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THE GROUP OF AUTOMORPHISMS OF A DIFFERENTIAL ALGEBRAIC FUNCTION FIELD

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Abstract

Consider a one-dimensional differential algebraic function field K over an algebraically closed ordinary differential field k of characteristic 0. We shall prove the following theorem:

Suppose that the group of all automorphisms of K over k is infinite. Then, K is either a differential elliptic function field over k or $K = k(v)$ with $v' = \xi$ or $v' = \eta v$, where $\xi, \eta \in k$.

It will not be assumed that the field of constants of K is the same as that of k. If we set this additional assumption, then our result is contained in a theorem due to Kolchin [4, p. 809].

§ 0. Introduction

Let k be an algebraically closed ordinary differential field of characteristic 0, and K be a one-dimensional algebraic function field over k. We shall assume that K is a differential extension of k. Then, K is called a *differential algebraic function field* over k if there exists an element y of K such that $K = k(y, y')$. Let F be an algebraically irreducible element of the differential polynomial algebra $k\{y\}$ of the first order. Then, there exists a differential algebraic function field K over k such that $K = k(y, y')$ and $F(y, y') = 0$. Throughout this note K will denote a differential algebraic function field over k.

We call K a *differential elliptic function field* over k if there exists an element z of K such that $K = k(z, z')$ and

$$(z')^2 = \lambda z(z^2 - 1)(z - \delta);\ \lambda, \delta \in k;\ \lambda \neq 0;\ \delta^2 \neq 0, 1;$$

here δ is a constant.

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THEOREM. *Suppose that the group of all automorphisms of K over k is infinite. Then, K is either a differential elliptic function field over k or $K = k(v)$ with*

$$v' = \xi \quad \text{or} \quad v' = \eta v; \qquad \xi, \eta \in k . \tag{1}$$

We do not assume that the field of constants of K is the same as that of k. If this assumption is set, then our result is contained in a theorem due to Kolchin [4, p. 809].

The author [6] gave a differential-algebraic definition for K to be free from parametric singularities. Some results obtained there will be applied to prove our theorem.

§ 1. Parametric singularities

Let P be a prime divisor of K, and ν_P be the normalized valuation belonging to P. Then, K is said to be *free from parametric singularities* if we have $\nu_P(\tau') \geqq 0$ for each P, where τ is a prime element in P. Let τ_1, τ_2 be two prime elements in P. Then, $\nu_P(\tau_1') \geqq 0$ if and only if $\nu_P(\tau_2') \geqq 0$. We have $\nu_P(\tau_1') > 0$ if and only if $\nu_P(\tau_2') > 0$.

We shall say that K is of *Riccati type* over k if there exists an element t of K such that $K = k(t)$ and

$$t' = a + bt + ct^2; \, a, b, c \in k . \tag{2}$$

If K is either of Riccati type or a differential elliptic function field over k, then it is free from parametric singularities. The following two lemmas are due to the author [6]:

LEMMA 1. *Suppose that K is free from parametric singularities, and that the genus of K is* 0. *Then, K is of Riccati type over k.*

LEMMA 2. *Suppose that K is free from parametric singularities, and the genus of K is* 1. *Then K is a differential elliptic function field over k.*

PROPOSITION 1. *Let Γ be the set of all prime divisors P of K such that $\nu_P(\tau') < 0$. Then, Γ is finite unless it is empty.*

Proof. We shall suppose that $K = k(y, y')$, and that y and y' satisfy an irreducible algebraic equation $F(y, y') = 0$ over k. Assume that P is an element of Γ satisfying $\nu_P(y) \geqq 0$. We have $\nu_P(y - \zeta) > 0$ for a certain element ζ of k. Let $A(y)$ and $D(y)$ denote respectively the leading coefficient of F and the discriminant of F with respect to y'. Then, either $A(\zeta) = 0$ or $D(\zeta) = 0$, because $\nu_P(\tau') < 0$.

§ 2. Riccati's equation

Let Q be a prime divisor of K, and $\Sigma(Q)$ denote the group of all automorphisms of K over k which leave Q invariant.

PROPOSITION 2. *Suppose that the genus of K is* 0, *and that there exists a prime divisor Q of K such that $\Sigma(Q)$ is infinite. Then, $K = k(v)$ with* (1).

Proof. We may take an element t of K such that the principal divisor (t) of K takes the form PQ^{-1}, where P is a prime divisor of K different from Q. Then, $K = k(t)$ and $t' = R(t)/S(t)$, where $R, S \in k[t]$. We assume that $(R, S) = 1$, and that the leading coefficient of S is 1. Let Φ be an element of $\Sigma(Q)$. Then, $\Phi(t) = \alpha t + \beta; \alpha, \beta \in k$. Because of $\{\Phi(t)\}' = \Phi(t')$, we have the identity in t:

$$\alpha' t + \beta' + \alpha R(t)/S(t) = R(\alpha t + \beta)/S(\alpha t + \beta) .$$

Since $\Sigma(Q)$ is infinite, S can not have two roots different from each other. Hence, $S = (t - d)^s; d \in k$. Suppose that $s > 0$. Let u denote $t - d$, and $R^*(u), S^*(u)$ be $R(u + d), S(u + d)$ respectively. Then, $\alpha d + \beta = d$, and

$$\alpha R^*(u) + u^s\{\alpha' u + (1 - \alpha)d'\} = \alpha^{-s} R^*(\alpha u) . \tag{3}$$

Let e be the constant term of R^*. It is not 0, since $(R, S) = 1$. From (3) we have $\alpha e = \alpha^{-s} e$, and $\alpha^{s+1} = 1$. This contradicts our assumption that $\Sigma(Q)$ is infinite. Hence, $s = 0$ and $S = 1$. We have

$$\alpha R(t) + \alpha' t + \beta' = R(\alpha t + \beta) . \tag{4}$$

Suppose that the degree r of R is greater than 1. Then, from (4) we have $\alpha = \alpha^r$. Let c_0, c_1 be the coefficients of t^r, t^{r-1} in R respectively. Then,

$$\alpha c_1 = \alpha^{r-1}(r c_0 \beta + c_1) .$$

This contradicts our assumption. Hence, $r \leq 1$, and $R(t) = a + bt; a, b \in k$. From (4) we have

$$\alpha' = 0 , \qquad \beta' = a(1 - \alpha) + b\beta .$$

If $\alpha = 1$ for any Φ, then $\beta \neq 0$ for a certain Φ and $(t/\beta)' = a/\beta$. If $\alpha \neq 1$ for some Φ, then

$$(t-\gamma)' = b(t-\gamma)\,, \qquad \gamma = \beta(1-\alpha)^{-1}\,.$$

PROPOSITION 3. *Assume that K is of Riccati type over k, and that $K = k(t)$ with* (2). *Suppose that we have an automorphism Φ of K over k taking the form*:

$$\Phi(t) = (\alpha t + \beta)/(t + \varepsilon)\,;\ \alpha, \beta, \varepsilon \in k\,. \tag{5}$$

Then, there exists in k a solution of (2).

Proof. From the identity

$$\{\Phi(t)\}' = a + b\Phi(t) + c\Phi(t)^2$$

in t we have

$$\begin{cases} \alpha' = a + b\alpha + c(\alpha^2 - \alpha\varepsilon + \beta)\,; \\ \beta' = a(\varepsilon - \alpha) + 2b\beta - c(\varepsilon - \alpha)\beta\,; \\ \varepsilon' = -a + b\varepsilon + c(\alpha\varepsilon - \varepsilon^2 - \beta)\,. \end{cases} \tag{6}$$

Let us define an element σ of k as a root of the quadratic equation:

$$\sigma^2 + (\varepsilon - \alpha)\sigma - \beta = 0\,.$$

If the discriminant Δ is not 0, then

$$\sigma' = (2\sigma + \varepsilon - \alpha)^{-1}\{(\alpha' - \varepsilon')\sigma + \beta'\}\,.$$

If $\Delta = 0$, then $\sigma' = (\alpha' - \varepsilon')/2$. Because of (6), σ is a solution of (2) in any case.

§ 3. Proof of Theorem

Consider K as an algebraic function field over k free from the differentiation. Then, the following two theorems are well known (cf. Hurwitz [2], and Iwasawa [3, pp. 117–118], Kolchin [4, pp. 818–819] respectively):

LEMMA 3. *The group of all automorphisms of K over k is finite if the genus of K is greater than* 1.

LEMMA 4. *If the genus of K is* 1, *then $\Sigma(Q)$ is finite for any Q.*

Proof of Theorem. By Lemma 3, we may assume that the genus of

K is either 0 or 1. Let Γ be the set in Proposition 1. We shall prove that Γ is empty. To the contrary suppose that Γ is not empty. Then, it is finite by Proposition 1. The set Γ is left invariant by any automorphism of K over k. Hence, there exists an element Q of Γ such that $\Sigma(Q)$ is infinite. By Lemma 4, the genus of K is 0. By Proposition 2, $K=k(v)$ with (1). It is of Riccati type over k and free from parametric singularities. This contradicts our assumption. Hence, Γ is empty. By Lemma 1 and Lemma 2, K is either of Riccati type or a differential elliptic function field over k. Assume that $K=k(t)$ with (2). We have $(t)=PQ^{-1}$ with certain prime divisors P, Q of K. Suppose that any automorphism Φ of K over k does not take the form (5). Then, $\Sigma(Q)$ is infinite. Hence, in this case, we have $K=k(v)$ with (1) by Proposition 2. Suppose that some automorphism Φ of K over k takes the form (5). Then, by Proposition 3, there exists an element σ of k which satisfies (2). For each element η of k, let $P(\eta)$ denote the prime divisor of K determined by

$$\nu_{P(\eta)}(t-\eta)>0\,.$$

Then, we have

$$\nu_{P(\eta)}(\tau(\eta)')>0$$

if and only if η is a solution of (2), where $\tau(\eta)$ is a prime element in $P(\eta)$. We shall define the set Λ as that of all prime divisors P^* of K such that $\nu_{P^*}(\tau^{*\prime})>0$, where τ^* is a prime element in P^*. It is not empty, because $P(\sigma)\in\Lambda$. Suppose that Λ is infinite. Then, there exist in k two solutions, σ_1,σ_2 of (2) different from each other. Hence, we have $K=k(v)$ with (1) (cf. Forsyth [1, pp. 192–193]). Suppose that Λ is finite. Then, there exists an element P^* of Λ such that $\Sigma(P^*)$ is infinite, because any automorphism of K over k leaves the set Λ invariant. By Proposition 2, we have $K=k(v)$ with (1).

§4. Automorphisms of a differential elliptic function field

Assume that K is a differential elliptic function field $k(z,z')$ over k with

$$(z')^2=4S(z)=4z(1-z)(1-\kappa^2z);$$

here κ^2 is a constant of k different from 0 and 1. For a pair of elements

a, b of k satisfying $b^2 = S(a)$, let us define $\phi(z, z'; a, b)$ by

$$\phi = \{a(1 - z)(1 - \kappa^2 z) + bz' + z(1 - a)(1 - \kappa^2 a)\}/(1 - \kappa^2 az)^2 .$$

For a pair of (∞, ∞) we shall define ϕ by

$$\phi(z, z'; \infty, \infty) = \kappa^{-2} z^{-1} .$$

Consider K as an elliptic function field over k free from the differentiation. Let Φ be an automorphism of K over k. Then, $\Phi(z)$ takes the form (cf. [4, p. 804], [9, Chap. 3, § 3]:)

$$\Phi(z) = \omega\phi(z, z'; a, b) + \gamma :$$

Here, (ω, γ) is either $(1, 0)$ or the following pair: $(-1, 0)$ if $\kappa^2 = -1$; $(-1, 1)$ if $\kappa^2 = 2$; $(-1, 2)$ if $\kappa^2 = 1/2$; $(-\kappa^2, 1)$, $(-\kappa^2, \kappa^{-2})$ if $\kappa^4 - \kappa^2 + 1 = 0$. Let y denote $\Phi(z)$. Then, $S(y) = \omega^3 S(\phi)$, and $[K: k(y)] = 2$.

Suppose that $a = \infty$ or $b = 0$. Then, $(y')^2 = 4S(y)$ if and only if $(\omega, \gamma) = (1, 0)$.

Proposition 4. *Suppose that $a \neq \infty$ and $b \neq 0$. Then, $(y')^2 = 4S(y)$ if and only if we have*

$$4(1 - \omega)S(a) + 4ba' + (a')^2 = 0 . \tag{7}$$

Proof. Let us define $\psi(z, z'; a, b)$ by

$$\psi = \phi_z z' + 2\phi_{z'} S_z(z);$$

here $(z')^2$ is replaced by $4S(z)$ and ψ is linear in z'. Then, we obtain the identity in z, z', a, b (cf. [6]):

$$\psi^2 = 4S(\phi) . \tag{8}$$

Let us define $\chi(z, z'; a, b)$ by

$$\chi = 2b\phi_a + \phi_b S_a(a);$$

here b^2 is replaced by $S(a)$ and χ is linear in b. Then, we have the identity in z, z', a, b:

$$\chi = \psi . \tag{9}$$

By the definition of ψ and χ,

$$y' = \omega\{\psi + a'\chi/(2b)\} .$$

Because of (8) and (9), $(y')^2 = 4S(y)$ if and only if we have (7).

COROLLARY. *The group of all automorphisms of K over k is infinite if K is a differential elliptic function field over k.*

In fact we have (7) if $a' = 0$ and $\omega = 1$.

§ 5. Remarks

In the previous section let us assume that the field of constants of K is the same as that of k. Then, $(y')^2 = 4S(y)$, if and only if $(\omega, \gamma) = (1, 0)$ and a is either a constant or ∞. This result is due to Kolchin [4, p. 807].

Suppose that $K = k(y, y')$ with $F(y, y') = 0$, and that K has a transcendental constant c over k. Let P be a prime divisor of K satisfying $\nu_P(c) > 0$, and τ be a prime element in P. Then, $\nu_P(\tau') > 0$. For any constant a of k, $c + a$ is a transcendental constant over k. Hence, infinitely many prime divisors P of K satisfy $\nu_P(\tau') > 0$, and there exist in k infinitely many solutions of $F = 0$ (cf. [8]).

Suppose that $K = k(v)$ with (1). Then, the following four conditions are equivalent (cf. [7], [5] on (iv)):

(i) K has a transcendental constant over k:

(ii) In K we have a solution of $v' = \xi$ or a nontrivial solution $v' = \eta v$:

(iii) There exists an element w of K such that $K = k(w)$ and $w' = 0$:

(iv) We have two elements w_1, w_2 of K such that $K = k(w_1) = k(w_2)$ and $w_1' = \zeta_1$, $w_2' = \zeta_2 w_2$, where $\zeta_1, \zeta_2 \in k$.

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