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Fourier–Mukai transforms and canonical divisors

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Abstract

Let X be a smooth projective variety. We study a relationship between the derived category of X and that of a canonical divisor. As an application, we study Fourier–Mukai transforms when $\kappa(X) = \dim X - 1$.

1. Introduction

Let X be a smooth projective variety and $D(X)$ the bounded derived category of coherent sheaves on X. Recently, $D(X)$ has drawn much attention from many aspects, especially mirror symmetry, moduli spaces of stable sheaves, and birational geometry. Kontsevich [MK94] conjectured the existence of equivalence between the derived category of X and the derived Fukaya category of its mirror. From the physical viewpoint, we cannot distinguish the mirror pair by observations or experiments, so this gives a motivation for the new concept of 'spaces'. In this respect, the properties that are invariant under the Fourier–Mukai transform (i.e. categorical invariant) can be considered as the essential properties of the 'spaces'. For example, the Serre functor $S_X = \otimes\omega_X[\dim X]$ is such a categorical invariant.

On the other hand, there are many works concerning the derived equivalent varieties. Let $FM(X)$ be a set of isomorphism class of smooth projective varieties that have equivalent derived categories to X. In [Muk81], Mukai showed that if A is an abelian variety and $\hat{A}$ is its dual variety, then $\hat{A}$ belongs to $FM(A)$. This fact implies that $D(X)$ does not completely determine X. However, if we assume that K_X or $-K_X$ is ample, Bondal and Orlov [BD01] showed that $FM(X)$ consists of X itself. When X is a minimal surface, Bridgeland and Maciocia [BM01] described $FM(X)$, and the non-minimal case was treated by Kawamata [Kaw02]. In these cases, we can see the following common phenomenon:

> 'if more information about K_X is given, then $FM(X)$ is smaller; for example, the greater $\kappa(X, \pm K_X)$ is, the smaller is $FM(X)$.'

The main purpose of this paper is to explain why this phenomenon occurs. The idea is to extract information concerning Serre functors. Here we state the main theorem. Let $Y \in FM(X)$ and $\Phi : D(X) \to D(Y)$ an equivalence of triangulated categories. Let $\mathcal{P} \in D(X \times Y)$ be a kernel of Φ. Here the definition of kernel will be given in Definition 2.1. Let $\Psi : D(Y) \to D(X)$ be a quasi-inverse of Φ, and $\mathcal{E} \in D(X \times Y)$ be a kernel of Ψ. Then we prove the following.

- Φ induces an isomorphism of vector spaces, $H^0(X, mK_X) \to H^0(Y, mK_Y)$ for $m \in \mathbb{Z}$; this is also proved in [Cal03]. Let $E \in |mK_X|$ correspond to $E^\dagger \in |mK_Y|$.
- Φ induces a bijection between $\pi_0(\bigcap_{i=1}^n E_i)$ and $\pi_0(\bigcap_{i=1}^n E_i^\dagger)$. Here $E_i \in |m_i K_X|$ for $i = 1, \dots, n$ and $m_i \in \mathbb{Z}$; n and m_i are arbitrary, and π_0 means connected component. Let $C \in \pi_0(\bigcap_{i=1}^n E_i)$ correspond to $C^\dagger \in \pi_0(\bigcap_{i=1}^n E_i^\dagger)$.

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Then the main theorem is the following.

Theorem 1.1. *Assume that C and $C^\dagger$ satisfy the following conditions:*

- *C and $C^\dagger$ are complete intersections;*
- *$\mathcal{P} \overset{\boldsymbol{L}}{\otimes} \mathcal{O}_{C\times Y}$, $\mathcal{E} \overset{\boldsymbol{L}}{\otimes} \mathcal{O}_{C\times Y}$ are sheaves, up to shift.*

Then there exists an equivalence of triangulated categories $\Phi_C : D(C) \to D(C^\dagger)$ such that the following diagram is 2-commutative.

$$\begin{tikzcd} D(X) \arrow[r, "\mathbf{L}i_C^*"] \arrow[d, "\Phi"'] & D(C) \arrow[r, "i_{C*}"] \arrow[d, "\Phi_C"'] & D(X) \arrow[d, "\Phi"'] \\ D(Y) \arrow[r, "\mathbf{L}i_{C^\dagger}^*"] & D(C^\dagger) \arrow[r, "i_{C^\dagger *}"] & D(Y) \end{tikzcd}$$

The assumptions are satisfied if $|m_iK_X|$ are free, $E_i \in |m_iK_X|$ are generic members, and $\mathcal{P}$ is a sheaf, up to shift. The above theorem says that 'If there are many members in $|mK_X|$, then we can reduce the problem of describing $FM(X)$ to the lower dimensional case'. As an application, we study Fourier–Mukai transforms when $\kappa(X) = \dim X - 1$. Using this method, we give a generalization of the theorem of Bondal and Orlov [BD01], and determine $FM(X)$ when $\dim X = 3$ and $\kappa(X) = 2$.

From the viewpoint of birational geometry, there are some works concerning derived categories and birational geometry. For example, Bridgeland [Bri02] constructed smooth three-dimensional flops as a moduli space of perverse point sheaves, which are objects in derived category. Surprisingly his method gives an equivalence of derived categories under flops simultaneously. This result was generalized by Chen [Che02] and Kawamata [Kaw02]. The existence of flops and flips is a very difficult problem in birational geometry, and Bridgeland's result gives a possibility of treating the problem by a moduli theoretic method.

2. Derived categories and Serre functors

Notation and conventions

- Throughout this paper, we assume all the varieties are defined over $\mathbb{C}$.
- For smooth projective variety X, let $D(X) := D^b(\mathrm{Coh}(X))$, i.e. bounded derived category of coherent sheaves on X. The translation functor is written [1], and the symbol $E[m]$ means the object E shifted to the left by m places.
- ω_X means canonical bundle, and K_X means canonical divisor. For a Cartier divisor D, we write the global section of $\mathcal{O}_X(D)$ as $H^0(X, D)$, $|D|$ means linear system, and $\mathrm{Bs}\,|D|$ is a base locus as usual.
- For the derived functors, we omit $\mathbf{R}$ or $\mathbf{L}$ if the functors we want to derive are exact.
- For another variety Y, we denote by p_i the projections $p_1 : X \times Y \to X$, $p_2 : X \times Y \to Y$.
- For a closed point $x \in X$, $\mathcal{O}_x$ means a skyscraper sheaf supported at x.

In this section we recall some definitions and properties concerning derived categories.

Definition 2.1. For an object $\mathcal{P} \in D(X \times Y)$, we define a functor $\Phi^{\mathcal{P}}_{X\to Y} : D(X) \to D(Y)$ by

$$\Phi^{\mathcal{P}}_{X\to Y}(E) := \mathbf{R}p_{2*}\Big(p_1^*E \overset{\mathbf{L}}{\otimes} \mathcal{P}\Big).$$

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The object $\mathcal{P}$ is called the kernel of $\Phi^{\mathcal{P}}_{X\to Y}$. For a morphism $\mu : \mathcal{P}_1 \to \mathcal{P}_2$ in $D(X \times Y)$, we also denote by $\Phi^{\mu}_{X\to Y}$ the natural transform:

$$\Phi^{\mu}_{X\to Y} : \Phi^{\mathcal{P}_1}_{X\to Y} \longrightarrow \Phi^{\mathcal{P}_2}_{X\to Y},$$

induced by μ.

The functor of the form $\Phi^{\mathcal{P}}_{X\to Y}$ is called an integral functor. If an integral functor gives an equivalence of categories, then it is called a Fourier–Mukai transform. The following theorem is fundamental in this paper.

Theorem 2.2 (Orlov [Orl97]). *Let $\Phi : D(X) \to D(Y)$ give an equivalence of $\mathbb{C}$-linear triangulated categories. Then there exists an object $\mathcal{P} \in D(X \times Y)$ such that Φ is isomorphic to the functor $\Phi^{\mathcal{P}}_{X\to Y}$. Moreover, $\mathcal{P}$ is uniquely determined up to isomorphism.*

Next we introduce the notion of Fourier–Mukai partners.

Definition 2.3. We define $FM(X)$ as the set of isomorphism classes of smooth projective varieties Y, which has an equivalence of $\mathbb{C}$-linear triangulated categories, $\Phi : D(X) \to D(Y)$. If $Y \in FM(X)$, Y is called a Fourier–Mukai partner of X.

By Theorem 2.2, if $Y \in FM(X)$, then $D(Y)$ is related to $D(X)$ by a Fourier–Mukai transform. To study the relation between derived categories and canonical divisors, the following Serre functor plays an important role.

Definition 2.4. Let $\mathcal{T}$ be a $\mathbb{C}$-linear triangulated category of finite type. An exact equivalence $S : \mathcal{T} \to \mathcal{T}$ is called a Serre functor if there exists a bifunctorial isomorphism

$$\mathrm{Hom}(E, F) \longrightarrow \mathrm{Hom}(F, S(E))^*$$

for $E, F \in \mathcal{T}$.

As in [BD01, Proposition 1.5], if a Serre functor exists, then it is unique up to canonical isomorphism. If X is a smooth projective variety and $\mathcal{T} = D(X)$, then Serre duality implies that the Serre functor S_X is given by $S_X(E) = E \otimes \omega_X[\dim X]$.

Proposition–Definition 2.5. *Let X, Y, Z be varieties, and p_{ij} be projections from $X \times Y \times Z$ onto corresponding factors. Let us take $\mathcal{F} \in D(X \times Y)$, $\mathcal{G} \in D(Y \times Z)$. We define $\mathcal{G} \circ \mathcal{F} \in D(X \times Z)$ as*

$$\mathcal{G} \circ \mathcal{F} := \mathbf{R}p_{13*}\Big(p_{12}^*\mathcal{F} \overset{\boldsymbol{L}}{\otimes} p_{23}^*\mathcal{G}\Big).$$

Then we have the isomorphism of functors: $\Phi^{\mathcal{G}}_{Y\to Z} \circ \Phi^{\mathcal{F}}_{X\to Y} \cong \Phi^{\mathcal{G}\circ\mathcal{F}}_{X\to Z}$, and for a morphism $\mu : \mathcal{F}_1 \to \mathcal{F}_2$ in $D(X \times Y)$, the isomorphism of natural transforms:

$$\Phi^{\mathcal{G}}_{Y\to Z} \circ \Phi^{\mu}_{X\to Y} \cong \Phi^{\mathcal{G}\circ\mu}_{X\to Z} : \Phi^{\mathcal{G}\circ\mathcal{F}_1}_{X\to Z} \longrightarrow \Phi^{\mathcal{G}\circ\mathcal{F}_1}_{X\to Z}.$$

Moreover, the operation $\circ$ is associative, i.e. $(\mathcal{H} \circ \mathcal{G}) \circ \mathcal{F} \cong \mathcal{H} \circ (\mathcal{G} \circ \mathcal{F})$.

Proof. The proof of $\Phi^{\mathcal{G}}_{Y\to Z} \circ \Phi^{\mathcal{F}}_{X\to Y} \cong \Phi^{\mathcal{G}\circ\mathcal{F}}_{X\to Z}$ is seen in several references. For example, see [Che02, Proposition 2.3]. The same proof works for natural transforms, formally replacing $\mathcal{F}$ by μ. We can check that the operation $\circ$ is associative by the same method, but we would like to give the proof for the lack of references. Let X, Y, Z, W be varieties, and take $\mathcal{F} \in D(X \times Y)$, $\mathcal{G} \in D(Y \times Z)$ and $\mathcal{H} \in D(Z \times W)$. We change the index p_{ij} to p_{XY} etc. Let p_{**}, q_{**}, r_{**} and s_{**} be projections, given as in the following diagrams.

$$
\begin{tikzcd}[column sep=small]
 & X \times Y \times Z \arrow[dl, "p_{XY}"'] \arrow[d, "p_{XZ}"] \arrow[dr, "p_{YZ}"] & \\
X \times Y & X \times Z & Y \times Z
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
 & X \times Z \times W \arrow[dl, "q_{XZ}"'] \arrow[d, "q_{XW}"] \arrow[dr, "q_{ZW}"] & \\
X \times Z & X \times W & Z \times W
\end{tikzcd}
$$

$$
\begin{tikzcd}[column sep=small]
 & X \times Y \times W \arrow[dl, "r_{XY}"'] \arrow[d, "r_{XW}"] \arrow[dr, "r_{YW}"] & \\
X \times Y & X \times W & Y \times W
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
 & Y \times Z \times W \arrow[dl, "s_{YZ}"'] \arrow[d, "s_{YW}"] \arrow[dr, "s_{ZW}"] & \\
Y \times Z & Y \times W & Z \times W
\end{tikzcd}
$$

Let π_{**} or π_{***} be projections from $X \times Y \times Z \times W$ onto corresponding factors, for example as in the following diagram.

$$
\begin{tikzcd}[column sep=small]
 & & X \times Y \times Z \times W \arrow[dll, "\pi_{XYZ}"'] \arrow[dl, "\pi_{XYW}"] \arrow[d, "\pi_{XZW}"] \arrow[dr, "\pi_{YZW}"] & \\
X \times Y \times Z & X \times Y \times W & X \times Z \times W & Y \times Z \times W
\end{tikzcd}
$$

Then $\mathcal{H} \circ (\mathcal{G} \circ \mathcal{F})$ is calculated as

$$
\begin{aligned}
\mathcal{H} \circ (\mathcal{G} \circ \mathcal{F}) &\cong \mathbf{R}q_{XW*}\Big(q_{XZ}^*(\mathcal{G} \circ \mathcal{F}) \overset{\mathbf{L}}{\otimes} q_{ZW}^*\mathcal{H}\Big) \\
&\cong \mathbf{R}q_{XW*}\Big(q_{XZ}^*\mathbf{R}p_{XZ*}\Big(p_{XY}^*\mathcal{F} \overset{\mathbf{L}}{\otimes} p_{YZ}^*\mathcal{G}\Big) \overset{\mathbf{L}}{\otimes} q_{ZW}^*\mathcal{H}\Big) \\
&\cong \mathbf{R}q_{XW*}\Big(\mathbf{R}\pi_{XZW*}\pi_{XYZ}^*\Big(p_{XY}^*\mathcal{F} \overset{\mathbf{L}}{\otimes} p_{YZ}^*\mathcal{G}\Big) \overset{\mathbf{L}}{\otimes} q_{ZW}^*\mathcal{H}\Big) \\
&\cong \mathbf{R}q_{XW*}\mathbf{R}\pi_{XZW*}\Big(\pi_{XYZ}^*\Big(p_{XY}^*\mathcal{F} \overset{\mathbf{L}}{\otimes} p_{YZ}^*\mathcal{G}\Big) \overset{\mathbf{L}}{\otimes} \pi_{XZW}^*q_{ZW}^*\mathcal{H}\Big) \\
&\cong \mathbf{R}\pi_{XW*}\Big(\pi_{XY}^*\mathcal{F} \overset{\mathbf{L}}{\otimes} \pi_{YZ}^*\mathcal{G} \overset{\mathbf{L}}{\otimes} \pi_{ZW}^*\mathcal{H}\Big).
\end{aligned}
$$

Here the third isomorphism follows from flat base change, and fourth isomorphism from projection formula. Similarly, $(\mathcal{H} \circ \mathcal{G}) \circ \mathcal{F}$ is calculated as

$$
\begin{aligned}
(\mathcal{H} \circ \mathcal{G}) \circ \mathcal{F} &\cong \mathbf{R}r_{XW*}\Big(r_{YW}^*(\mathcal{H} \circ \mathcal{G}) \overset{\mathbf{L}}{\otimes} r_{XY}^*\mathcal{F}\Big) \\
&\cong \mathbf{R}r_{XW*}\Big(r_{YW}^*\mathbf{R}s_{YW*}\Big(s_{YZ}^*\mathcal{G} \overset{\mathbf{L}}{\otimes} s_{ZW}^*\mathcal{H}\Big) \overset{\mathbf{L}}{\otimes} r_{XY}^*\mathcal{F}\Big) \\
&\cong \mathbf{R}r_{XW*}\Big(\mathbf{R}\pi_{XYW*}\pi_{YZW}^*\Big(s_{YZ}^*\mathcal{G} \overset{\mathbf{L}}{\otimes} s_{ZW}^*\mathcal{H}\Big) \overset{\mathbf{L}}{\otimes} r_{XY}^*\mathcal{F}\Big) \\
&\cong \mathbf{R}\pi_{XW*}\Big(\pi_{YZ}^*\mathcal{G} \overset{\mathbf{L}}{\otimes} \pi_{ZW}^*\mathcal{H} \overset{\mathbf{L}}{\otimes} \pi_{XY}^*\mathcal{F}\Big).
\end{aligned}
$$

Therefore, we obtain the isomorphism $\mathcal{H} \circ (\mathcal{G} \circ \mathcal{F}) \cong (\mathcal{H} \circ \mathcal{G}) \circ \mathcal{F}$. □

Here we give one remark. The category $D(X \times Y)$ is like a category of functors from $D(X)$ to $D(Y)$. In fact, an object $\mathcal{F} \in D(X \times Y)$ corresponds to a functor $\Phi_{X \to Y}^{\mathcal{F}}$, and a morphism $\mathcal{F} \to \mathcal{G}$ gives a natural transform $\Phi_{X \to Y}^{\mathcal{F}} \to \Phi_{X \to Y}^{\mathcal{G}}$. However, as remarked in [Cal03], this correspondence is not faithful, i.e. the non-trivial morphism $\mathcal{F} \to \mathcal{G}$ may induce a trivial natural transform. Although natural transform is a categorical concept, it is not useful for our purpose. So sometimes we use the objects of $D(X \times Y)$ instead of functors, and treat their morphisms as if they are natural transforms.

3. Moduli spaces of stable sheaves

In this section, we introduce the notation of the moduli spaces of stable sheaves, and recall some properties. These are used for the applications of Theorem 1.1. The details are given in [DM97]. Let X be a projective scheme and H be a polarization. For a non-zero object $E \in \mathrm{Coh}(X)$, its Hilbert polynomial has the following form:

$$\chi(E \otimes H^{\otimes m}) = \sum_{i=0}^{d} \frac{\alpha_i(E)}{i!} m^i \quad (\alpha_i(E) \in \mathbb{Z}, d = \dim(\operatorname{Supp} E)).$$

We define a rank of E and its reduced Hilbert polynomial by

$$\mathrm{rk}(E) := \alpha_d(E)/\alpha_d(\mathcal{O}_X), \quad p(E,H) := \chi(E \otimes H^{\otimes m})/\alpha_d(E).$$

Now let us introduce the order on $\mathbb{Q}[m]$ as follows: if $p, p' \in \mathbb{Q}[m]$, then $p \leqslant p'$ if and only if $p(m) \leqslant p'(m)$ for sufficiently large m. We denote $p < p'$ if $p(m) < p'(m)$ for sufficiently large m.

Definition 3.1. A non-zero object $E \in \mathrm{Coh}(X)$ is said to be H-semistable if E is pure, i.e. there exists no subsheaf of dimension lower than d, and for all subsheaves $F \subsetneq E$, we have $p(F,H) \leqslant p(E,H)$. An object E is said to be H-stable if E is H-semistable and for all subsheaves $F \subsetneq E$, we have $p(F,H) < p(E,H)$.

Using the above stability, we can consider the moduli spaces of stable (semistable) sheaves. Also we can consider the relative version of the moduli spaces of such sheaves, under the projective morphism $f : X \to S$ and f-ample divisor H. Let T be an S-scheme, and $p_X : X \times_S T \to X$ and $p_T : X \times_S T \to T$ be projections. We define a contravariant functor $\overline{\mathcal{M}}^H(X/S) : (Sch/S)^{\circ} \to (Sets)$ as follows:

$$\overline{\mathcal{M}}^H(X/S)(T) := \left\{ \begin{array}{l} \mathcal{F} \in \mathrm{Coh}(X \times_S T), \text{ which are flat over } T, \\ \text{and for all geometric points } \operatorname{Spec} k(t) \to T, \\ \mathcal{F}|_{X \times \operatorname{Spec} k(t)} \text{ is } p_X^* H|_{X \times \operatorname{Spec} k(t)}\text{-semistable.} \end{array} \right\} \Big/ \sim .$$

Here for $E, E' \in \mathrm{Coh}(X \times_S T)$, the equivalence relation $\sim$ is the following:

$$E \sim E' \quad \overset{\text{def}}{\Leftrightarrow} \quad E \cong E' \otimes p_T^* \mathcal{L} \quad \text{for some } \mathcal{L} \in \mathrm{Pic}(T).$$

Then there exists a projective scheme

$$\overline{M}^H(X/S) \longrightarrow S,$$

which corepresents $\overline{\mathcal{M}}^H(X/S)$. Let $M^H(X/S) \subset \overline{M}^H(X/S)$ be a subset that corresponds to stable sheaves. It is known that $M^H(X/S)$ is an open subscheme of $\overline{M}^H(X/S)$, for example see [DM97].

Definition 3.2. Let $M \subset M^H(X/S)$ be an irreducible component. The component M is called fine if it is projective over S and there exists a universal sheaf on $X \times_S M$.

The following theorem is due to Mukai [Muk87].

Theorem 3.3 (Mukai [Muk87]). *For $x \in M$, we denote by E_x the corresponding stable sheaf. Then there exists a universal family on $X \times_S M$ if*

$$\gcd\{\chi(E_x \otimes \mathcal{N}) \mid \mathcal{N} \text{ is a vector bundle on } X\} = 1$$

holds.

We have the following criteria to find the fine moduli scheme.

Lemma 3.4. *If $\gcd\{\chi(E_x \otimes H^{\otimes n}) \mid n \in \mathbb{Z}\} = 1$, then M is projective over S, i.e. there exists no properly semistable boundary. Hence, M is fine by Theorem 3.3.*

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Proof. Indeed if there exists some $x \in \overline{M} \setminus M$, then there exists a subsheaf $F \subsetneq E_x$ such that $p(F, H) = p(E_x, H)$. If we take $n_i, \omega_i \in \mathbb{Z}$ such that $\sum \omega_i \cdot \chi(E_x \otimes H^{\otimes n_i}) = 1$, then

$$\sum \omega_i \cdot \chi(F \otimes H^{\otimes n_i}) = \alpha_d(F)/\alpha_d(E_x).$$

As the left-hand side is an integer and $0 < \alpha_d(F)/\alpha_d(E_x) < 1$, we have a contradiction. So by the above theorem M is fine. □

Finally, we recall the significant result on the moduli spaces of stable sheaves and derived categories, established by Bridgeland and Maciocia [BM02]. We say that a family of sheaves $\{\mathcal{U}_p\}_{p\in M}$ on X is complete if the Kodaira–Spencer map

$$T_pM \longrightarrow \operatorname{Ext}^1_X(\mathcal{U}_p, \mathcal{U}_p)$$

is bijective.

Theorem 3.5 (Bridgeland and Maciocia [BM02]). *Let X be a smooth projective variety of dimension n and $\{\mathcal{U}_p\}_{p\in M}$ be a complete family of simple sheaves on X parameterized by an irreducible projective scheme M of dimension n. Suppose that $\operatorname{Hom}_X(\mathcal{U}_{p_1}, \mathcal{U}_{p_2}) = 0$ for $p_i \in M$, $p_1 \neq p_2$ and the set*

$$\Gamma(\mathcal{U}) := \{(p_1, p_2) \in M \times M \mid \operatorname{Ext}^i_X(\mathcal{U}_{p_1}, \mathcal{U}_{p_2}) \neq 0 \quad \text{for some } i \in \mathbb{Z}\}$$

has $\dim \Gamma(\mathcal{U}) \leqslant n + 1$. Suppose also that $\mathcal{U}_p \otimes \omega_X \cong \mathcal{U}_p$ for all $p \in M$. Then M is a nonsingular projective variety and $\Phi^{\mathcal{U}}_{M\to X} : D(M) \to D(X)$ is an equivalence.

4. Correspondences of canonical divisors

In this section we fix two smooth projective varieties X and Y, such that $Y \in FM(X)$. The purpose of this section is to establish the relation between the canonical divisors of X and Y, and state our main theorem. We fix the following notation:

- $\Phi : D(X) \to D(Y)$ gives an equivalence and $\mathcal{P} \in D(X \times Y)$ is a kernel of Φ;
- $\Psi : D(Y) \to D(X)$ is a quasi-inverse of Φ and $\mathcal{E} \in D(X \times Y)$ is a kernel of Ψ;
- $S_X := \otimes\omega_X[\dim X] : D(X) \to D(X)$ is a Serre functor of $D(X)$.

As Serre functor is categorical, we have the isomorphism of functors,

$$\tau : \Phi \circ S_X \xrightarrow{\sim} S_Y \circ \Phi.$$

Note that the kernel of left-hand side is $\mathcal{P} \otimes p_1^*\omega_X[\dim X]$ and right-hand side is $\mathcal{P} \otimes p_2^*\omega_Y[\dim Y]$. So by Theorem 2.2, we have the isomorphism,

$$\rho : \mathcal{P} \otimes p_1^*\omega_X[\dim X] \xrightarrow{\sim} \mathcal{P} \otimes p_2^*\omega_Y[\dim Y].$$

Therefore, $\dim X = \dim Y$, and there exists an isomorphism for all $m \in \mathbb{Z}$,

$$\rho_m : \mathcal{P} \otimes p_1^*\omega_X^{\otimes m} \xrightarrow{\sim} \mathcal{P} \otimes p_2^*\omega_Y^{\otimes m}.$$

Therefore, we can see the following proposition.

Proposition 4.1. *The isomorphism of graded $\mathbb{C}$-algebras is induced by $\{\rho_m\}_{m\in\mathbb{Z}}$:*

$$\{\rho'_m\} : \bigoplus_{m\in\mathbb{Z}} \operatorname{Hom}_{X\times Y}(\mathcal{P}, \mathcal{P} \otimes p_1^*\omega_X^{\otimes m}) \xrightarrow{\sim} \bigoplus_{m\in\mathbb{Z}} \operatorname{Hom}_{X\times Y}(\mathcal{P}, \mathcal{P} \otimes p_2^*\omega_Y^{\otimes m}).$$

Proof. Clear by the above argument. □

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Next we compare the vector spaces $H^0(X, mK_X)$ and $\mathrm{Hom}_{X\times Y}(\mathcal{P}, \mathcal{P}\otimes p_1^*\omega_X^{\otimes m})$. As Φ gives an identification of categories, Φ must give the bijection between functors $D(X)\to D(X)$ and functors $D(X)\to D(Y)$. In this respect, the following lemma is obvious.

Lemma 4.2. *The following functor,*

$$\mathcal{P}\circ : D(X\times X)\ni a \longmapsto \mathcal{P}\circ a\in D(X\times Y),$$

gives equivalence.

Proof. Let Ψ be a quasi-inverse of Φ, and $\mathcal{E}\in D(X\times Y)$ be a kernel of Ψ. Let $\Delta_X\subset X\times X$ and $\Delta_Y\subset Y\times Y$ be diagonals. Note that the operations $\mathcal{O}_{\Delta_X}\circ$, $\mathcal{O}_{\Delta_Y}\circ$ induce identities. As $\mathcal{E}\circ\mathcal{P}\cong\mathcal{O}_{\Delta_X}$, $\mathcal{P}\circ\mathcal{E}\cong\mathcal{O}_{\Delta_Y}$, the following functor:

$$\mathcal{E}\circ : D(X\times Y)\ni b\longmapsto \mathcal{E}\circ b\in D(X\times X)$$

gives a quasi-inverse by Proposition–Definition 2.5. $\square$

In the same way, we have equivalence of categories:

$$\circ\mathcal{P} : D(Y\times Y)\ni a\longmapsto a\circ\mathcal{P}\in D(X\times Y).$$

We have the following lemma.

Lemma 4.3. *The following diagrams are 2-commutative.*

$$\begin{tikzcd} D(X\times X) \arrow[r, "\mathcal{P}\circ"] & D(X\times Y) \\ D(X) \arrow[u, "\Delta_*"] \arrow[r, equal] & D(X) \arrow[u, "p_1^*(*)\overset{\mathbf{L}}{\otimes}\mathcal{P}"'] \end{tikzcd} \qquad \begin{tikzcd} D(Y\times Y) \arrow[r, "\circ\mathcal{P}"] & D(X\times Y) \\ D(Y) \arrow[u, "\Delta_*"] \arrow[r, equal] & D(Y) \arrow[u, "p_2^*(*)\overset{\mathbf{L}}{\otimes}\mathcal{P}"'] \end{tikzcd}$$

Here Δ means diagonal embedding.

Proof. Let us check the left diagram commutes. Let p_{ij} be projections from $X\times X\times Y$ onto corresponding factors. Take $a\in D(X)$. Then

$$\begin{aligned}\mathcal{P}\circ(\Delta_* a) &\cong \mathbf{R}p_{13*}\Big(p_{12}^*\Delta_* a\overset{\mathbf{L}}{\otimes} p_{23}^*\mathcal{P}\Big)\\ &\cong \mathbf{R}p_{13*}\Big((\Delta\times\mathrm{id}_Y)_* p_1^* a\overset{\mathbf{L}}{\otimes} p_{23}^*\mathcal{P}\Big)\\ &\cong \mathbf{R}p_{13*}(\Delta\times\mathrm{id}_Y)_*\Big(p_1^* a\overset{\mathbf{L}}{\otimes}(\Delta\times\mathrm{id}_Y)^* p_{23}^*\mathcal{P}\Big)\\ &\cong p_1^* a\overset{\mathbf{L}}{\otimes}\mathcal{P}.\end{aligned}$$

The second isomorphism follows from the flat base change of the diagram below

$$\begin{tikzcd} X\times Y \arrow[r, "\Delta\times\mathrm{id}_Y"] \arrow[d, "p_1"] & X\times X\times Y \arrow[d, "p_{12}"] \\ X \arrow[r, "\Delta"] & X\times X \end{tikzcd}$$

and the third isomorphism follows from projection formula. $\square$

As the immediate corollary, we have the following.

Corollary 4.4. *The isomorphism of graded $\mathbb{C}$-algebras is induced by Φ:*

$$\{\phi_m\}_{m\in\mathbb{Z}} : \bigoplus_{m\in\mathbb{Z}} H^0(X, mK_X)\xrightarrow{\sim}\bigoplus_{m\in\mathbb{Z}} H^0(Y, mK_Y).$$

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Proof. By Lemma 4.3, we have the isomorphism of graded $\mathbb{C}$-algebras:

$$\bigoplus_{m\in\mathbb{Z}} \mathrm{Hom}_{X\times X}(\Delta_*\mathcal{O}_X, \Delta_*\omega_X^{\otimes m}) \xrightarrow{\mathcal{P}\circ} \bigoplus_{m\in\mathbb{Z}} \mathrm{Hom}_{X\times Y}(\mathcal{P}, \mathcal{P}\otimes p_1^*\omega_X^{\otimes m}),$$
$$\bigoplus_{m\in\mathbb{Z}} \mathrm{Hom}_{Y\times Y}(\Delta_*\mathcal{O}_Y, \Delta_*\omega_Y^{\otimes m}) \xrightarrow{\circ\mathcal{P}} \bigoplus_{m\in\mathbb{Z}} \mathrm{Hom}_{X\times Y}(\mathcal{P}, \mathcal{P}\otimes p_2^*\omega_Y^{\otimes m}).$$

As $H^0(X, mK_X) = \mathrm{Hom}_{X\times X}(\Delta_*\mathcal{O}_X, \Delta_*\omega_X^{\otimes m})$, combining ρ'_m given in Proposition 4.1, we obtain the corollary. □

Now let us interpret the isomorphism $\phi_m : H^0(X, mK_X) \to H^0(Y, mK_Y)$ categorically. Take $\sigma \in H^0(X, mK_X)$ and $\sigma^\dagger := \phi_m(\sigma) \in H^0(Y, mK_Y)$. Let $d := \dim X = \dim Y$. Then we can think of σ and $\sigma^\dagger$ as natural transforms,

$$\sigma : \mathrm{id}_X \longrightarrow S_X^m[-md], \quad \sigma^\dagger : \mathrm{id}_Y \longrightarrow S_Y^m[-md]$$

Here $S_X^m[-md]$ is an m-times composition of the shifted Serre functor, $S_X[-d] = \otimes\omega_X$. Let

$$\tau_m : \Phi\circ S_X^m[-md] \xrightarrow{\sim} S_Y^m[-md]\circ\Phi$$

be the isomorphism of functors, induced by $\tau : \Phi\circ S_X \xrightarrow{\sim} S_Y\circ\Phi$ naturally.

Lemma 4.5. *The following composition is equal to $\sigma^\dagger$:*

$$\mathrm{id}_Y = \Phi\circ\mathrm{id}_X\circ\Phi^{-1} \xrightarrow{\mathrm{id}\circ\sigma\circ\mathrm{id}} \Phi\circ S_X^m[-dm]\circ\Phi^{-1} \xrightarrow{\tau_m\circ\mathrm{id}} S_Y^m[-dm]\circ\Phi\circ\Phi^{-1} = S_Y^m[-dm].$$

Proof. This follows from Proposition–Definition 2.5 for natural transforms, and the construction of $\sigma^\dagger$. □

Let $E := \mathrm{div}(\sigma) \in |mK_X|$, $E^\dagger := \mathrm{div}(\sigma^\dagger) \in |mK_Y|$. For the closed subscheme $Z \hookrightarrow X$, we define the full subcategory $D_Z(X) \subset D(X)$ as follows:

$$D_Z(X) := \{a \in D(X) \mid \mathrm{Supp}\, a := \cup\,\mathrm{Supp}\, H^i(a) \subset Z\}.$$

We can observe the following.

Lemma 4.6. *In the above situation, Φ takes $D_E(X)$ to $D_{E^\dagger}(Y)$.*

Proof. Take $a \in \mathrm{Coh}(X)\cap D_E(X)$. Let $\sigma^l : \mathrm{id}_X \to S_X^{lmd}[-lmd] = \otimes\omega_X^{\otimes lm}$ be an l-times composition of σ. Then

$$\sigma^l(a) : a \longrightarrow a\otimes\omega_X^{\otimes lm}$$

are zero-maps for sufficiently large l. Then by the above categorical interpretation of $\sigma^\dagger$, we have that

$$(\sigma^\dagger)^l(\Phi(a)) : \Phi(a) \longrightarrow \Phi(a)\otimes\omega_Y^{\otimes lm}$$

are also zero-maps. As $(\sigma^\dagger)^l$ is a natural transform, locally multiplying the defining equation of $lE^\dagger$, we have $\mathrm{Supp}\,\Phi(a) \subset E^\dagger$. As $D_E(X)$ is generated by $\mathrm{Coh}(X)\cap D_E(X)$, the lemma follows. □

For the sake of applications, it is convenient to generalize the above lemma to the intersections of canonical divisors.

Corollary 4.7. *Take $E_i \in |m_iK_X|$ and their corresponding divisors $E_i^\dagger \in |m_iK_Y|$ for $i = 1, 2, \ldots, n$. There exists a one-to-one correspondence,*

$$\pi_0\Big(\bigcap_{i=1}^n E_i\Big) \ni C \longmapsto C^\dagger \in \pi_0\Big(\bigcap_{i=1}^n E_i^\dagger\Big),$$

such that Φ takes $D_C(X)$ to $D_{C^\dagger}(Y)$. Here π_0 means connected component.

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Proof. Lemma 4.6 shows that Φ takes $D_{\cap E_i}(X)$ to $D_{\cap E_i^{\dagger}}(Y)$. Take a connected component $C \subset \bigcap_{i=1}^n E_i$. As

$$\mathrm{Hom}_Y(\Phi(\mathcal{O}_{C_{\mathrm{red}}}), \Phi(\mathcal{O}_{C_{\mathrm{red}}})) = \mathrm{Hom}_X(\mathcal{O}_{C_{\mathrm{red}}}, \mathcal{O}_{C_{\mathrm{red}}}) = \mathbb{C},$$

$\mathrm{Supp}\,\Phi(\mathcal{O}_{C_{\mathrm{red}}})$ is connected. Therefore, there exists a unique connected component $C^{\dagger} \subset \bigcap_{i=1}^n E_i^{\dagger}$ such that $\mathrm{Supp}\,\Phi(\mathcal{O}_{C_{\mathrm{red}}}) \subset C^{\dagger}$. We show that Φ takes $D_C(X)$ to $D_{C^{\dagger}}(Y)$. It suffices to show that Φ takes $\mathrm{Coh}(\mathcal{O}_C)$ to $D_{C^{\dagger}}(Y)$. Take a closed point $x \in C$. Then $\mathrm{Supp}(\Phi(\mathcal{O}_x))$ is connected by the same reason. As there exists a non-trivial morphism $\mathcal{O}_{C_{\mathrm{red}}} \to \mathcal{O}_x$, we have $\Phi(\mathcal{O}_x) \in D_{C^{\dagger}}(Y)$. Let us take a simple $\mathcal{O}_C$-module $\mathcal{F}$. Then as $\mathrm{Supp}(\Phi(\mathcal{F}))$ is connected and there exists a non-trivial morphism $\mathcal{F} \to \mathcal{O}_x$ for some closed point $x \in C$, we have $\Phi(\mathcal{F}) \in D_{C^{\dagger}}(Y)$. The lemma follows by taking Harder–Narasimhan filtrations. □

Unfortunately, the natural functor $D(C) \to D_C(X)$ does not give an equivalence. (In general, the latter has larger Ext-groups.) However, the existence of equivalence between $D_C(X)$ and $D_{C^{\dagger}}(Y)$ leads us to the speculation that $D(C)$ and $D(C^{\dagger})$ may be equivalent. If $D(C)$ and $D(C^{\dagger})$ are equivalent, then the relation between C and $C^{\dagger}$ will give us information of the relation between X and Y. One of the purposes of this paper is to claim that this speculation is true, under some technical conditions. We assume the following conditions on C, $C^{\dagger}$ and $\mathcal{P}, \mathcal{E} \in D(X \times Y)$. Recall that $\mathcal{P}, \mathcal{E}$ are kernels of Φ and Φ^{-1}.

- C and $C^{\dagger}$ are complete intersections.
- $\mathcal{P} \overset{\mathbf{L}}{\otimes} \mathcal{O}_{C \times Y}$ and $\mathcal{E} \overset{\mathbf{L}}{\otimes} \mathcal{O}_{C \times Y}$ are sheaves, up to shift.

These conditions are satisfied, for example, if the linear systems $|m_i K_X|$ are free and E_i are generic members and $\mathcal{P}$ is a sheaf. Now we can state our main theorem.

Theorem 4.8. *Under the above conditions, there exists equivalence $\Phi_C : D(C) \to D(C^{\dagger})$ such that the following diagram is 2-commutative.*

$$\begin{tikzcd} D(X) \arrow[r, "\mathbf{L}i_C^*"] \arrow[d, "\Phi"'] & D(C) \arrow[r, "i_{C*}"] \arrow[d, "\Phi_C"'] & D(X) \arrow[d, "\Phi"'] \\ D(Y) \arrow[r, "\mathbf{L}i_{C^{\dagger}}^*"] & D(C^{\dagger}) \arrow[r, "i_{C^{\dagger}*}"] & D(Y) \end{tikzcd}$$

Here i_C, $i_{C^{\dagger}}$ are inclusions of C, $C^{\dagger}$ into X and Y, respectively.

5. Proof of Theorem 4.8

In this section, we give the proof of Theorem 4.8. We use the notation of the previous section. First, we explain the plan of the proof. We divide the proof into four steps. In Steps 1 and 2, we show there exists an isomorphism, $\mathcal{P} \overset{\mathbf{L}}{\otimes} \mathcal{O}_{C \times Y} \cong \mathcal{P} \overset{\mathbf{L}}{\otimes} \mathcal{O}_{X \times C^{\dagger}}$. Using this and the assumptions, we find a $\mathcal{P}_C \in D(C \times C^{\dagger})$, and construct a functor $\Phi_C : D(C) \to D(C^{\dagger})$. In Steps 3 and 4, we will show that Φ_C gives the desired equivalence.

Step 1. There exists an isomorphism $\mathcal{P} \overset{\mathbf{L}}{\otimes} \mathcal{O}_{E_i \times Y} \cong \mathcal{P} \overset{\mathbf{L}}{\otimes} \mathcal{O}_{X \times E_i^{\dagger}}$.

Proof. We omit the index i, and write E_i as E, etc. We have the following exact sequences:

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{\sigma} \omega_X^{\otimes m} \longrightarrow \mathcal{O}_E \otimes \omega_X^{\otimes m} \longrightarrow 0$$

$$0 \longrightarrow \mathcal{O}_Y \xrightarrow{\sigma^{\dagger}} \omega_Y^{\otimes m} \longrightarrow \mathcal{O}_{E^{\dagger}} \otimes \omega_Y^{\otimes m} \longrightarrow 0.$$

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Applying $p_1^*(*) \overset{\mathbf{L}}{\otimes} \mathcal{P}$ and $p_2^*(*) \overset{\mathbf{L}}{\otimes} \mathcal{P}$ respectively, we obtain the distinguished triangles:

$$\mathcal{P} \xrightarrow{\mathrm{id}\otimes p_1^*\sigma} \mathcal{P}\otimes p_1^*\omega_X^{\otimes m} \longrightarrow \mathcal{P} \overset{\mathbf{L}}{\otimes} \mathcal{O}_{E\times Y}\otimes p_1^*\omega_X^{\otimes m} \longrightarrow \mathcal{P}[1]$$
$$\mathcal{P} \xrightarrow{\mathrm{id}\otimes p_2^*\sigma^\dagger} \mathcal{P}\otimes p_2^*\omega_Y^{\otimes m} \longrightarrow \mathcal{P} \overset{\mathbf{L}}{\otimes} \mathcal{O}_{X\times E^\dagger}\otimes p_2^*\omega_Y^{\otimes m} \longrightarrow \mathcal{P}[1].$$

On the other hand, by Lemma 4.3 and the definition of ϕ_m given in Corollary 4.4, we obtain the following commutative diagram.

$$\begin{tikzcd}
\mathcal{P} \arrow[r, "\mathrm{id}\otimes p_1^*\sigma"] \arrow[d, equal] & \mathcal{P}\otimes p_1^*\omega_X^{\otimes m} \arrow[r] \arrow[d, "\rho_m"] & \mathcal{P} \overset{\mathbf{L}}{\otimes} \mathcal{O}_{E\times Y}\otimes p_1^*\omega_X^{\otimes m} \\
\mathcal{P} \arrow[r, "\mathrm{id}\otimes p_2^*\sigma^\dagger"] & \mathcal{P}\otimes p_2^*\omega_Y^{\otimes m} \arrow[r] & \mathcal{P} \overset{\mathbf{L}}{\otimes} \mathcal{O}_{X\times E^\dagger}\otimes p_2^*\omega_Y^{\otimes m}
\end{tikzcd}$$

Here ρ_m is an isomorphism constructed in the previous section. Therefore, there exists an (not necessarily unique) isomorphism,

$$\mathcal{P} \overset{\mathbf{L}}{\otimes} \mathcal{O}_{E\times Y}\otimes p_1^*\omega_X^{\otimes m} \cong \mathcal{P} \overset{\mathbf{L}}{\otimes} \mathcal{O}_{X\times E^\dagger}\otimes p_2^*\omega_Y^{\otimes m}.$$

As $\mathcal{P}\otimes p_1^*\omega_X^{\otimes m} \cong \mathcal{P}\otimes p_2^*\omega_Y^{\otimes m}$, we have an isomorphism, $\mathcal{P} \overset{\mathbf{L}}{\otimes} \mathcal{O}_{E\times Y} \cong \mathcal{P} \overset{\mathbf{L}}{\otimes} \mathcal{O}_{X\times E^\dagger}$. □

Step 2. There exists an isomorphism,

$$\mathcal{P} \overset{\mathbf{L}}{\otimes} \mathcal{O}_{C\times Y} \cong \mathcal{P} \overset{\mathbf{L}}{\otimes} \mathcal{O}_{X\times C^\dagger}.$$

Proof. By using the isomorphism of Step 1 n-times, we can get the isomorphism:

$$\mathcal{P} \overset{\mathbf{L}}{\otimes} \Bigg(\overset{\mathbf{L}}{\bigotimes_{1\leqslant i\leqslant n}} \mathcal{O}_{E_i\times Y}\Bigg) \cong \mathcal{P} \overset{\mathbf{L}}{\otimes} \Bigg(\overset{\mathbf{L}}{\bigotimes_{1\leqslant i\leqslant n}} \mathcal{O}_{X\times E_i^\dagger}\Bigg).$$

On the other hand, we have

$$\overset{\mathbf{L}}{\bigotimes_{1\leqslant i\leqslant n}} \mathcal{O}_{E_i\times Y} = \bigoplus_{C\in\pi_0(\bigcap_{i=1}^n E_i)} p_1^*\mathcal{A}_C, \quad \overset{\mathbf{L}}{\bigotimes_{1\leqslant i\leqslant n}} \mathcal{O}_{X\times E_i^\dagger} = \bigoplus_{C'\in\pi_0(\bigcap_{i=1}^n E_i^\dagger)} p_2^*\mathcal{B}_{C'},$$

for some $\mathcal{A}_C \in D_C(X)$, $\mathcal{B}_{C'} \in D_{C'}(Y)$. Therefore, we have the following isomorphism:

$$\bigoplus_{C\in\pi_0(\bigcap_{i=1}^n E_i)} \mathcal{P} \overset{\mathbf{L}}{\otimes} p_1^*\mathcal{A}_C \cong \bigoplus_{C'\in\pi_0(\bigcap_{i=1}^n E_i^\dagger)} \mathcal{P} \overset{\mathbf{L}}{\otimes} p_2^*\mathcal{B}_{C'}.$$

Now we have the following lemma.

Lemma 5.1. *The objects $\mathcal{P} \overset{\mathbf{L}}{\otimes} p_1^*\mathcal{A}_C$, $\mathcal{P} \overset{\mathbf{L}}{\otimes} p_2^*\mathcal{B}_{C^\dagger}$ are supported on $C\times C^\dagger$.*

Proof. We show that $\mathcal{P} \overset{\mathbf{L}}{\otimes} p_1^*\mathcal{A}_C$ is supported on $C\times C^\dagger$. The other case follows similarly. We can write,

$$\mathcal{P} \overset{\mathbf{L}}{\otimes} p_1^*\mathcal{A}_C \cong \bigoplus_{C'\in\pi_0(\bigcap_{i=1}^n E_i^\dagger)} \mathcal{R}_{C'},$$

where $\mathcal{R}_{C'}$ is supported on $C\times C'$. Take $C' \neq C \in \pi_0(\cap E_i)$ and assume that $\mathcal{R}_{C'}$ is not zero. Let us take a sufficiently ample line bundle $\mathcal{L}$ on X. As $\Phi(\mathcal{A}_C\otimes\mathcal{L}) \in D_{C^\dagger}(Y)$, we have $\mathbf{R}p_{2*}(\mathcal{R}_{C'}\otimes p_1^*\mathcal{L}) = 0$.

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On the other hand, if $\mathcal{L}$ is sufficiently ample and $H^q(\mathcal{R}_{C'}) \neq 0$, then $p_{2*}(H^q(\mathcal{R}_{C'}) \otimes p_1^*\mathcal{L}) \neq 0$ and $R^p p_{2*}(H^q(\mathcal{R}_{C'}) \otimes p_1^*\mathcal{L}) = 0$ for $p > 0$. As there exists the following spectral sequence:

$$E_2^{p,q} = R^p p_{2*}(H^q(\mathcal{R}_{C'}) \otimes p_1^*\mathcal{L}) \Rightarrow \mathbf{R}^{p+q} p_{2*}(\mathcal{R}_{C'} \otimes p_1^*\mathcal{L}),$$

we have $\mathbf{R}p_{2*}(\mathcal{R}_{C'} \otimes p_1^*\mathcal{L}) \neq 0$. However, this is a contradiction. $\square$

By the lemma above we have $\mathcal{P} \overset{\mathbf{L}}{\otimes} p_1^*\mathcal{A}_C \cong \mathcal{P} \overset{\mathbf{L}}{\otimes} p_2^*\mathcal{B}_{C^\dagger}$. As we have assumed C and $C^\dagger$ are complete intersections, we have $\mathcal{A}_C = \mathcal{O}_C$, $\mathcal{B}_{C^\dagger} = \mathcal{O}_{C^\dagger}$ in our case. Combining these, we have the desired isomorphism:

$$\mathcal{P} \overset{\mathbf{L}}{\otimes} \mathcal{O}_{C\times Y} \cong \mathcal{P} \overset{\mathbf{L}}{\otimes} \mathcal{O}_{X\times C^\dagger}. \qquad \square$$

By the assumptions, the object $\mathcal{P} \overset{\mathbf{L}}{\otimes} \mathcal{O}_{C\times Y} \cong \mathcal{P} \overset{\mathbf{L}}{\otimes} \mathcal{O}_{X\times C^\dagger}$ is a sheaf, up to shift. This sheaf is $\mathcal{O}_{C\times Y}$-module and also $\mathcal{O}_{X\times C^\dagger}$-module. Hence, this object is a $\mathcal{O}_{C\times C^\dagger}$-module, so there exists an object $\mathcal{P}_C \in D(C \times C^\dagger)$, such that

$$\mathcal{P} \overset{\mathbf{L}}{\otimes} \mathcal{O}_{C\times Y} \cong \mathcal{P} \overset{\mathbf{L}}{\otimes} \mathcal{O}_{X\times C^\dagger} \cong i_{C\times C^\dagger *}\mathcal{P}_C.$$

Let $\Phi_C := \Phi_{C\to C^\dagger}^{\mathcal{P}_C} : D(C) \to D(C^\dagger)$. In what follows, we do not use the fact these are sheaves up to shift, and show that Φ_C gives a desired equivalence.

Step 3. In the diagram of Theorem 4.8 we have the following isomorphisms of functors:

$$\Phi_C \circ \mathbf{L}i_C^* \cong \Phi_{X\to C^\dagger}^{(i_C\times \mathrm{id}_{C^\dagger})_*\mathcal{P}_C}, \quad \mathbf{L}i_{C^\dagger}^* \circ \Phi \cong \Phi_{X\to C^\dagger}^{\mathbf{L}(\mathrm{id}_X \times i_{C^\dagger})^*\mathcal{P}},$$
$$i_{C^\dagger *} \circ \Phi_C \cong \Phi_{C\to Y}^{(\mathrm{id}_C \times i_{C^\dagger})_*\mathcal{P}_C}, \quad \Phi \circ i_{C*} \cong \Phi_{C\to Y}^{\mathbf{L}(i_C\times \mathrm{id}_Y)^*\mathcal{P}}.$$

See the following diagram.

$$\begin{tikzcd}
 & X \times C^\dagger \arrow[dr, "\mathrm{id}_X \times i_{C^\dagger}"] & \\
C \times C^\dagger \arrow[ur, "i_C\times \mathrm{id}_{C^\dagger}"] \arrow[rr, "i_{C\times C^\dagger}"] \arrow[dr, "\mathrm{id}_C \times i_{C^\dagger}"'] & & X \times Y \\
 & C \times Y \arrow[ur, "i_C\times \mathrm{id}_Y"'] &
\end{tikzcd}$$

Proof. Let us calculate $\Phi_C \circ \mathbf{L}i_C^*$ by using Proposition–Definition 2.5. The other formulas follow similarly. Let $q_{12} : X \times C \times C^\dagger \to X \times C$, $q_{23} : X \times C \times C^\dagger \to C \times C^\dagger$, $q_{13} : X \times C \times C^\dagger \to X \times C^\dagger$ be projections. Let $\Gamma_C \subset X \times C$ be the graph of the inclusion i_C. Let j be the inclusion of $\Gamma_C \times C^\dagger$ into $X \times C \times C^\dagger$. As $\mathbf{L}i_C^* = \Phi_{X\to C}^{\mathcal{O}_{\Gamma_C}}$, we can compute the kernel of $\Phi_C \circ \mathbf{L}i_C^*$ as follows:

$$\begin{aligned}
\mathbf{R}q_{13*}\Big(q_{12}^*\mathcal{O}_{\Gamma_C} \overset{\mathbf{L}}{\otimes} q_{23}^*\mathcal{P}_C\Big) &\cong \mathbf{R}q_{13*}\Big(\mathcal{O}_{\Gamma_C\times C^\dagger} \overset{\mathbf{L}}{\otimes} q_{23}^*\mathcal{P}_C\Big) \\
&\cong \mathbf{R}q_{13*}j_*\mathbf{L}j^*\mathbf{L}q_{23}^*\mathcal{P}_C \\
&\cong (i_C \times \mathrm{id}_{C^\dagger})_*\mathbf{R}q_{23*}j_*\mathbf{L}j^*\mathbf{L}q_{23}^*\mathcal{P}_C \\
&\cong (i_C \times \mathrm{id}_{C^\dagger})_*\mathbf{R}(q_{23}\circ j)_*\mathbf{L}(q_{23}\circ j)^*\mathcal{P}_C \\
&\cong (i_C \times \mathrm{id}_{C^\dagger})_*\mathcal{P}_C.
\end{aligned}$$

Here the third isomorphism follows from $q_{13} \circ j = (i_C \times \mathrm{id}_{C^\dagger}) \circ q_{23} \circ j$ and the last isomorphism follows as $q_{23} \circ j$ is identity. $\square$

Step 4. A desired equivalence is given by Φ_C.

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By Step 3, to prove that the diagram of Theorem 4.8 commutes, we only have to check that the following hold:

$$(i_C \times \mathrm{id}_{C^\dagger})_* \mathcal{P}_C \cong \mathbf{L}(\mathrm{id}_X \times i_{C^\dagger})^* \mathcal{P}, \quad (\mathrm{id}_C \times i_{C^\dagger})_* \mathcal{P}_C \cong \mathbf{L}(i_C \times \mathrm{id}_Y)^* \mathcal{P}.$$

There exists the following morphism:

$$\mathcal{P} \longrightarrow \mathcal{P} \overset{\mathbf{L}}{\otimes} \mathcal{O}_{X\times C^\dagger} \cong i_{C\times C^\dagger *}\mathcal{P}_C = (\mathrm{id}_X \times i_{C^\dagger})_*(i_C \times \mathrm{id}_{C^\dagger})_* \mathcal{P}_C.$$

Taking its adjoint, we have a morphism $\mathbf{L}(\mathrm{id}_X \times i_{C^\dagger})^* \mathcal{P} \to (i_C \times \mathrm{id}_{C^\dagger})_* \mathcal{P}_C$. Let us take its distinguished triangle:

$$\mathcal{Q} \longrightarrow \mathbf{L}(\mathrm{id}_X \times i_{C^\dagger})^* \mathcal{P} \longrightarrow (i_C \times \mathrm{id}_{C^\dagger})_* \mathcal{P}_C \longrightarrow \mathcal{Q}[1].$$

By applying $(\mathrm{id}_X \times i_{C^\dagger})_*$, we get the distinguished triangle,

$$(\mathrm{id}_X \times i_{C^\dagger})_* \mathcal{Q} \longrightarrow \mathcal{P} \overset{\mathbf{L}}{\otimes} \mathcal{O}_{X\times C^\dagger} \xrightarrow{\cong} i_{C\times C^\dagger *}\mathcal{P}_C \longrightarrow (\mathrm{id}_X \times i_{C^\dagger})_* \mathcal{Q}[1].$$

So, we have $(\mathrm{id}_X \times i_{C^\dagger})_* \mathcal{Q} = 0$. Therefore, $\mathcal{Q} = 0$ and the morphism $\mathbf{L}(\mathrm{id}_X \times i_{C^\dagger})^* \mathcal{P} \to (i_C \times \mathrm{id}_{C^\dagger})_* \mathcal{P}_C$ is an isomorphism. We can prove the isomorphism $(\mathrm{id}_C \times i_{C^\dagger})_* \mathcal{P}_C \cong \mathbf{L}(i_C \times \mathrm{id}_Y)^* \mathcal{P}$ similarly.

Finally, we prove that Φ_C gives an equivalence. Let us define $\Psi_C : D(C^\dagger) \to D(C)$ as in the same way of Φ_C, from $\Psi = \Phi^{-1}$. Then the following diagram commutes.

$$\begin{tikzcd}
D(X) \arrow[r, "\mathbf{L}i_C^*"] \arrow[d, "\Phi"] & D(C) \arrow[r, "i_{C*}"] \arrow[d, "\Phi_C"] & D(X) \arrow[d, "\Phi"] \\
D(Y) \arrow[r, "\mathbf{L}i_{C^\dagger}^*"] \arrow[d, "\Psi"] & D(C^\dagger) \arrow[r, "i_{C^\dagger *}"] \arrow[d, "\Psi_C"] & D(Y) \arrow[d, "\Psi"] \\
D(X) \arrow[r, "\mathbf{L}i_C^*"] & D(C) \arrow[r, "i_{C*}"] & D(X)
\end{tikzcd}$$

Take a closed point $x \in C$. Then by the diagram above, $i_{C*} \circ \Psi_C \circ \Phi_C(\mathcal{O}_x) \cong i_{C*}(\mathcal{O}_x)$, so $\Psi_C \circ \Phi_C(\mathcal{O}_x) \cong \mathcal{O}_x$. Then, by [Bri99, Lemma 4.3], the kernel of $\Psi_C \circ \Phi_C$ is a sheaf on $C \times C$, therefore it must be a line bundle on its diagonal. Hence $\Psi_C \circ \Phi_C \cong \otimes \mathcal{L}_C$ for some line bundle $\mathcal{L}_C$ on C. However, again by the diagram above, we have $\Psi_C \circ \Phi_C(\mathcal{O}_C) \cong \mathcal{O}_C$. This implies that $\mathcal{L}_C \cong \mathcal{O}_C$ and $\Psi_C \circ \Phi_C \cong \mathrm{id}$. Similarly, $\Phi_C \circ \Psi_C \cong \mathrm{id}$. Therefore, Φ_C is an equivalence and the proof of Theorem 4.8 is completed.

Remark 5.2. The conditions of kernels are required to find the object $\mathcal{P}_C$ which satisfies

$$\mathcal{P} \overset{\mathbf{L}}{\otimes} \mathcal{O}_{C\times Y} \cong \mathcal{P} \overset{\mathbf{L}}{\otimes} \mathcal{O}_{X\times C^\dagger} \cong i_{C\times C^\dagger *}\mathcal{P}_C.$$

In fact, if we can find such a $\mathcal{P}_C$, then our theorem holds by using $\mathcal{P}_C$. In Steps 3 and 4, we did not use the fact that these are sheaves.

6. Fourier–Mukai transforms of varieties of $\kappa(X) = \dim X - 1$

In this section we explain the important situation to which Theorem 4.8 can be applied. Let us consider the situation when K_X (or $-K_X$) is semi-ample, i.e. $|mK_X|$ is free for some $m > 0$ (or $m < 0$). When K_X is semi-ample, we have the following morphism, called the Iitaka fibration:

$$\pi_X : X \longrightarrow Z := \mathrm{Proj} \bigoplus_{m \geqslant 0} H^0(X, mK_X).$$

The Kodaira dimension of its generic fiber is zero. Let $Y \in FM(X)$ and $\Phi : D(X) \to D(Y)$ be an equivalence. Note that K_Y is also semi-ample by Corollary 4.7. By Corollary 4.4, the target of its

Iitaka fibration is also Z. Let $\pi_Y : Y \to Z$ be the Iitaka fibration. Let us take a general closed point $p \in Z$. Let $X_p := \pi_X^{-1}(p)$ and $Y_p := \pi_Y^{-1}(p)$. Assume that the kernel of Φ satisfies the condition as in Theorem 4.8, for example the kernel of Φ is a sheaf. Then Theorem 4.8 states that there exists equivalence $\Phi_p : D(X_p) \to D(Y_p)$ such that the following diagram commutes.

$$\begin{tikzcd} D(X) \arrow[r, "\mathbf{L}i_p^*"] \arrow[d, "\Phi"'] & D(X_p) \arrow[r, "i_{p*}"] \arrow[d, "\Phi_p"'] & D(X) \arrow[d, "\Phi"'] \\ D(Y) \arrow[r, "\mathbf{L}j_p^*"] & D(Y_p) \arrow[r, "j_{p*}"] & D(Y) \end{tikzcd} \qquad (\diamondsuit)$$

Here i_p and j_p are inclusions, $i_p : X_p \hookrightarrow X$, $j_p : Y_p \hookrightarrow Y$. The conditions of kernels are satisfied if $\kappa(X) = \dim X - 1$. Note that Fourier–Mukai partners of the varieties of $\kappa(X) = \dim X$ are studied in [Kaw02].

Theorem 6.1. *Let X be a smooth projective variety such that K_X is semi-ample, and $\kappa(X) = \dim X - 1$. Let $Y \in FM(X)$ and $\Phi : D(X) \to D(Y)$ be an equivalence. Then in the above notation, there exists an equivalence $\Phi_p : D(X_p) \to D(Y_p)$ such that the diagram $(\diamondsuit)$ commutes.*

Proof. Let $\mathcal{P} \in D(X \times Y)$ be a kernel of Φ. It suffices to show that $\mathcal{P} \overset{\mathbf{L}}{\otimes} \mathcal{O}_{X_p \times Y}$ is a sheaf, up to shift. Note that

$$\mathcal{P} \overset{\mathbf{L}}{\otimes} \mathcal{O}_{X_p \times Y} \cong \mathcal{P} \overset{\mathbf{L}}{\otimes} \mathcal{O}_{X \times Y_p},$$

by Step 2 of Theorem 4.8. By taking the functors whose kernels are the left-hand side and right-hand side, respectively, we can obtain the isomorphism of functors:

$$\Phi(* \overset{\mathbf{L}}{\otimes} \mathcal{O}_{X_p}) \cong \Phi(*) \overset{\mathbf{L}}{\otimes} \mathcal{O}_{Y_p}.$$

Note that the above isomorphism can be also applied to derived categories of quasi-coherent sheaves. Let us consider $\Phi(\mathcal{O}_x)$ for $x \in X_p$. Take a general morphism:

$$v_x : \operatorname{Spec} \mathbb{C}[[t_1, \dots, t_{d-1}]] \longrightarrow X,$$

which takes a closed point of $\operatorname{Spec} \mathbb{C}[[t_1, \dots, t_{d-1}]]$ to $x \in X_p$. Here $d := \dim X$. Let $R_x := v_{x*}\mathbb{C}[[t_1, \dots, t_{d-1}]] \in \mathrm{QCoh}(X)$. Then $R_x \overset{\mathbf{L}}{\otimes} \mathcal{O}_{X_p} \cong \mathcal{O}_x$, and

$$\begin{aligned} \Phi(\mathcal{O}_x) &\cong \Phi(R_x) \overset{\mathbf{L}}{\otimes} \mathcal{O}_{Y_p} \\ &\cong j_{p*}\mathbf{L}j_p^*\Phi(R_x). \end{aligned}$$

As Y_p is one-dimensional, $\mathbf{L}j_p^*\Phi(R_x)$ is a direct sum of its cohomologies. As

$$\mathrm{Hom}_X(\mathcal{O}_x, \mathcal{O}_x) \cong \mathrm{Hom}_Y(\Phi(\mathcal{O}_x), \Phi(\mathcal{O}_x)) \cong \mathbb{C},$$

we can conclude $\Phi(\mathcal{O}_x)$ is a coherent $\mathcal{O}_{Y_p}$-module, up to shift. We may assume that $\Phi(\mathcal{O}_x)$ is a sheaf for general $x \in X_p$. Then for all $x \in X_p$, $\Phi(\mathcal{O}_x)$ is a sheaf. Hence,

$$\mathcal{P} \overset{\mathbf{L}}{\otimes} \mathcal{O}_{x \times Y} \cong \mathcal{P} \overset{\mathbf{L}}{\otimes} \mathcal{O}_{X_p \times Y} \overset{\mathbf{L}}{\otimes} p_1^*\mathcal{O}_{R_x},$$

is a sheaf. The above object is calculated by the spectral sequence:

$$E_2^{p,q} = \mathcal{T}or_{-p}^{\mathcal{O}_{X \times Y}}(H^q(A), p_1^*\mathcal{O}_{R_x}) \Rightarrow H^{p+q}(\mathcal{P} \overset{\mathbf{L}}{\otimes} \mathcal{O}_{x \times Y}).$$

Here $A := \mathcal{P} \overset{\mathbf{L}}{\otimes} \mathcal{O}_{X_p \times Y}$. The above spectral sequence degenerates at E_2-terms, as $E_2^{p,q} = 0$ for $p \leqslant -2$. Therefore, if $k \neq 0$, $H^k(A) \otimes p_1^*\mathcal{O}_{R_x} = 0$ for $x \in X_p$. This implies that $H^k(A) = 0$ for $k \neq 0$. $\square$

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As the immediate application, we generalize the theorem of Bondal and Orlov [BD01].

Theorem 6.2. *Let C be an elliptic curve and Z be a smooth projective variety. Assume that K_Z or $-K_Z$ is ample. Then $FM(C \times Z) = \{C \times Z\}$.*

Proof. We show the theorem when K_Z is ample. The other case follows similarly. Let us take $Y \in FM(C \times Z)$, and let $\Phi : D(C \times Z) \to D(Y)$ be an equivalence. As C is an elliptic curve, the projection $C \times Z \to Z$ gives Iitaka fibration. Note that K_Y is also semi-ample, and let $\pi : Y \to Z$ be its Iitaka fibration. Take a general closed point $p \in Z$ and fix it. Let $C^\dagger := \pi^{-1}(p)$. Then we can find an object $\mathcal{U} \in D(C \times C^\dagger)$ such that $\Phi^{\mathcal{U}}_{C \to C^\dagger} : D(C) \to D(C^\dagger)$ gives equivalence by Theorem 6.1. Note that $C \cong C^\dagger$, as Fourier–Mukai partners of a curve consists of itself. On the other hand, as in Lemma 4.2, the following functor gives equivalence:

$$\circ\mathcal{U} : D(C \times Z) \ni a \longmapsto a \circ \mathcal{U} \in D(C^\dagger \times Z).$$

Let us compose the above equivalence with $\Psi := \Phi^{-1}$. We obtain the equivalence:

$$(\circ\mathcal{U}) \circ \Psi : D(Y) \longrightarrow D(C \times Z) \longrightarrow D(C^\dagger \times Z),$$

which takes $\mathcal{O}_x$ to $\mathcal{O}_{(x,p)}$ for all $x \in C^\dagger$. Therefore, we obtain the birational map over Z by Lemma 7.3 below,

$$f : Y \dashrightarrow C^\dagger \times Z.$$

Note that f is defined on the neighborhood of $C^\dagger$. As Y and $C^\dagger \times Z$ are both minimal models, f is isomorphic in codimension one. We show that f is in fact an isomorphism. Let us take an ample divisor $H \subset Y$, and its strict transform $H^\dagger \subset C^\dagger \times Z$. It suffices to show that $H^\dagger$ is nef. However, this is clear as $H^\dagger$ is effective and we can deform $H^\dagger$ freely using translations of $C^\dagger$. □

7. Fourier–Mukai partners of 3-folds of $\kappa(X) = 2$

In this section, we study $FM(X)$ when $\dim X = 3$ and $\kappa(X) = 2$. The relative moduli spaces of stable sheaves for three-dimensional Calabi–Yau fibrations are studied in [BM02]. Combining Theorem 4.8 with their results, we can study $FM(X)$ in this case. Before that, we recall some terminology of birational geometry, and give some useful lemmas.

Definition 7.1. Let X and Y be projective varieties with only canonical singularities. A birational map $\alpha : X \dashrightarrow Y$ is called crepant, if there exists a smooth projective variety Z and birational morphisms $f : Z \to X$, $g : Z \to Y$, such that $\alpha \circ f = g$, and $f^*K_X = g^*K_Y$. In this case, we say that X and Y are K-equivalent under α.

The following birational transform called a 'flop' is a special kind of crepant birational map.

Definition 7.2. Let X and Y be projective varieties with only canonical singularities. A birational map $\alpha : X \dashrightarrow Y$ is called a flop, if there exist a normal projective variety W and crepant birational morphisms $\phi : X \to W$, $\psi : Y \to W$ that satisfy the following:

- $\phi = \psi \circ \alpha$;
- ϕ and ψ are isomorphisms in codimension one;
- relative Picard numbers of ϕ, ψ are one;
- let H be a ϕ-ample divisor on X and H' be its strict transform on Y; then $-H'$ is ψ-ample.

In this paper, we will only use flops of smooth 3-folds. In dimension three, crepant birational maps are connected by a finite number of flops [Kaw02, Theorem 4.6]. Next we give some useful lemmas.

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Lemma 7.3. *Let X and Y be smooth projective varieties, and $\Phi : D(X) \to D(Y)$ be an equivalence. Assume for some closed point $x \in X$, we have $\dim \operatorname{Supp} \Phi(\mathcal{O}_x) = 0$. Then there exists an open neighborhood U of x, and $r \in \mathbb{Z}$, such that for $x' \in U$, there exists $f(x') \in Y$ which satisfies $\Phi(\mathcal{O}_{x'}) = \mathcal{O}_{f(x')}[r]$. Moreover, X and Y are K-equivalent under birational map $f : X \dashrightarrow Y$.*

Proof. As Φ gives an equivalence, we have

$$\operatorname{Ext}^i_Y(\Phi(\mathcal{O}_x), \Phi(\mathcal{O}_x)) = \begin{cases} 0 & i < 0, \\ \mathbb{C} & i = 0. \end{cases}$$

Then using the same argument as in [BD01, Proposition 2.2], there exists a point $y \in Y$ and $r \in \mathbb{Z}$ such that $\Phi(\mathcal{O}_x) \cong \mathcal{O}_y[r]$. Then as in [BM01, Theorem 2.5], we can find a desired U and a birational map f. Let $\mathcal{P} \in D(X \times Y)$ be a kernel of Φ. As

$$\mathcal{P} \otimes p_1^* \omega_X \cong \mathcal{P} \otimes p_2^* \omega_Y,$$

as in § 4, $p_1^*\omega_X$ and $p_2^*\omega_Y$ are numerically equal on $\operatorname{Supp} \mathcal{P}$. By the construction of f, we have $\Gamma_f \subset \operatorname{Supp} \mathcal{P}$, where Γ_f is a graph of f. Therefore, X and Y are K-equivalent under f. □

Lemma 7.4. *Let X and Y be smooth projective varieties, and $\Phi : D(X) \to D(Y)$ gives equivalence. Then the following hold.*

(i) *For a closed point $x \in X$, ω_Y is numerically zero on $\operatorname{Supp} \Phi(\mathcal{O}_x)$.*

(ii) *If $x \in \operatorname{Bs} |mK_X|$, then $\operatorname{Supp} \Phi(\mathcal{O}_x) \subset \operatorname{Bs} |mK_Y|$.*

(iii) *If $x \notin \operatorname{Bs} |mK_X|$, then $\operatorname{Supp} \Phi(\mathcal{O}_x) \cap \operatorname{Bs} |mK_Y| = \emptyset$.*

Proof. (i) As Φ and the Serre functor commute, we have $\Phi(\mathcal{O}_x) \otimes \omega_Y \cong \Phi(\mathcal{O}_x)$. Part (i) follows from this.

(ii) This follows from Lemma 4.6 immediately.

(iii) Take $x \notin \operatorname{Bs} |mK_X|$ and assume that there exists $y \in \operatorname{Supp} \Phi(\mathcal{O}_x) \cap \operatorname{Bs} |mK_Y|$. Then there exists a non-zero map $\Phi(\mathcal{O}_x) \to \mathcal{O}_y[i]$ for some i. Therefore, there exists a non-zero map $\mathcal{O}_x \to \Psi(\mathcal{O}_y)[i]$. As $\Psi(\mathcal{O}_y)[i]$ is supported on $\operatorname{Bs} |mK_X|$, this is a contradiction. □

Now we state the main theorem of this section.

Theorem 7.5. *Let X be a smooth projective 3-fold of $\kappa(X) = 2$. Then $Y \in FM(X)$ if and only if one of the following holds.*

(1) *X and Y are connected by a finite number of flops.*

(2) *There exists the following diagram.*

$$\begin{tikzcd} Y \arrow[r, dotted, "\text{flops}"] & J^H(d) \arrow[dr, "\pi'"'] & & M \arrow[dl, "\pi"] & X \arrow[l, dotted, "\text{flops}"'] \\ & & S & & \end{tikzcd}$$

where $\pi : M \to S$ is an elliptic fibration with $\omega_M \equiv_\pi 0$, $H \in \operatorname{Pic}(M)$ is a polarization and $d \in \mathbb{Z}$. $J^H(d) \subset M^H(M/S)$ is an irreducible component which is fine, and contains a point corresponding to line bundles of degree d on smooth fibers of π.

The 'if' direction is already proved in [BM02, Theorem 8.3] and [Bri02], when S is smooth. We can check that the assumption 'S is smooth' is not required in their proof, hence the 'if' direction holds. We prove the 'only if' direction. Let us take $Y \in FM(X)$. We use the same notation as in the previous sections. In particular, $\Phi : D(X) \to D(Y)$ gives equivalence, $\mathcal{P} \in D(X \times Y)$ is a kernel of Φ, and Ψ is a quasi-inverse of Φ. Note that, by Lemma 7.3, we may assume that $\dim \operatorname{Supp} \Phi(\mathcal{O}_x) \geqslant 1$ for all $x \in X$. In this situation, we construct a diagram (2) or show that (1) holds. We divide the proof into five steps.

Step 1. Application of Theorem 4.8

First, we apply Theorem 4.8, and give the preparation for the proof. As $\dim X = \dim Y = 3$, we can run minimal model programs and obtain birational minimal models $X_{\min}$ and $Y_{\min}$.

$$\begin{tikzcd} X \arrow[r, dotted, "\text{MMP}"] & X_{\min} \arrow[dr, "\pi_X"'] & & Y_{\min} \arrow[dl, "\pi_Y"] & Y \arrow[l, dotted, "\text{MMP}"'] \\ & & Z & & \end{tikzcd}$$

Here π_X, π_Y are Iitaka fibrations. Note that $\dim Z = 2$, and generic fibers of π_X, π_Y are elliptic curves. Then, for sufficiently large m, we obtain isomorphisms,

$$X \setminus \operatorname{Bs}|mK_X| \xrightarrow{\cong} X_{\min} \setminus C_X, \quad Y \setminus \operatorname{Bs}|mK_Y| \xrightarrow{\cong} Y_{\min} \setminus C_Y,$$

for some closed subsets $C_X \subset X_{\min}$, $C_Y \subset Y_{\min}$ with $\dim C_X \leqslant 1$, $\dim C_Y \leqslant 1$. Let us take general members $E_i \in |mK_X|$, for $i = 1, 2$. By Corollary 4.4, we have the isomorphism of linear systems:

$$|mK_X| \xrightarrow{\sim} |mK_Y|.$$

Let $E_i^{\dagger} \in |mK_Y|$ correspond to E_i. Also note that we have the isomorphisms:

$$H^0(X, mK_X) \cong H^0(X_{\min}, mK_{X_{\min}}), \quad H^0(X, mK_Y) \cong H^0(Y_{\min}, mK_{Y_{\min}}),$$

for sufficiently divisible m. Let

$$E_i' \in |mK_{X_{\min}}|, \quad E_i'^{\dagger} \in |mK_{Y_{\min}}|,$$

correspond to E_i, $E_i^{\dagger}$ under the above isomorphisms, respectively. Then, if we choose E_i sufficiently general, then we have

$$E_1' \cap E_2' \cap C_X = \emptyset.$$

Therefore we have the following decompositions:

$$E_1 \cap E_2 = (E_1' \cap E_2') \coprod \operatorname{Bs}|mK_X|, \quad E_1^{\dagger} \cap E_2^{\dagger} = (E_1'^{\dagger} \cap E_2'^{\dagger}) \coprod \operatorname{Bs}|mK_Y|.$$

Now let us take $C \in \pi_0(E_1' \cap E_2')$. We can consider C as a curve on X. Using Corollary 4.7 and Lemma 7.4, we can find $C^{\dagger} \in \pi_0(E_1'^{\dagger} \cap E_2'^{\dagger})$ such that Φ takes $D_C(X)$ to $D_{C^{\dagger}}(Y)$. Now using the same argument as in Theorem 6.1, we can see $\mathcal{P} \overset{\mathbf{L}}{\otimes} \mathcal{O}_{C \times Y}$ is a sheaf, up to shift. Then we can apply Theorem 4.8, so there exists an equivalence $\Phi_C : D(C) \to D(C^{\dagger})$, such that the diagram of Theorem 4.8 commutes.

Step 2. Construction of M

In this step, we construct a desired M. We construct M as a moduli space of stable sheaves on Y. Let us take $x \in C$, and consider $\Phi_C(\mathcal{O}_x) \in D(C^{\dagger})$. As in Theorem 6.1, we may assume that $\Phi_C(\mathcal{O}_x)$ is a simple sheaf on $C^{\dagger}$. As $C^{\dagger}$ is an elliptic curve, $\Phi_C(\mathcal{O}_x)$ is a stable sheaf on $C^{\dagger}$. Let $\operatorname{rk} \Phi_C(\mathcal{O}_x) = a$ and $\deg \Phi_C(\mathcal{O}_x) = b$. By the commutative diagram of Theorem 4.8, $\Phi(\mathcal{O}_x)$ is a stable sheaf on Y supported on $C^{\dagger}$, with respect to any polarization. Then take a polarization $H' \in \operatorname{Pic}(Y)$, and consider the moduli space of stable sheaves $M^{H'}(Y/\operatorname{Spec}\mathbb{C})$. Let

$$M \subset M^{H'}(Y/\operatorname{Spec}\mathbb{C})$$

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be an irreducible component, which contains a point corresponding to $\Phi(\mathcal{O}_x) \in \mathrm{Coh}(Y)$. Note that there exists a birational map:

$$f_1 : X \dashrightarrow M,$$

which takes a general point $x \in X$ to a point of M, corresponding to a stable sheaf $\Phi(\mathcal{O}_x)$. We show that M is a fine moduli scheme or (1) holds. For $E, F \in D(X)$, we define $\chi(E, F)$ as follows:

$$\chi(E, F) := \sum (-1)^i \dim \mathrm{Ext}^i_X(E, F).$$

As $\chi(\Phi(\mathcal{O}_X), \Phi(\mathcal{O}_x)) = \chi(\mathcal{O}_X, \mathcal{O}_x) = 1$, Riemann–Roch imply that

$$b \cdot \mathrm{ch}_0 \, \Phi(\mathcal{O}_X)^* + a(c_1(\Phi(\mathcal{O}_X)^*) \cdot C^\dagger) = 1.$$

Here $\Phi(\mathcal{O}_X)^*$ means that derived dual of $\Phi(\mathcal{O}_X)$. We divide into two cases.

Case 1. $b = 0$. If $b = 0$, then $a = c_1(\Phi(\mathcal{O}_X)^*) \cdot C^\dagger = 1$. Therefore, there exists an effective divisor E on Y such that $E \cdot C^\dagger = 1$. There exists a birational map

$$f_2 : Y \dashrightarrow M,$$

such that f_2 takes the general point $y \in Y$ to a point corresponding to $\mathcal{O}_{C_y}(E \cap C_y - y)$, a degree zero line bundle on C_y. Here C_y is a compact fiber of the Iitaka fibration $Y \dashrightarrow Z$, which contains y. Composing these we obtain a birational map,

$$f := f_2^{-1} \circ f_1 : X \dashrightarrow M \dashrightarrow Y,$$

which satisfies $f(x) \in \mathrm{Supp}\, \Phi(\mathcal{O}_x)$ for general $x \in X$. Therefore, $\Gamma_f \subset \mathrm{Supp}\, \mathcal{P}$, where Γ_f is a graph of f. As $p_1^* K_X \equiv p_2^* K_Y$ on $\mathrm{Supp}\, \mathcal{P}$, it is also true on Γ_f. Therefore, X and Y are K-equivalent under birational map f.

Case 2. $b \neq 0$. Let us replace H' to $\det \Phi(\mathcal{O}_X)^* \pm lbH'$ for $l \gg 0$. Then we may assume $\gcd(a(H' \cdot C^\dagger), b) = 1$. Then

$$\begin{aligned} \gcd\{\chi(\Phi(\mathcal{O}_x) \otimes H'^{\otimes m}) \mid m \in \mathbb{Z}\} &= \gcd\{ma(H' \cdot C^\dagger) + b \mid m \in \mathbb{Z}\} \\ &= 1. \end{aligned}$$

By Lemma 3.4, this implies that M is a fine moduli scheme.

Step 3

This scheme M is smooth and the universal sheaf $\mathcal{U} \in \mathrm{Coh}(Y \times M)$ gives an equivalence

$$\Phi_M := \Phi^{\mathcal{U}}_{M \to Y} : D(M) \longrightarrow D(Y).$$

Proof. For $p \in M$, let $\mathcal{U}_p \in \mathrm{Coh}(Y)$ be the corresponding stable sheaf. We check that the conditions of Theorem 3.5 are satisfied. First we show $\mathcal{U}_p \otimes \omega_Y \cong \mathcal{U}_p$. Let

$$\begin{tikzcd} \widetilde{X} \arrow[d, "g"'] \arrow[dr, "h"] & \\ X \arrow[r, dotted, "f_1"] & M \end{tikzcd}$$

be an elimination of indeterminacy. Consider morphisms

$$g \times \mathrm{id} : \widetilde{X} \times Y \longrightarrow X \times Y, \quad h \times \mathrm{id} : \widetilde{X} \times Y \longrightarrow M \times Y,$$

and objects

$$\mathbf{L}(g \times \mathrm{id})^* \mathcal{P} \in D(\widetilde{X} \times Y), \quad (h \times \mathrm{id})^* \mathcal{U} \in \mathrm{Coh}(\widetilde{X} \times Y).$$

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Take $x \in \widetilde{X}$ and let $i_{x\times Y} : x \times Y \hookrightarrow X \times Y$ be an inclusion. Then

$$\mathbf{L}i^*_{x\times Y} \circ \mathbf{L}(g \times \mathrm{id})^*\mathcal{P} = \mathbf{L}i^*_{g(x)\times Y}\mathcal{P} = \Phi(\mathcal{O}_{g(x)})$$

$$\mathbf{L}i^*_{x\times Y} \circ (h \times \mathrm{id})^*\mathcal{U} = \mathcal{U}_{h(x)}.$$

Take open subsets $X^0 \subset X$, $Y^0 \subset Y$, $Z^0 \subset Z$ such that the rational maps $X \dashrightarrow Z$, $Y \dashrightarrow Z$ are defined on X^0, Y^0, and $X^0 \to Z^0$, $Y^0 \to Z^0$ are smooth projective. From here, we shrink Z^0 if necessary. As f_1 is defined on X^0, we can think X^0 as an open subset of $\widetilde{X}$. So if $x \in X^0 \subset \widetilde{X}$, then $\Phi(\mathcal{O}_{g(x)}) = \mathcal{U}_{h(x)}$. This implies that

$$\mathrm{Supp}(h \times \mathrm{id})^*\mathcal{U} \cap (X^0 \times Y) = \mathrm{Supp}\, \mathbf{L}(g \times \mathrm{id})^*\mathcal{P} \cap (X^0 \times Y).$$

Therefore, by Lemma 7.6 below, we have

$$\mathrm{Supp}(h \times \mathrm{id})^*\mathcal{U} \subset \mathrm{Supp}\, \mathbf{L}(g \times \mathrm{id})^*\mathcal{P} \subset \widetilde{X} \times Y.$$

Therefore, for all $x \in \widetilde{X}$, we have

$$\mathrm{Supp}(h \times \mathrm{id})^*\mathcal{U} \cap (x \times Y) \subset \mathrm{Supp}\, \mathbf{L}(g \times \mathrm{id})^*\mathcal{P} \cap (x \times Y).$$

So $\mathrm{Supp}\,\mathcal{U}_{h(x)} \subset \mathrm{Supp}\,\Phi(\mathcal{O}_{g(x)})$ follows. As ω_Y is numerically zero on $\mathrm{Supp}\,\Phi(\mathcal{O}_{g(x)})$, this is also true on $\mathrm{Supp}\,\mathcal{U}_{h(x)}$, hence on $\mathrm{Supp}\,\mathcal{U}_p$ for all $p \in M$. Therefore, $\mathcal{U}_p \otimes \omega_Y$ is also H'-stable and its reduced Hilbert polynomial is equal to $\mathcal{U}_p$, i.e.

$$p(\mathcal{U}_p, H') = p(\mathcal{U}_p \otimes \omega_Y, H').$$

On the other hand, there exists a non-trivial map $\mathcal{U}_p \to \mathcal{U}_p \otimes \omega_Y$ by semi-continuity. So $\mathcal{U}_p \cong \mathcal{U}_p \otimes \omega_Y$ for all $p \in M$.

Secondly, we show that the set

$$\Gamma(\mathcal{U}) := \{(p_1, p_2) \in M \times M \mid \mathrm{Ext}^i_Y(\mathcal{U}_{p_1}, \mathcal{U}_{p_2}) \neq 0 \quad \text{for some } i \in \mathbb{Z}\}$$

has $\dim \Gamma(\mathcal{U}) \leqslant 4$. It suffices to show that if $(p_1, p_2) \in \Gamma(\mathcal{U}) \setminus \Delta_M$, where Δ_M is a diagonal, then $p_i \in M \setminus f_1(X^0)$. Assume that $p_1 \in f_1(X^0)$. As $\mathrm{Ext}^i_Y(\mathcal{U}_{p_1}, \mathcal{U}_{p_2}) \neq 0$, we have $\mathrm{Supp}\,\mathcal{U}_{p_1} \cap \mathrm{Supp}\,\mathcal{U}_{p_2} \neq \emptyset$. Take an irreducible component $l \subset \mathrm{Supp}\,\mathcal{U}_{p_2}$ such that $\mathrm{Supp}\,\mathcal{U}_{p_1} \cap l \neq \emptyset$. As we have assumed $p_1 \in f_1(X^0)$, we have

$$\mathrm{Supp}\,\mathcal{U}_{p_1} \cap \mathrm{Bs}\,|mK_Y| = \emptyset.$$

So it follows that l is not contained in $\mathrm{Bs}\,|mK_Y|$. Furthermore, $K_Y \cdot l = 0$, as K_Y is numerically zero on $\mathrm{Supp}\,\mathcal{U}_p$. Therefore, $l \cap \mathrm{Bs}\,|mK_Y| = \emptyset$ and l is contained in the fiber of the Iitaka fibration, $Y \setminus \mathrm{Bs}\,|mK_Y| \to Z$. This implies that $l = \mathrm{Supp}\,\mathcal{U}_{p_1}$ and therefore $\mathrm{Supp}\,\mathcal{U}_{p_2} = \mathrm{Supp}\,\mathcal{U}_{p_1}$, as $\mathrm{Supp}\,\mathcal{U}_{p_2}$ is connected. Therefore, $\mathcal{U}_{p_2}$ is a stable sheaf on $\mathrm{Supp}\,\mathcal{U}_{p_1}$, so $p_2 \in f_1(X^0)$. Let $q_i \in X^0$ be points such that $p_i = f_1(q_i)$. Then $\mathrm{Ext}^i_Y(\mathcal{U}_{p_1}, \mathcal{U}_{p_2}) = \mathrm{Ext}^i_X(\mathcal{O}_{q_1}, \mathcal{O}_{q_2}) \neq 0$ implies that $q_1 = q_2$ and $p_1 = p_2$. However, this contradicts to $(p_1, p_2) \notin \Delta_M$. $\square$

In the above proof, we used the following lemma.

Lemma 7.6. *The support of* $(h \times \mathrm{id})^*\mathcal{U}$, $\mathrm{Supp}(h \times \mathrm{id})^*\mathcal{U}$, *is irreducible.*

Proof. Let $\widetilde{f} : \widetilde{X} \times Y \to \widetilde{X}$ be a projection. Note that a general fiber of the restriction of $\widetilde{f}$ to $\mathrm{Supp}(h \times \mathrm{id})^*\mathcal{U}$ is an elliptic curve. Therefore if $\mathrm{Supp}(h \times \mathrm{id})^*\mathcal{U}$ is not irreducible, then there exists $p \in \mathrm{Ass}((h \times \mathrm{id})^*\mathcal{U})$ such that $\dim \mathcal{O}_{\widetilde{X},\widetilde{f}(p)} \geqslant 1$. Take a non-zero element of the maximal ideal $t \in m_{\widetilde{f}(p)} \subset \mathcal{O}_{\widetilde{X},\widetilde{f}(p)}$. Then $\mathcal{O}_{\widetilde{X},\widetilde{f}(p)} \stackrel{\times t}{\to} \mathcal{O}_{\widetilde{X},\widetilde{f}(p)}$ is injective. Since $(h \times \mathrm{id})^*\mathcal{U}$ is flat over $\widetilde{X}$, we have an injection,

$$((h \times \mathrm{id})^*\mathcal{U})_p \xrightarrow{\times \widetilde{f}^* t} ((h \times \mathrm{id})^*\mathcal{U})_p,$$

and $\widetilde{f}^* t \in m_p \mathcal{O}_{\widetilde{X}\times Y,p}$. But this contradicts the statement $p \in \mathrm{Ass}((h \times \mathrm{id})^*\mathcal{U})$. $\square$

Step 4

The schemes X and M are connected by finite number of flops and M has an elliptic fibration $\pi : M \to S$ with $\omega_M \equiv_\pi 0$.

Proof. Consider the following composition:

$$\Psi \circ \Phi_M : D(M) \longrightarrow D(Y) \longrightarrow D(X).$$

This is an equivalence and for general points $p \in M$, we have

$$\dim \mathrm{Supp}\, \Psi \circ \Phi_M(\mathcal{O}_p) = 0.$$

Therefore, X and M are connected by a finite number of flops. As $\mathrm{Supp}\mathcal{U} \subset Y \times M$ is irreducible and all of the fibers of the projection $\mathrm{Supp}\mathcal{U} \to M$ are one-dimensional, this is a well-defined family of proper algebraic cycles in the sense of [JK96]. Therefore, there exists a morphism $M \to \mathrm{Chow}(Y)$, which takes $p \in M$ to an algebraic cycle whose support is equal to $\mathrm{Supp}\mathcal{U}_p$. Let

$$M \xrightarrow{\pi} S \longrightarrow \mathrm{Chow}(Y)$$

be a Stein factorization. We show that $\omega_M \equiv_\pi 0$. Let us take $p, p' \in M$ such that $\pi(p) = \pi'(p)$. Then by the definition of π, it follows that

$$\mathrm{Supp}\, \Phi_M(\mathcal{O}_p) = \mathrm{Supp}\, \Phi_M(\mathcal{O}_{p'}).$$

Let us take $q \in \mathrm{Supp}\, \Phi_M(\mathcal{O}_p)$. Then $p' \in \mathrm{Supp}(\Phi_M)^{-1}(\mathcal{O}_q)$. Therefore, $\pi^{-1}\pi(p) \subset \mathrm{Supp}(\Phi_M)^{-1}(\mathcal{O}_q)$. This implies that $\omega_M \equiv_\pi 0$. $\square$

Step 5

There exists a polarization $H \subset M$, $d \in \mathbb{Z}$, such that $J^H(d)$ is fine and smooth. Moreover, Y and $J^H(d)$ are connected by a finite number of flops.

Proof. We continue the same argument. Let us take a general closed point $y \in Y$. The object

$$(\Phi_M)^{-1}(\mathcal{O}_y) \in D(M)$$

is a stable sheaf on a general fiber of π. Let its rank and degree be c and d, respectively. Let $H \in \mathrm{Pic}(M)$ be a polarization and take an irreducible component $M^+ \subset M^H(M/S)$ that contains a point corresponding to $(\Phi_M)^{-1}(\mathcal{O}_y)$. Similarly, take an irreducible component $J^H(d) \subset M^H(M/S)$ that contains a point corresponding to line bundles of degree d on smooth fibers of π. By the same argument as before, we can choose H such that

$$\pi'' : M^+ \longrightarrow S, \quad \pi' : J^H(d) \longrightarrow S,$$

are fine moduli schemes (or X and Y are connected by finite number of flops if $d = 0$). By [BM02, Theorem 8.3], M^+ and $J^H(d)$ are smooth, $\omega_{M^+} \equiv_{\pi''} 0$, $\omega_{J^H(d)} \equiv_{\pi'} 0$, and the universal sheaf $\mathcal{V} \in \mathrm{Coh}(M^+ \times_S M)$ gives an equivalence

$$\Phi_{M^+} := \Phi^{\mathcal{V}}_{M^+ \to M} : D(M^+) \xrightarrow{\sim} D(M).$$

As the composition

$$\Phi_M \circ \Phi_{M^+} : D(M^+) \xrightarrow{\sim} D(M) \xrightarrow{\sim} D(Y)$$

takes general points to general points, Y and M^+ are connected by a finite number of flops. By [Ati57, Theorem 6], there exists the following birational map over S:

$$M^+ \ni E \longmapsto \wedge^c E \in J^H(d).$$

As they are both minimal over S, $M^\dagger$ and $J^H(d)$ are connected by a finite number of flops. Now we have obtained the diagram (2). $\square$

If X is minimal we have a better result. By the abundance theorem in dimension three, K_X is semi-ample. Let $\pi_X : X \to Z$ be its Iitaka fibration. We define $\lambda_X > 0$ as follows:

$$\lambda_X := \gcd\{c_1(E) \cdot f_X \mid E \in D(X)\},$$

where f_X is a cohomology class of a general fiber of π_X. For a polarization H on X, let $J^H(b) \subset M^H(X/Z)$ be as in the Theorem 7.5. The proof of the following theorem is almost the same as in the previous theorem and is left to the reader.

Theorem 7.7. *Let X be a smooth minimal 3-fold with $\kappa(X) = 2$. Then $Y \in FM(X)$ if and only if there exists some $b \in \mathbb{Z}$ that is co-prime to λ_X, and there exists a polarization H on X, for which $J^H(b)$ is a fine moduli scheme, such that Y and $J^H(b)$ are connected by finite number of flops.*

As $J^H(b + \lambda_X) \cong J^H(b)$, birational classes of $FM(X)$ are finite in the above case. By [Kaw97], the number of three-dimensional minimal model in a fixed birational class is finite. So we obtain the following corollary.

Corollary 7.8. *Let X be a smooth minimal 3-fold with $\kappa(X) = 2$. Then $\sharp FM(X) < \infty$.*

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