CORRESPONDENCE.

ON THE FORMULAS FOR THE APPROXIMATE DETERMINATION OF THE RATE OF INTEREST OF AN ANNUITY.

To the Editor of the Journal of the Institute of Actuaries.

Sir,—Having recently had occasion to examine the ordinary formulas for the determination, with the help of tables, of the rate of interest involved in the value of a term annuity, I venture to send you an account of certain modifications of them which appear to be desirable, in the hope that it may not be without interest to readers of the *Journal.*

The problem to be considered is as follows. Let a be the present value at an unknown rate of interest, x, of an annuity of 1 for n years. Let a' be the value of a similar annuity at a rate of interest, i, being the tabulated value which approaches most nearly to a. Let $\rho = x - i$. Required, an expression for the value of ρ in terms of the above known quantities, or others depending upon them.

The best known formulas for this purpose are, that to be found at page 42 of Mr. Jones's work on annuities, previously given by Baily, and the more exact one given by Professor De Morgan at page 67 of the eighth volume of the *Journal.* The well known expression for the value of an annuity for n years, at a rate of interest of $i+\rho$, is $\frac{1-(1+i+\rho)^{-n}}{i+\rho}$. Equating this to a, and multiplying both sides of the equation by $i+\rho$, we obtain

$$(i+\rho)a = 1-(1+i+\rho)^{-n};$$

and from this form of the equation, Mr. Jones, and apparently Professor De Morgan, have derived the formulas given by them. Professor De Morgan's formula may be thus obtained. Expanding the right hand side of the equation by the binomial theorem, and rejecting terms involving powers of ρ above the second, we have

$$(i+\rho)a = 1-(1+i)^{-n} + n(1+i)^{-(n+1)}\rho - \frac{n(n+1)}{2}(1+i)^{-(n+2)}\rho^2.$$

Transposing, and writing v for $(1+i)^{-1}$, we have

$$ai = 1 - v^n - (a - nv^{n+1})\rho - \frac{n(n+1)}{2}v^{n+2}\rho^2,$$

that is,

$$\frac{n(n+1)}{2}v^{n+2}\rho^2 + (a - nv^{n+1})\rho - (a'-a)i = 0.$$

One of the roots of this equation is

$$\frac{-(a-nv^{n+1}) + \sqrt{(a-nv^{n+1})^2 + 4i(a'-a)\frac{n(n+1)}{2}v^{n+2}}}{2\left\{\frac{n(n+1)}{2}v^{n+2}\right\}}.$$

Multiplying numerator and denominator of this fraction by

$$(a-nv^{n+1}) + \sqrt{(a-nv^{n+1})^2 + 4i(a'-a)\frac{n(n+1)}{2}v^{n+2}},$$

it is reduced to

$$\frac{2i(a'-a)}{(a-nv^{n+1}) + \sqrt{(a-nv^{n+1})^2 + 4i(a'-a)\frac{n(n+1)}{2}v^{n+2}}}.$$

Putting the second term of the denominator into the form

https://doi.org/10.1017/S2046167400044785 Published online by Cambridge University Press

$$\sqrt{(a-nv^{n+1})^2+4(a-nv^{n+1})\frac{n(n+1)}{2}\cdot\frac{i(a'-a)v^{n+2}}{a-nv^{n+1}}},$$

we see that an approximate value is

$$(a-nv^{n+1})+2\frac{n(n+1)}{2}\cdot\frac{i(a'-a)v^{n+2}}{a-nv^{n+1}}.$$

Substituting this value, the root becomes

$$\frac{i(a'-a)}{(a-nv^{n+1})+\frac{n(n+1)}{2}\cdot\frac{i(a'-a)v^{n+2}}{a-nv^{n+1}}} \quad . \quad . \quad . \quad (1)$$

This is the formula given by Professor De Morgan, while, as pointed out by him, if the second term of the denominator be neglected, we obtain a formula which by a simple transformation may be shown to be equivalent to that given by Mr. Jones.

It is to be observed, however, that the multiplication of both sides of the fundamental equation, $a=\frac{1-(1+i+\rho)^{-n}}{i+\rho}$ by $i+\rho$, which forms the first step in the above process, is objectionable, as involving a considerable sacrifice of accuracy. Its effect is to increase the coefficient of each power of ρ by $\frac{1}{i}$th part of the coefficient of the next lower power, a portion of the coefficient of ρ^2 thus becoming involved in that of ρ^3, and left out of account in obtaining the solution of the equation. A more correct result will be obtained by expanding the right hand side of the equation in ascending powers of ρ. This may be done by the binomial theorem, thus,

$$\frac{1-(1+i+\rho)^{-n}}{i+\rho}=\{1-(1+i+\rho)^{-n}\}(i+\rho)^{-1}$$

$$=\left\{1-v^n+nv^{n+1}\rho-\frac{n(n+1)}{2}v^{n+2}\rho^2\right\}\left\{\frac{1}{i}-\frac{\rho}{i^2}+\frac{\rho^2}{i^3}\right\}$$

$$=\frac{1-v^n}{i}+\frac{nv^{n+1}}{i}\rho-\frac{1-v^n}{i^2}\rho-\frac{n(n+1)v^{n+2}}{2i}\rho^2-\frac{nv^{n+1}}{i^2}\rho^2+\frac{1-v^n}{i^3}\rho^3$$

$$=a'-\frac{1}{i}(a'-nv^{n+1})\rho-\frac{1}{i^2}\left\{\frac{n(n+1)iv^{n+2}}{2}-(a'-nv^{n+1})\right\}\rho^2$$

This gives for the complete equation,

$$\frac{1}{i^2}\left\{\frac{n(n+1)iv^{n+2}}{2}-(a'-nv^{n+1})\right\}\rho^2+\frac{1}{i}(a'-nv^{n+1})\rho-(a'-a)=0.$$

It will be satisfactory to show how this result may be obtained by a different method.

Expanding by the binomial theorem, the common expression for the value of an annuity for n years at a rate of interest, i, becomes

$$\frac{1}{i}-\frac{1}{i}\left\{1-ni+\frac{n(n+1)}{2}i^2-\frac{n(n+1)(n+2)}{2.3}i^3+\ldots.\right\}$$

$$=n-\frac{n(n+1)}{2}i+\frac{n(n+1)(n+2)}{2.3}i^2-\ldots.$$

and the difference between such value and the value of a similar annuity at a rate of interest, $i+\rho$, is seen to be

$$-\frac{n(n+1)}{2}\{i-(i+\rho)\}+\frac{n(n+1)(n+2)}{2.3}\{i^2-(i+\rho)^2\}-\ldots\ldots$$

$$=\frac{n(n+1)}{2}\{(i+\rho)-i\}-\frac{n(n+1)(n+2)}{2.3}\{(i+\rho)^2-i^2\}+\ldots\ldots$$

ρ being a small quantity, its powers of a higher order than the second may be neglected, and the expression last obtained then becomes

$$\frac{n(n+1)}{2}\rho-\frac{n(n+1)(n+2)}{2.3}(2i\rho+\rho^2)+\frac{n(n+1)(n+2)(n+3)}{2.3.4}(3i^2\rho+3i\rho^2)$$
$$-\frac{n(n+1)(n+2)(n+3)(n+4)}{2.3.4.5}(4i^3\rho+6i^2\rho^2)+\ldots\ldots$$

The coefficient of ρ in this series is

$$\frac{n(n+1)}{2}-\frac{n(n+1)(n+2)}{2.3}2i+\frac{n(n+1)(n+2)(n+3)}{2.3.4}3i^2$$
$$-\frac{n(n+1)(n+2)(n+3)(n+4)}{2.3.4.5}4i^3+\ldots\ldots$$

which may be written

$$\frac{n(n+1)}{2}2-\frac{n(n+1)(n+2)}{2.3}3i+\frac{n(n+1)(n+2)(n+3)}{2.3.4}4i^2$$
$$-\frac{n(n+1)(n+2)(n+3)(n+4)}{2.3.4.5}5i^3+\ldots\ldots$$

$$-\frac{n(n+1)}{2}+\frac{n(n+1)(n+2)}{2.3}i-\frac{n(n+1)(n+2)(n+3)}{2.3.4}i^2$$
$$+\frac{n(n+1)(n+2)(n+3)(n+4)}{2.3.4.5}i^3-\ldots\ldots$$

Adding to this the neutral quantity

$$\frac{1}{i^2}-\frac{n}{i}-\frac{1}{i^2}+\frac{n}{i},$$

we obtain

$$\frac{1}{i^2}-\frac{n}{i}+n(n+1)-\frac{n(n+1)(n+2)}{2}i+\frac{n(n+1)(n+2)(n+3)}{2.3}i^2$$
$$-\frac{n(n+1)(n+2)(n+3)(n+4)}{2.3.4}i^3+\ldots\ldots$$

$$-\frac{1}{i^2}+\frac{n}{i}-\frac{n(n+1)}{2}+\frac{n(n+1)(n+2)}{2.3}i-\frac{n(n+1)(n+2)(n+3)}{2.3.4}i^2$$
$$+\frac{n(n+1)(n+2)(n+3)(n+4)}{2.3.4.5}i^3-\ldots\ldots$$

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$$=\frac{1}{i^2}-\frac{n}{i}\left\{1-(n+1)i+\frac{(n+1)(n+2)}{2}i^2-\frac{(n+1)(n+2)(n+3)}{2.3}i^3+\ldots.\right\}$$

$$-\frac{1}{i^2}\left\{1-ni+\frac{n(n+1)}{2}i^2-\frac{n(n+1)(n+2)}{2.3}i^3+\ldots.\right\}$$

$$=\frac{1}{i^2}-\frac{nv^{n+1}}{i}-\frac{v^n}{i^2}=\frac{1}{i}\left\{\frac{1-v^n}{i}-nv^{n+1}\right\}$$

$$=\frac{1}{i}(a'-nv^{n+1}).$$

The coefficient of ρ^2 in the original equation is

$$-\frac{n(n+1)(n+2)}{2.3}+\frac{n(n+1)(n+2)(n+3)}{2.3.4}3i$$
$$-\frac{n(n+1)(n+2)(n+3)(n+4)}{2.3.4.5}6i^2+\ldots.$$

Bearing in mind that the first order of differences of a series of figurate numbers is the series of the next lower order, we may write this expression in the form

$$-\frac{n(n+1)(n+2)}{2.3}3+\frac{n(n+1)(n+2)(n+3)}{2.3.4}6i$$
$$-\frac{n(n+1)(n+2)(n+3)(n+4)}{2.3.4.5}10i^2+\ldots.$$

$$+\frac{n(n+1)(n+2)}{2.3}2-\frac{n(n+1)(n+2)(n+3)}{2.3.4}3i$$
$$+\frac{n(n+1)(n+2)(n+3)(n+4)}{2.3.4.5}4i^2-\ldots.$$

$$=-\frac{n(n+1)(n+2)}{2.2.3}6+\frac{n(n+1)(n+2)(n+3)}{2.2.3.4}12i$$
$$-\frac{n(n+1)(n+2)(n+3)(n+4)}{2.2.3.4.5}20i^2+\ldots.$$

$$+\frac{n(n+1)(n+2)}{2.3}3-\frac{n(n+1)(n+2)(n+3)}{2.3.4}4i$$
$$+\frac{n(n+1)(n+2)(n+3)(n+4)}{2.3.4.5}5i^2-\ldots.$$

$$-\frac{n(n+1)(n+2)}{2.3}+\frac{n(n+1)(n+2)(n+3)}{2.3.4}i$$
$$-\frac{n(n+1)(n+2)(n+3)(n+4)}{2.3.4.5}i^2+\ldots.$$

Adding the neutral quantity,

$$\frac{n(n+1)}{2i}+\frac{n}{i^2}-\frac{n(n+1)}{i}+\frac{1}{i^3}-\frac{n}{i^2}+\frac{n(n+1)}{2i}-\frac{1}{i^3},$$

we obtain

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$$\frac{n(n+1)}{2i}-\frac{n(n+1)(n+2)}{2}+\frac{n(n+1)(n+2)(n+3)}{2.2}i$$
$$-\frac{n(n+1)(n+2)(n+3)(n+4)}{2.2.3}i^2+\ldots\ldots$$
$$+\frac{n}{i^2}-\frac{n(n+1)}{i}+\frac{n(n+1)(n+2)}{2}-\frac{n(n+1)(n+2)(n+3)}{2.3}i+\ldots\ldots$$
$$+\frac{1}{i^3}-\frac{n}{i^2}+\frac{n(n+1)}{2i}-\frac{n(n+1)(n+2)}{2.3}i+\ldots\ldots-\frac{1}{i^3}$$

that is

$$\frac{n(n+1)}{2i}\left\{1-(n+2)i+\frac{(n+2)(n+3)}{2}i^2-\frac{(n+2)(n+3)(n+4)}{2.3}i^3+\ldots\ldots\right\}$$
$$+\frac{n}{i^2}\left\{1-(n+1)i+\frac{(n+1)(n+2)}{2}i^2-\frac{(n+1)(n+2)(n+3)}{2.3}i^3+\ldots\ldots\right\}$$
$$+\frac{1}{i^3}\left\{1-ni+\frac{n(n+1)}{2}i^2-\frac{n(n+1)(n+2)}{2.3}i^3+\ldots\ldots\right\}-\frac{1}{i^3}$$
$$=\frac{n(n+1)v^{n+2}}{2i}+\frac{nv^{n+1}}{i^2}+\frac{v^n}{i^3}-\frac{1}{i^3}$$
$$=\frac{1}{i^2}\left\{\frac{n(n+1)iv^{n+2}}{2}-(a'-nv^{n+1})\right\}.$$

The required expression for the difference between the values of the annuities, when the difference between the rates of interest involved is ρ, is thus

$$\frac{1}{i^2}\left\{\frac{n(n+1)iv^{n+2}}{2}-(a'-nv^{n+1})\right\}\rho^2+\frac{1}{i}(a'-nv^{n+1})\rho.$$

Equating this to $a'-a$, we have

$$\frac{1}{i^2}\left\{\frac{n(n+1)iv^{n+2}}{2}-(a'-nv^{n+1})\right\}\rho^2+\frac{1}{i}(a'-nv^{n+1})\rho-(a'-a)=0,$$

which is the result obtained by the former process.

Applying to this equation the method of solution applied to the former, we obtain as the required root

$$\frac{i(a'-a)}{a'-nv^{n+1}+\left\{\frac{n(n+1)i(a'-a)v^{n+2}}{2(a'-nv^{n+1})}-(a'-a)\right\}}$$
$$=\frac{i(a'-a)}{a-nv^{n+1}+\frac{n(n+1)}{2}\cdot\frac{i(a'-a)v^{n+2}}{a'-nv^{n+1}}}\quad . \quad . \quad . \quad (2)$$

This, it will be observed, differs from Professor De Morgan's formula in the second term of the denominator, in which $a'-nv^{n+1}$ takes the place of $a-nv^{n+1}$.

If, following the example of Mr. Jones, we neglect all powers of ρ above the first, we obtain the formula

$$\frac{i(a'-a)}{a'-nv^{n+1}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

in place of that given by him, namely

$$\frac{i(a'-a)}{a-nv^{n+1}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

To ascertain the effect of these modifications, let us take the example given by Professor De Morgan. He takes 21·924788, the value of an annuity for 98 years at $4\frac{1}{2}$ percent, and employs his formula to deduce the rate of interest involved in it from the value of a similar annuity at 4 percent; obtaining as the result ·0449808. The application of the formula (2) gives ·0449944, from which we see that the error resulting from its use is little more than $\frac{1}{4}$th of that which results from the use of Professor De Morgan's formula.

Mr. Jones's formula (4) can only be applied with advantage when the difference between the value of the given annuity and some one of the tabulated values is small. Given 19·474356, the value of an annuity for 30 years, the rate of interest involved when determined by (4) from the value of a similar annuity at 3 percent, comes out ·030506. The formula (3) gives ·030498, which is a considerably closer approximation to the true rate ·0305.

These alterations may seem inconsiderable. I am not, however, aware of the existence of any published notice of them, and I hope the growing importance of the problem to which the formulas are applicable will be held to justify the length of this communication.

I am, Sir,

Your most obedient servant,

26 *St. Andrew Square, Edinburgh.* J. J. McLAUCHLAN.
March 1874.

P.S.—Since the above was in type, Mr. Makeham's paper "On the Solution of Problems connected with Loans repayable by Instalments" has appeared, in which he refers to the above mentioned improvement in formula (4), as due to Barrett; and points out that it is a particular case of a new and more general formula demonstrated by himself. Prof. De Morgan, who was the author of the "Account of a Correspondence between Barrett and Baily", referred to by Mr. Makeham, did not apparently think it necessary, in his subsequent paper, to investigate the principles upon which Barrett's suggestion depended, or their bearing upon formula (1), which latter has the great advantage of giving a practically correct result by one application; and the present must be considered as an attempt in that direction.

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