$, K is a field, and $\mathrm{Rep}_K(Q)$ denotes the Grothendieck K-category of all K-linear representations of Q, then the full K-subcategory of $\mathrm{Rep}_K(Q)$ consisting of all locally finite-dimensional representations (that is, directed unions of finite-dimensional representations) has no nonzero projective object (see [14]). On the other hand, it is well known that every Krull–Schmidt ring is semiperfect. We shall see that Krull–Schmidt categories are suitable for replacing semiperfectness in order to generalise results to finitely accessible additive categories with products. Our main theorem will be the following one: if $\mathcal{C}$ is a finitely accessible additive category with products, and $(U_i)_{i\in I}$ is a family of representative classes of finitely presented objects in $\mathcal{C}$ such that each object U_i is pure-injective, then $\mathcal{C}$ is a Krull–Schmidt category if and only if every pure epimorphic image of the objects U_i is pure-injective.

2. Finitely accessible categories

Throughout all categories and functors will be additive. We recall, mainly from [1, 17], some terminology on finitely accessible categories. An additive category $\mathcal{C}$ is called *finitely accessible* if it has direct limits, the class of finitely presented objects is skeletally small, and every object is a direct limit of finitely presented objects. Also, $\mathcal{C}$ is called *locally finitely presented* if it is finitely accessible and cocomplete (that is, it has all colimits), or equivalently, it is finitely accessible and complete (that is, it has all limits). For instance, the category $\mathrm{Mod}(A)$ of unitary right modules over a ring A with enough idempotents is a locally finitely presented Grothendieck category.

Let $\mathcal{C}$ be a finitely accessible additive category. By a *sequence*

$$0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$$

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in $\mathcal{C}$ we mean a pair of composable morphisms $f : X \to Y$ and $g : Y \to Z$ such that $gf = 0$. The framework of a finitely accessible additive category is a natural one in which to consider and study purity-related notions. The above sequence in $\mathcal{C}$ is called *pure exact* if it induces an exact sequence of abelian groups $0 \to \mathrm{Hom}_{\mathcal{C}}(P, X) \to \mathrm{Hom}_{\mathcal{C}}(P, Y) \to \mathrm{Hom}_{\mathcal{C}}(P, Z) \to 0$ for every finitely presented object P of $\mathcal{C}$. This implies that f and g form a kernel–cokernel pair, that f is a monomorphism and g an epimorphism. In such a pure exact sequence f is said to be a *pure monomorphism* and g a *pure epimorphism*. Pure-injectivity in $\mathcal{C}$ is defined in the usual way.

As already mentioned above, any finitely accessible additive category $\mathcal{C}$ may be embedded as a full subcategory of $\mathrm{Mod}(A)$, where A is a ring with enough idempotents, called the *functor ring* of $\mathcal{C}$, such that the pure exact sequences in $\mathcal{C}$ are those which become exact sequences in $\mathrm{Mod}(A)$ through the embedding. Then $\mathcal{C}$ may be seen as being equivalent to the full subcategory $\mathrm{Fl}(\mathrm{Mod}(A))$ of the category $\mathrm{Mod}(A)$ consisting of flat right A-modules. The functor ring A of $\mathcal{C}$ is constructed as follows. If $(U_i)_{i\in I}$ is a representative set of finitely presented objects of $\mathcal{C}$, then

$$A = \bigoplus_{i\in I} \bigoplus_{j\in J} \mathrm{Hom}_{\mathcal{C}}(U_i, U_j)$$

as abelian group, with multiplication given by the rule: if $f \in \mathrm{Hom}_{\mathcal{C}}(U_i, U_j)$ and $g \in \mathrm{Hom}_{\mathcal{C}}(U_k, U_l)$, then $fg = f \circ g$ if $i = l$ and zero otherwise. Then A is a ring with enough idempotents, say $A = \bigoplus_{i\in I} e_i A = \bigoplus_{i\in I} Ae_i$, where the idempotents e_i are the elements of A which are the identity on U_i and zero elsewhere, and they form a complete family of pairwise orthogonal idempotents. The equivalence between $\mathcal{C}$ and $\mathrm{Fl}(\mathrm{Mod}(A))$ is induced by the (Yoneda) functor $H : \mathcal{C} \to \mathrm{Mod}(A)$, given on objects by

$$H(X) = \bigoplus_{i\in I} \mathrm{Hom}_{\mathcal{C}}(U_i, X).$$

Note that $H(U_i) \cong e_i A$ for each $i \in I$, and $(e_i A)_{i\in I}$ is a family of finitely generated projective generators of the category $\mathrm{Mod}(A)$. By a module we shall always understand a unitary module.

We begin with two categorical preliminary results.

LEMMA 2.1. *Let $F : \mathcal{A} \to \mathcal{B}$ be a functor between locally finitely presented additive categories such that F is left or right exact and preserves direct limits. Then F preserves pure exact sequences.*

PROOF. Let $0 \to X \to Y \to Z \to 0$ be a pure exact sequence in $\mathcal{A}$. Then it is a direct limit of split exact sequences

$$0 \to X_i \xrightarrow{\alpha_i} Y_i \xrightarrow{\beta_i} Z_i \to 0$$

in $\mathcal{A}$ for $i \in I$ [17, Theorem 5.2]. Assume that F is right exact, the case when F is left exact being analogous. Then we have the induced exact sequence

$$F(X_i) \xrightarrow{F(\alpha_i)} F(Y_i) \xrightarrow{F(\beta_i)} F(Z_i) \to 0$$

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in $\mathcal{B}$. Since α_i is a section (split monomorphism) in $\mathcal{A}$, $F(\alpha_i)$ is a section in $\mathcal{B}$. Moreover, since each (finitely) accessible category has split idempotents [1, 2.4], $\mathcal{B}$ must be weakly idempotent complete, in the sense that every section is a kernel (for example see [3]). Hence $F(\alpha_i)$ is a kernel in $\mathcal{B}$. Now using [19, Ch. IV, Proposition 2.4], which holds in additive categories, we must have split exact sequences

$$0 \to F(X_i) \xrightarrow{F(\alpha_i)} F(Y_i) \xrightarrow{F(\beta_i)} F(Z_i) \to 0$$

in $\mathcal{B}$ for each $i \in I$. Now take the direct limit of these split exact sequences. Note that the direct limit functor is exact by [17, Corollaries 3.7 and 3.13] and F preserves direct limits. Hence we have the exact sequence

$$0 \to F(X) \xrightarrow{F(\alpha)} F(Y) \xrightarrow{F(\beta)} F(Z) \to 0.$$

Again by [17, Theorem 5.2], the direct limit of split exact sequences is a pure exact sequence in $\mathcal{B}$, which finishes the proof. □

LEMMA 2.2. *Let $F : \mathcal{A} \to \mathcal{B}$ be a functor between locally finitely presented additive categories having a right adjoint $R : \mathcal{B} \to \mathcal{A}$. Then R preserves pure-injective objects.*

PROOF. Let B be a pure-injective object in $\mathcal{B}$ and let $0 \to X \to Y \to Z \to 0$ be a pure exact sequence in $\mathcal{A}$. The adjoint pair (F, R) ensures that F is right exact and preserves direct limits, hence Lemma 2.1 yields the pure exact sequence $0 \to F(X) \to F(Y) \to F(Z) \to 0$ in $\mathcal{B}$. This induces the exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{B}}(F(Z), B) \to \mathrm{Hom}_{\mathcal{B}}(F(Y), B) \to \mathrm{Hom}_{\mathcal{B}}(F(X), B) \to 0.$$

Using the adjointness we have the exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{A}}(Z, R(B)) \to \mathrm{Hom}_{\mathcal{A}}(Y, R(B)) \to \mathrm{Hom}_{\mathcal{A}}(X, R(B)) \to 0$$

which shows that $R(B)$ is pure-injective in $\mathcal{A}$. □

3. Complete pure-injectivity

Let $\mathcal{C}$ be a finitely accessible additive category. In the same manner as in [12], let us call an object C of $\mathcal{C}$ *completely pure-injective* if every pure epimorphic image of C is pure-injective. We shall need the following lemma.

LEMMA 3.1. *Let ${}_BM_A$ be a bimodule which is completely pure-injective as a right A-module. Consider the adjoint functors $T = -\bigotimes_B M : \mathrm{Mod}(B) \to \mathrm{Mod}(A)$ and $H = \mathrm{Hom}_A(M, -) : \mathrm{Mod}(A) \to \mathrm{Mod}(B)$. If the restriction of HT to the full subcategory of $\mathrm{Mod}(B)$ consisting of flat objects yields the identity, then B is a completely pure-injective right B-module.*

PROOF. Let $B \to Z$ be a pure epimorphism in $\mathrm{Mod}(B)$. Since T is right exact and preserves direct limits, Lemma 2.1 yields a pure epimorphism $M \cong T(B) \to T(Z)$

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in Mod(A). Then $T(Z)$ is pure-injective in Mod(A) by hypothesis. Since B is a flat right B-module, so is Z. Then by Lemma 2.2, $Z \cong HT(Z)$ is a pure-injective right B-module. Hence B is completely pure-injective in Mod(B). □

Now we are in a position to show a generalisation of [12, Corollary 2.3] from rings with identity to rings with enough idempotents. Note that the original proof uses the fact that the ring has identity, so that we need a different approach.

THEOREM 3.2. *Let $A = \bigoplus_{i \in I} e_i A = \bigoplus_{i \in I} Ae_i$ be a ring with enough idempotents such that each $e_i A$ is right pure-injective. Then the following are equivalent.*

(i) *Each $e_i A$ is right completely pure-injective.*
(ii) *A is semiperfect.*

PROOF. (i) ⇒ (ii). Assume that each $e_i A$ is right completely pure-injective. For each $i \in I$, denote $S_i = \mathrm{End}(e_i A)$ and consider the adjoint functors

$$T_i = - \bigotimes_{S_i} e_i A : \mathrm{Mod}(S_i) \to \mathrm{Mod}(A),$$
$$H_i = \mathrm{Hom}_A(e_i A, -) : \mathrm{Mod}(A) \to \mathrm{Mod}(S_i).$$

Since $e_i A$ is finitely presented, the restriction of $H_i T_i$ to the full subcategory of Mod(S_i) consisting of flat objects yields the identity [12, Lemma 2.4]. Then by Lemma 3.1, each $S_i = \mathrm{End}(e_i A)$ is completely pure-injective, and consequently, semiperfect by [12, Corollary 2.3]. Now by [5, Theorem 4.4], A is semiperfect.

(ii) ⇒ (i). Assume that A is semiperfect. Each $e_i A$ is finitely generated, and so semiperfect by [20, 49.10]. Now let $e_i A \to N$ be a pure epimorphism in Mod(A). Then N has a projective cover, say $P \to N$. However, since N is flat and is generated by a finitely presented module, it follows that $P \cong N$ by [20, 36.4]. Hence N is a direct summand of $e_i A$, and so N is pure-injective. Thus each $e_i A$ is completely pure-injective in Mod(A). □

COROLLARY 3.3. *Let A be a right pure-injective ring with enough idempotents. Then the following are equivalent.*

(i) *A is right completely pure-injective.*
(ii) *A is semiperfect.*

PROOF. (i) ⇒ (ii). If $A = \bigoplus_{i \in I} e_i A = \bigoplus_{i \in I} Ae_i$ is right (completely) pure-injective, then clearly so is each $e_i A$. Now use Theorem 3.2.

(ii) ⇒ (i). If A is semiperfect, then by an argument similar to that in the implication (ii) ⇒ (i) from Theorem 3.2, it follows that A is right completely pure-injective. □

REMARK 3.4. As already observed in [12], the classical Osofsky theorem is a consequence of a result of type Corollary 3.3. Indeed, if every cyclic right R-module is injective, then the ring R is von Neumann regular [15, Lemma 1], and so every submodule of a module is pure, and R is completely injective. Then R is semiperfect

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by Corollary 3.3, and so every (pure) right ideal of R is a direct summand, which shows that R is semisimple.

In what follows we shall prove a corresponding result for finitely accessible additive categories with products. But first let us see how (completely) pure-injective objects in such a category relate to (completely) pure-injective right modules over its functor ring.

THEOREM 3.5. *Let $\mathcal{C}$ be a finitely accessible additive category with products, and let A be its functor ring. The equivalence between $\mathcal{C}$ and* $\mathrm{Fl}(\mathrm{Mod}(A))$ *restricts to an equivalence between the full subcategories of:*

(i) *pure-injective objects of $\mathcal{C}$ and pure-injective flat objects of* $\mathrm{Mod}(A)$*;*

(ii) *completely pure-injective objects of $\mathcal{C}$ and completely pure-injective flat objects of* $\mathrm{Mod}(A)$.

PROOF. (i) By [13, Lemma 3], there is an equivalence between the pure-injective objects of $\mathcal{C}$ and the cotorsion flat objects of $\mathrm{Mod}(A)$, induced by the usual functor $H:\mathcal{C}\to\mathrm{Mod}(A)$. Recall that a module C is called cotorsion if $\mathrm{Ext}^1_A(F,C)=0$ for every flat module F [21]. Since $\mathcal{C}$ has products, the category $\mathrm{Mod}(A^{\mathrm{op}})$ of unitary left modules over the functor ring A of $\mathcal{C}$ is locally coherent (for example, see [17, Theorem 6.1]), and one shows in the usual way that any cotorsion flat right A-module is pure-injective (for example, see [21, Lemma 3.2.3]).

(ii) First let Y be a completely pure-injective object in $\mathcal{C}$. Let $H(Y)\to N$ be a pure epimorphism in $\mathrm{Mod}(A)$. Then N is flat in $\mathrm{Mod}(A)$, and so $N\cong H(Z)$ for some object Z of $\mathcal{C}$. Moreover, we have an induced pure exact sequence $0\to X\to Y\to Z\to 0$ in $\mathcal{C}$. Since Z is pure-injective in $\mathcal{C}$, $N\cong H(Z)$ is pure-injective in $\mathrm{Mod}(A)$ by (i). This shows that $H(Y)$ is completely pure-injective in $\mathrm{Mod}(A)$.

Now let $N\cong H(Y)$ be a completely pure-injective flat object in $\mathrm{Mod}(A)$ for some object Y of $\mathcal{C}$. Let $0\to X\to Y\to Z\to 0$ be a pure exact sequence in $\mathcal{C}$. Then the sequence $0\to H(X)\to H(Y)\to H(Z)\to 0$ is pure exact in $\mathrm{Mod}(A)$, hence $H(Z)$ is pure-injective in $\mathrm{Mod}(A)$ by hypothesis. Thus Z is pure-injective in $\mathcal{C}$ by (i), and so Y is completely pure-injective in $\mathcal{C}$. □

Recall that a finitely accessible additive category with products is called *Krull–Schmidt* if every finitely presented object is a finite direct sum of indecomposable objects with local endomorphism ring [4].

THEOREM 3.6. *Let $\mathcal{C}$ be a finitely accessible additive category with products, and let $(U_i)_{i\in I}$ be a representative set of finitely presented objects of $\mathcal{C}$. Assume that each object U_i is pure-injective. Then the following are equivalent.*

(i) *Each U_i is completely pure-injective.*

(ii) *$\mathcal{C}$ is Krull–Schmidt.*

PROOF. Let $A=\bigoplus_{i\in I} e_iA=\bigoplus_{i\in I} Ae_i$ be the functor ring of $\mathcal{C}$. Consider the usual functor $H:\mathcal{C}\to\mathrm{Mod}(A)$. Then each $e_iA\cong H(U_i)$ is pure-injective in $\mathrm{Mod}(A)$ by Theorem 3.5.

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(i) $\Rightarrow$ (ii). Assume that each U_i is completely pure-injective. By Theorem 3.5, each $e_i A \cong H(U_i)$ is now completely pure-injective in $\mathrm{Mod}(A)$. Then A is semiperfect by Theorem 3.2, and so $\mathcal{C}$ is Krull–Schmidt (for example [4, Theorem 4.1.15]).

(ii) $\Rightarrow$ (i). Assume that $\mathcal{C}$ is Krull–Schmidt. By [4, Theorem 4.1.15], A is semiperfect. Then each $e_i A$ is right completely pure-injective in $\mathrm{Mod}(A)$ by Theorem 3.2. Finally, each U_i is completely pure-injective in $\mathcal{C}$ by Theorem 3.5. □

As a consequence, we may extend [12, Corollary 2.6].

Corollary 3.7. *Let A be a ring with enough idempotents such that each finitely presented right A-module is pure-injective. Then the following are equivalent.*

(i) *Each finitely presented right A-module is completely pure-injective.*
(ii) *A is Krull–Schmidt.*
(iii) *A is semiperfect.*

Proof. (i) $\Rightarrow$ (ii). This implication follows from Theorem 3.6.

(ii) $\Rightarrow$ (iii). This is well known for rings with identity (for example see [10, p. 97]), without any further assumption on A. We recall a short proof for completeness. Let P be a finitely generated projective right A-module. Since A is Krull–Schmidt, we may write $P = \bigoplus_{i=1}^{n} P_i$ for some modules P_i with local endomorphism ring. However, since each P_i is projective, each P_i is local (for example [2, Proposition 3.7]). Hence A is semiperfect by [20, 49.10].

(iii) $\Rightarrow$ (i). The proof of this implication is similar to that for the implication (ii) $\Rightarrow$ (i) from Theorem 3.2. □

We end the paper with the following related application (see also [11, Theorem 1]).

Theorem 3.8. *Let $\mathcal{C}$ be a finitely accessible Grothendieck category, and let $(U_i)_{i\in I}$ be a representative set of finitely presented objects of $\mathcal{C}$. Assume that each object U_i is completely pure-injective and the functor ring A of $\mathcal{C}$ is von Neumann regular. Then A is semisimple.*

Proof. Let $A = \bigoplus_{i\in I} e_i A = \bigoplus_{i\in I} A e_i$ and consider the usual functor $H : \mathcal{C} \to \mathrm{Mod}(A)$. By Theorem 3.5, each $e_i A \cong H(U_i)$ is completely pure-injective, and consequently completely injective, because A is von Neumann regular. Hence $(e_i A)_{i\in I}$ is a family of completely injective finitely generated generators of $\mathrm{Mod}(A)$, and so A is semisimple by [7, Theorem 2.10]. □

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SEPTIMIU CRIVEI, Faculty of Mathematics and Computer Science,
'Babeş-Bolyai' University, Str. M. Kogălniceanu 1, 400084 Cluj-Napoca, Romania
e-mail: crivei@math.ubbcluj.ro

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