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ON AN INTEGRAL OPERATOR FOR CONVEX UNIVALENT FUNCTIONS

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Let $K(m,M)$ denote the class of functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ regular and satisfying $|1 + zf''(z)/f'(z) - m| < M$ in $|z| < 1$, where $|m-1| < M \le m$. Recently, R.K. Pandey and G. P. Bhargava have shown that if $f \in K(m,M)$, then the function $F(z) = \int_0^z \{f'(u)\}^\alpha du$ also belongs to $K(m,M)$ provided α is a complex number satisfying the inequality $|\alpha| \le (1-b)/2$, where $b = (m-1)/M$. In this paper we show by a counterexample that their inequality is in general wrong, and prove a corrected version of their result. We show that $F \in K(m,M)$ provided that α is a real number satisfying $-\phi \le \alpha \le 1$, $\phi = (M-|m-1|)/(M + |m-1|)$, or a complex number satisfying $|\alpha| \le \phi$. In both cases the bounds for α are sharp.

1. Introduction

Let S denote the class of functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ which

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are regular in the unit disc $U = \{z : |z| < 1\}$. A function f of S belongs to the class $S(m,M)$ if $|zf'(z)/f(z) - m| < M$ for $z \in U$, where $|m-1| < M \le m$. A function f of S belongs to the class $K(m,M)$ if $|1 + zf''(z)/f'(z) - m| < M$ for $z \in U$. Evidently, the functions in $S(m,M)$ and $K(m,M)$ are starlike and convex univalent respectively. These classes were introduced by Jakubowski [1],[2].

Recently, Kumar and Shukla [4] have studied class preserving integral operators for $S(m,M)$, and called them 'Jakubowski starlike integral operators'. It is now natural to consider a similar problem for the class $K(m,M)$. We define:

An integral operator which maps the class $K(m,M)$ into or onto itself is called a Jakubowski convex integral operator.

The object of this paper is to show that, for suitable choices of constant α , the integral operator

$$I(f) = \int_0^z \{f'(u)\}^{\alpha}\, du$$

maps $K(m,M)$ into itself. One of our results improves and corrects a recent result of Pandey and Bhargava [5].

2. Fundamental Lemmas.

In this section we prove two lemmas which play an important role in establishing a theorem concerning the Jakubowski convex integral operators. The first one is equivalent to Lemma 2.3 in [4] when $\beta = 1$. However we present the details of the proof since we require it in the discussion which follows.

LEMMA 2.1. *Let* α, m *and* M *be real numbers such that* $0 < \alpha \le 1$ *and* $|m-1| < M \le m$. *If*

(2.1) $$t = 1-\alpha + \alpha m \quad \text{and} \quad T = \alpha M ,$$

then $K(t,T) \subseteq K(m,M)$.

Proof. It suffices to show that

(2.2) $$m-M \le t-T \quad \text{and} \quad t+T \le m+M .$$

We need consider only the case when $0 < \alpha < 1$. Suppose $m-M > t-T$. Then, $m-M > 1-\alpha + \alpha(m-M)$, which implies that $m-M > 1$. But this is contrary to the assumption $|m-1| < M$. Next, suppose $t+T > m+M$.

Then, $1-\alpha+\alpha(m+M) > m+M$, which implies $m+M < 1$. This is also contrary to $|m-1| < M$. Therefore the inequalities (2.2) hold.

LEMMA 2.2 *Let* α, m *and* M *be real numbers such that*

$$\max\left\{-\frac{1-(m-M)}{m+M-1}, -\frac{m+M-1}{1-(m-M)}\right\} \le \alpha < 0 \tag{2.3}$$

and $|m-1| < M \le m$. *If*

$$\tilde{t} = 1-\alpha+\alpha m \quad \text{and} \quad \tilde{T} = -\alpha M, \tag{2.4}$$

then $K(\tilde{t},\tilde{T}) \subseteq K(m,M)$.

Proof. We need to show that

$$m-M \le \tilde{t}-\tilde{T} \quad \text{and} \quad \tilde{t}+\tilde{T} \le m+M. \tag{2.5}$$

Suppose $m-M > \tilde{t}-\tilde{T}$. Then, $m-M > 1-\alpha+\alpha(m+M)$, which implies that $\alpha < -(1-(m-M))/(m+M-1)$. Further, suppose $\tilde{t}+\tilde{T} > m+M$. Then, $1-\alpha+\alpha(m-M) > m+M$, which implies $\alpha < -(m+M-1)/(1-(m-M))$.
Now if

$$\alpha < \max\left\{-\frac{1-(m-M)}{m+M-1}, -\frac{m+M-1}{1-(m-M)}\right\},$$

then at least one of the inequalities in (2.5) does not hold. Whence the disc centred at $\tilde{t}$ and having radius $\tilde{T}$ is not contained in the disc centred at m and having radius M. Therefore $K(\tilde{t},\tilde{T}) \subseteq K(m,M)$ if α satisfies (2.3).

It is noticeable that the truth of $|\tilde{t}-1| < \tilde{T} \le \tilde{t}$ requires $\alpha \ge -1/m+M-1)$, which automatically holds since

$$\begin{aligned}\alpha &\ge \max\left\{-\frac{1-(m-M)}{m+M-1}, -\frac{m+M-1}{1-(m-M)}\right\}\\ &\ge -(1-(m-M))/(m+M-1)\\ &\ge -1/(m+M-1).\end{aligned}$$

NOTE. From now on t, T and $\tilde{t}, \tilde{T}$ will be as in (2.1) and (2.4) respectively.

3. Jakubowski Convex Integral Operators.

In this section we obtain the main results of this paper.

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THEOREM 3.1. *If* $f \in K(m,M)$, *then the function* F , *defined by*

$$(3.1) \qquad F(z) = \int_0^z \{f'(u)\}^\alpha \, du ,$$

also belongs to $K(m,M)$, *provided*

$$(3.2) \qquad -\phi \le \alpha \le 1$$

where $\phi = (M-|m-1|)/M+|m-1|)$.

The result is sharp.

Proof. From (3.1) we have

$$F'(z) = \{f'(z)\}^\alpha .$$

Logarithmic differentiation yields

$$1+z\frac{F''(z)}{F'(z)} = \alpha\,\{1+z\frac{f''(z)}{f'(z)}\} + (1-\alpha) .$$

Therefore

$$(3.3) \qquad 1+z\frac{F''(z)}{F'(z)} - (1-\alpha+\alpha m) = \alpha\{1+z\frac{f''(z)}{f'(z)} - m\} .$$

Now consider two cases, namely $\alpha > 0$ and $\alpha < 0$.

Case I. When $\alpha > 0$, it follows from (3.3) that

$$\left|1+z\frac{F''(z)}{F'(z)} - (1-\alpha+\alpha m)\right| = \alpha\left|1+z\frac{f''(z)}{f'(z)} - m\right| < \alpha M, \text{ since } f \in K(m,M) .$$

Thus $F \in K(t,T)$, and hence, by Lemma 2.1, $F \in K(m,M)$ provided

$$(3.4) \qquad 0 < \alpha \le 1 .$$

Case II. When $\alpha < 0$, from (3.3) we get

$$\left|1+z\frac{F''(z)}{F'(z)} - (1-\alpha+\alpha m)\right| = -\alpha\left|1+z\frac{f''(z)}{f'(z)} - m\right| < -\alpha M .$$

Thus $F \in K(\tilde{t},\tilde{T})$, and hence, by Lemma 2.2, $F \in K(m,M)$ provided

$$(3.5) \qquad \max\left\{-\frac{1-(m-M)}{m+M-1}, -\frac{m+M-1}{1-(m-M)}\right\} \le \alpha < 0 .$$

It is now easy to compute that

$$\max\left\{-\frac{1-(m-M)}{m+M-1}, -\frac{m+M-1}{1-(m-M)}\right\} = \begin{cases} -\frac{1-(m-M)}{m+M-1}, & \text{when } m > 1 \\ -1, & \text{when } m = 1 \\ -\frac{m+M-1}{1-(M-M)}, & \text{when } m < 1 . \end{cases}$$

The expressions on the right hand side are easily seen to be equal to $-\phi$.

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Hence, combining (3.4), (3.5) and the trivial case $\alpha = 0$, we conclude that $F \in K(m,M)$ if α satisfies (3.2).

To establish the sharpness of the result we take

$$f(z) = \begin{cases} \int_0^z (1-bu)^{-(a+b)/b}du, & \text{when } m \neq 1, \\ \int_0^z e^{Mu}\, du, & \text{when } m = 1, \end{cases}$$

where

(3.6) $$a = \frac{M^2-m^2+m}{M} \text{ and } b = \frac{m-1}{M}.$$

It is easy to see that $1+zf''(z)/f'(z)$ maps U onto the disc centred at m and having radius M. It follows then from (3.3) that $1+zF''(z)/F'(z)$ maps U onto the disc centred at t and having radius T, when $\alpha > 0$. Now if $\alpha > 1$, none of the inequalities in (2.2) holds, and consequently the disc centred at t and having radius T is not contained in the disc centred at m and having radius M. Therefore $F \notin K(m,M)$ if $\alpha > 1$. Similarly we can show that $F \notin K(m,M)$ if $\alpha < -\phi$. Hence the result is sharp.

In the particular case when $m = M$ and $m \to \infty$, $K(m,M)$ is equal to the well known class K of convex functions. Hence the following result of Kim and Merkes [3] follows from Theorem 3.1.

COROLLARY 3.1. If $f \in K$, then the function F defined by (3.1) also belongs to K, provided $0 \leq \alpha \leq 1$. The result is sharp with the extremal function $f(z) = \int_0^z (1-u)^{-2}du$.

Very recently, Pandey and Bhargava [5] have shown that if $f \in K(m,M)$, then the function F defined by (3.1) also belongs to $K(m,M)$ provided α is a complex number such that $|\alpha| \leq (1-b)/2$, where b is given by (3.6). We show below that their result is incorrect:

Let us consider the function

(3.7) $$f(z) = \int_0^z (1-bu)^{-(a+b)/b}du$$

with $m = .7$ and $M = .6$, so that $a = .95$ and $b = -.5$. Evidently, $f \in K(.7,.6)$ and $(1-b)/2 = .75$. Now, by the abovementioned result of Pandey and Bhargava, $F \in K(.7, .6)$ provided $|\alpha| \leq .75$. But, if we take $\alpha = -.75$, then from (3.1) and (3.7) we are led to

$$\left|1+z\frac{F''(z)}{F'(z)} - .7\right| = \left|-.75\left\{\frac{1+.95z}{1+.5z}\right\} + 1.05\right| ;$$

and at $z = -.8$, the right hand side of this equation is equal to $.75$. Thus we see that

$$\left| 1+z\frac{F''(z)}{F'(z)} -.7 \right| > .6$$

at some point in the unit disc U . Therefore $F \notin K(.7,.6)$. Hence the result of Pandey and Bhargava [5] is incorrect.

We now proceed to improve and correct the result of Pandey and Bhargava [5] . First we prove:

THEOREM 3.2. *If $f \in K(m,M)$, then the function F defined by (3.1) also belongs to $K(m,M)$ provided α is a complex number satisfying*

(3.8) $$|\alpha| < 1 , \frac{|1-\alpha|}{1-|\alpha|} \le \frac{M}{|m-1|} = \frac{1+\phi}{1-\phi} , \text{ when } m \neq 1$$

and

(3.9) $$|\alpha| \le 1 , \text{ when } m = 1 ,$$

where $\phi = (M-|m-1|)/(M+|m-1|)$.

Proof. We have (as in Theorem 3.1)

$$1+z\frac{F''(z)}{F'(z)} = \alpha\left\{1+z\frac{f''(z)}{f'(z)} - m\right\} + (1-\alpha) .$$

Therefore, when $m \neq 1$, we obtain

$$\begin{aligned}\left|1+z\frac{F''(z)}{F'(z)} -m\right| &= \left|\alpha\left\{1+z\frac{f''(z)}{f'(z)} -m\right\} -(1-\alpha)(m-1)\right| \\ &\le |\alpha|\left|1+z\frac{f''(z)}{f'(z)} -m\right| + |1-\alpha||m-1| \\ &< |\alpha|M+|1-\alpha||m-1| \\ &\le M ,\end{aligned}$$

provided

$$|\alpha| < 1 \text{ and } \frac{|1-\alpha|}{1-|\alpha|} \le \frac{M}{|m-1|} = \frac{1+\phi}{1-\phi} .$$

Further, when $m=1$, we obtain

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$$\left|1+z\frac{f''(z)}{f'(z)}-1\right| = \left|\alpha\{1+z\frac{f''(z)}{f'(z)}-1\}\right| ,$$
$$= |\alpha|\left|1+z\frac{f''(z)}{f'(z)}-1\right| ,$$
$$< |\alpha| M ,$$
$$\le M ,$$

provided $|\alpha| \le 1$. This completes the proof of the theorem.

Using the above theorem we now establish the following corollary which improves and corrects the result of Pandey and Bhargava [5]. It is worth noting that the technique employed by us is different; and the result is sharp also.

COROLLARY 3.2. If $f \in K(m,M)$, then the function F defined by (3.1) also belongs to $K(m,M)$ provided α is a complex number satisfying

$$|\alpha| \le \phi . \tag{3.10}$$

The result is sharp in the sense that the region given by (3.10) cannot be extended into any larger disc centred at the origin on the α-plane.

Proof. When $m=1$, the inequality (3.10) is identical with the inequality (3.9). We need therefore consider only the case when $m \ne 1$. In this case, from (3.8), we have

$$\frac{|1-\alpha|}{1-|\alpha|} \le \frac{1+\phi}{1-\phi} .$$

This inequality is implied by

$$\frac{1+|\alpha|}{1-|\alpha|} \le \frac{1+\phi}{1-\phi} ,$$

which is equivalent to

$$|\alpha| \le \phi .$$

Hence, by Theorem 3.2, $F \in K(m,M)$ if $|\alpha| \le \phi$.

To show the sharpness of the inequality (3.10), let us assume that $F \in K(m,M)$ for all values of α lying in a larger disc $|\alpha| \le \phi+s$, for some $s > 0$. Then, in particular, $F \in K(m,M)$ for all values of α lying in the annulus $\phi < |\alpha| \le \phi+s$. From this it follows that when α is real and negative, then $F \in K(m,M)$ even if $-(\phi+s) \le \alpha < -\phi$. But this is contrary to the sharpness of $\alpha \geq -\phi$ in (3.2). Therefore the region given by (3.10) cannot be extended into any larger disc centred at the origin on the α-plane. Hence the result is sharp.

Remark. We could not establish the sharpness of Theorem 3.2 when $m \neq 1$. However the result is sharp for $m=1$ (as shown in Corollary 3.2). Therefore, it would be interesting to establish the sharpness of the inequality (3.8).

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